Lecture 1B: jeu de taquin

Piotr Śniady

Polska Akademia Nauk

infinite version of RSK

Input:

- infinite word $\mathbf{w} = (w_1, w_2, \ldots)$

Output:

- ~~semistandard tableau P~~
- infinite standard tableau $Q \in \mathcal{T}$;

$\mathcal{T}$ denotes the set of infinite standard tableaux

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example:
$\mathbf{w} = (23,\ 53,\ 74,\ 16,\ \ldots)$

	$\vdots$		
8	9	12	
4	6	7	$\cdots$
1	2	3	

recording tableau $Q(\mathbf{w})$

infinite version of RSK

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	⋮		
8	9	12	
4	6	7	⋯
1	2	3	

recording tableau $Q(\mathbf{w})$

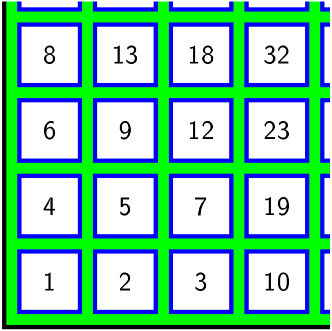
definition

if $w_1, w_1, \dots$ are iid $U(0,1)$ random variables then $Q(w_1, w_1, \dots) \overset{\text{distribution}}{=}$ Plancherel measure on $\mathcal{T}$

8	13	18	32
6	9	12	23
4	5	7	19
1	2	3	10

jeu de taquin

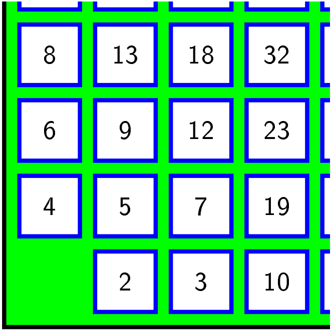
① start with $t \in \mathcal{T}$,



jeu de taquin

① start with $t \in \mathcal{T}$,

② remove corner box,



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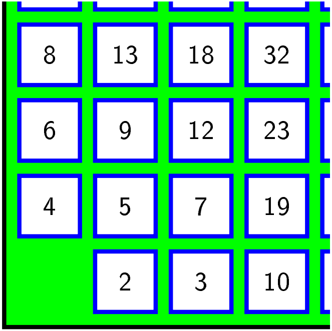

jeu de taquin

① start with $t \in \mathcal{T}$,
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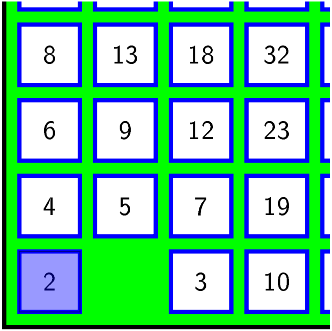
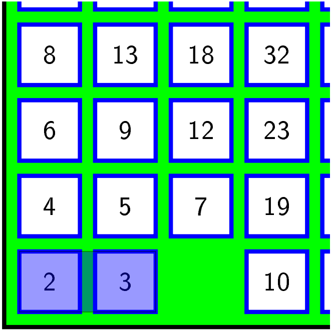
③ sliding,



jeu de taquin

1. start with $t \in \mathcal{T}$,
2. remove corner box,
3. sliding,



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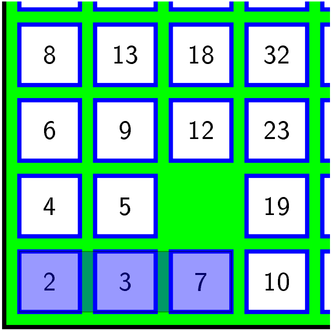

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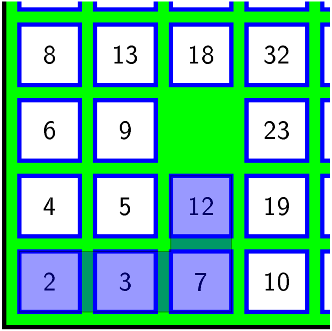

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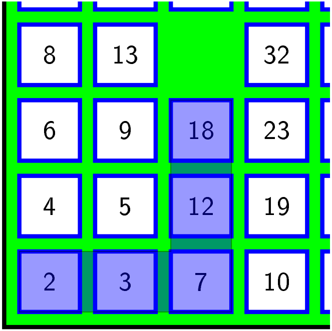

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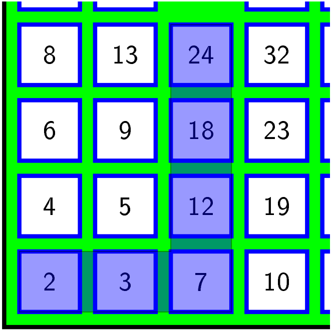

jeu de taquin

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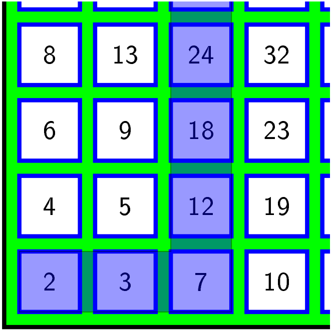

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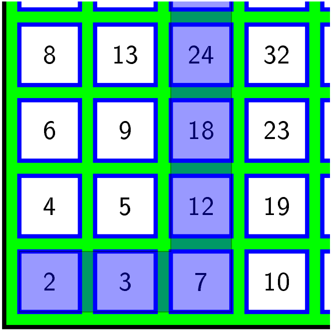

jeu de taquin

① start with $t \in \mathcal{T}$,
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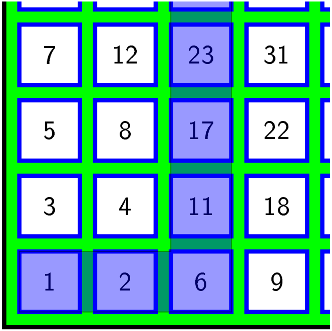
③ sliding,
④ subtract 1 from all boxes



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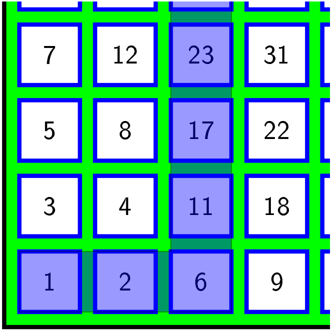

jeu de taquin

① start with $t \in \mathcal{T}$,
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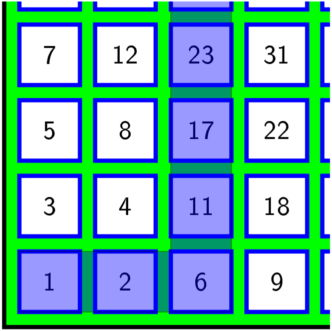
output:

- new tableau $J(t)$,
- blue trajectory $\mathbf{c}(t) = (c_1, c_2, \dots)$



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output:

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'how representation of $\mathfrak{S}_{\{1,2,3,\dots\}}$ is related to its restriction to $\mathfrak{S}_{\{2,3,\dots\}}$?'

jeu de taquin - overview

8	13	18	32
6	9	12	23
4	5	7	19
1	2	3	10

original tableau t

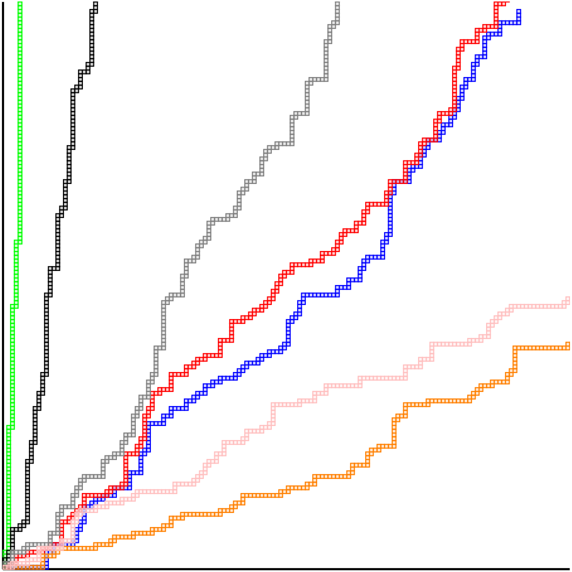
8	13	24	32
6	9	18	23
4	5	12	19
2	3	7	10

outcome of slidings

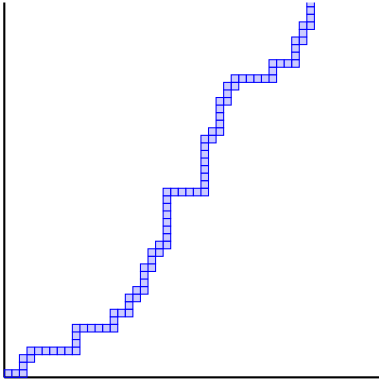
7	12	23	31
5	8	17	22
3	4	11	18
1	2	6	9

new tableau $J(t)$

trajectories of jeu de taquin



trajectories of jeu de taquin



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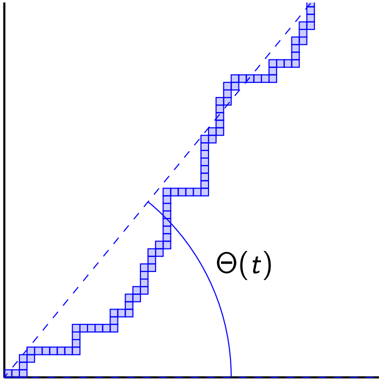
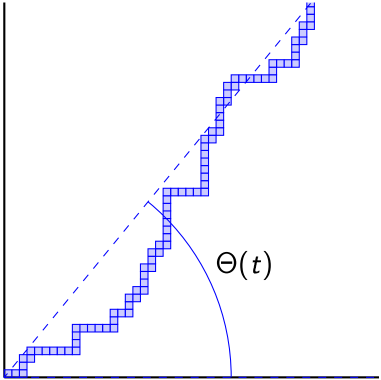


trajectories of jeu de taquin



Romik&Śniady 2015

if $t = Q(w_1, w_2, \dots) \in \mathcal{T}$ is random, Plancherel distributed

then its jdt trajectory $\mathbf{c}(t)$ is almost surely asymptotically a straight line,

i.e.

$$\lim_{k\to\infty} \frac{c_k}{\|c_k\|} = \big(\cos\Theta(t), \sin\Theta(t)\big)$$

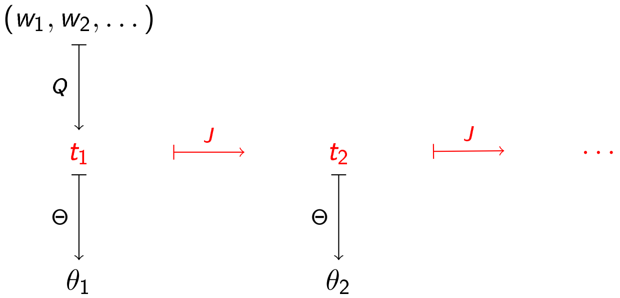
exists almost surely

$$\begin{array}{ccc} t_1 & \overset{J}{\longmapsto} & t_2 \\ \Theta \downarrow & & \\ \theta_1 & & \end{array}$$

$$
\begin{array}{ccccc}
t_1 & \xmapsto{J} & t_2 & \xmapsto{J} & \cdots \\
\Theta \big\downarrow & & \Theta \big\downarrow & & \\
\theta_1 & & \theta_2 & &
\end{array}
$$

$$\begin{array}{ccccc} t_1 & \overset{J}{\longmapsto} & t_2 & \overset{J}{\longmapsto} & \cdots \\ \Theta \big\downarrow & & \Theta \big\downarrow & & \\ \theta_1 & & \theta_2 & & \end{array}$$

jeu de taquin dynamical system $(\mathcal{T}, \text{Plancherel}, J)$



jeu de taquin dynamical system $(\mathcal{T}, \text{Plancherel}, J)$

$$
\begin{array}{ccccc}
(w_1, w_2, \dots) & \xmapsto{?} & ? & \xmapsto{?} & \dots \\
\Big\downarrow{\scriptstyle Q} & & \Big\downarrow{\scriptstyle Q} & & \\
{\color{red} t_1} & {\color{red} \xmapsto{J}} & {\color{red} t_2} & {\color{red} \xmapsto{J}} & {\color{red} \dots} \\
\Big\downarrow{\scriptstyle \Theta} & & \Big\downarrow{\scriptstyle \Theta} & & \\
\theta_1 & & \theta_2 & &
\end{array}
$$

jeu de taquin dynamical system $(\mathcal{T}, \text{Plancherel}, J)$

$$
\begin{array}{ccccc}
(w_1, w_2, \dots) & \overset{S}{\longmapsto} & (w_2, w_3, \dots) & \overset{S}{\longmapsto} & \cdots \\
\downarrow Q & & \downarrow Q & & \\
t_1 & \overset{J}{\longmapsto} & t_2 & \overset{J}{\longmapsto} & \cdots \\
\downarrow \Theta & & \downarrow \Theta & & \\
\theta_1 & & \theta_2 & &
\end{array}
$$

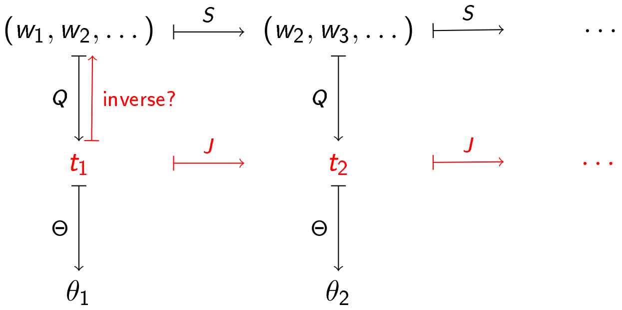
jeu de taquin dynamical system $(\mathcal{T}, \text{Plancherel}, J)$

i.i.d. shift dynamical system $\left([0,1]^{\mathbb{N}}, \prod \text{Lebesgue}, s\right)$

$$\begin{array}{ccccc}
(w_1, w_2, \dots) & \xrightarrow{S} & (w_2, w_3, \dots) & \xrightarrow{S} & \cdots \\
\downarrow Q & & \downarrow Q & & \\
t_1 & \xrightarrow{J} & t_2 & \xrightarrow{J} & \cdots \\
\downarrow \Theta & & \downarrow \Theta & & \\
\theta_1 & & \theta_2 & &
\end{array}$$

jeu de taquin dynamical system $\left(\mathcal{T}, \text{Plancherel}, J\right)$

i.i.d. shift dynamical system $\left([0,1]^{\mathbb{N}}, \prod \text{Lebesgue}, s\right)$



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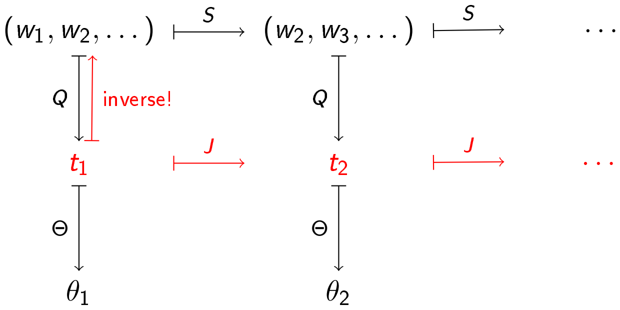


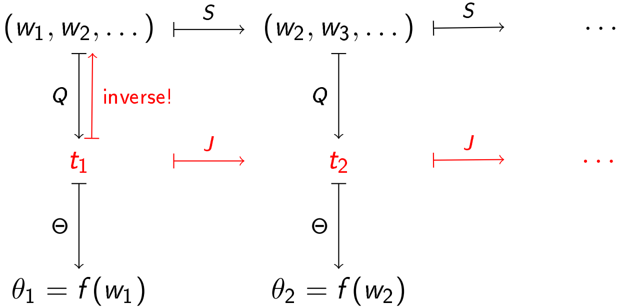
jeu de taquin dynamical system $\left(\mathcal{T}, \text{Plancherel}, J\right)$

i.i.d. shift dynamical system $\left([0,1]^{\mathbb{N}}, \prod \text{Lebesgue}, s\right)$

$$
\begin{array}{ccccc}
(w_1, w_2, \dots) & \xrightarrow{s} & (w_2, w_3, \dots) & \xrightarrow{s} & \cdots \\
\downarrow Q \;\uparrow \text{inverse!} & & \downarrow Q & & \\
t_1 & \xrightarrow{J} & t_2 & \xrightarrow{J} & \cdots \\
\downarrow \Theta & & \downarrow \Theta & & \\
\theta_1 = f(w_1) & & \theta_2 & &
\end{array}
$$

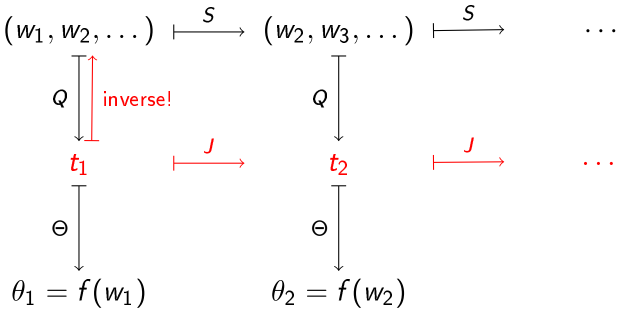
jeu de taquin dynamical system $\left(\mathcal{T}, \text{Plancherel}, J\right)$

i.i.d. shift dynamical system $\left([0,1]^{\mathbb{N}}, \prod \text{Lebesgue}, s\right)$



jeu de taquin dynamical system $(\mathcal{T}, \text{Plancherel}, J)$

i.i.d. shift dynamical system $\left([0,1]^{\mathbb{N}}, \prod \text{Lebesgue}, s\right)$



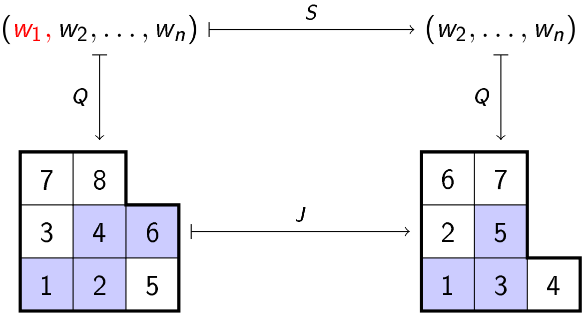
jeu de taquin dynamical system $\left(\mathcal{T}, \text{Plancherel}, J\right)$

> the jeu de taquin dynamical system is isomorphic to i.i.d. shift
>
> the inverse map is given by $w_i = f^{-1}(\theta_i)$

some consequences of the isomorphism:

- jdt is a measure-preserving transformation,
- jdt is ergodic,
- slope angles $\theta_1, \theta_2, \ldots$ are independent random variables (put paths $\mathbf{c}(t_1), \mathbf{c}(t_2), \ldots$ are not independent),
- generalizations to other probability measures on $\mathcal{T}$ and other representations of $\mathfrak{S}_\infty$, $\longrightarrow$the second lecture

why Θ exists and is a function of w_1?



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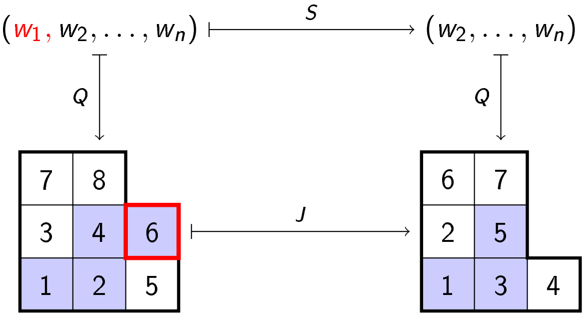

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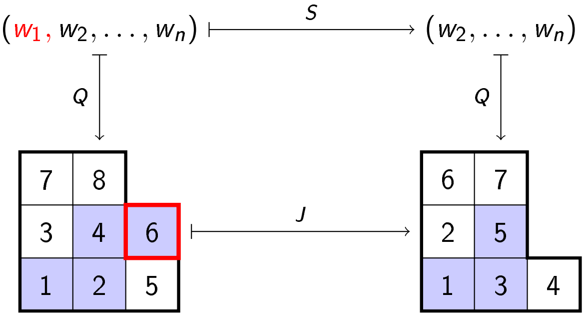
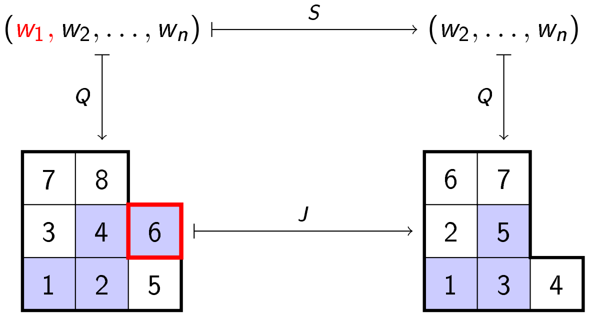


$\{\square\} = \{\text{the box on jdt trajectory with the biggest number} \leq n\} =$

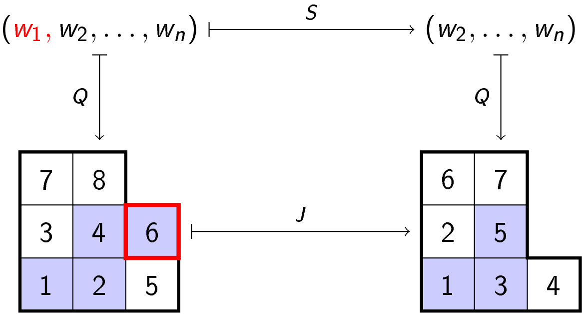
why Θ exists and is a function of w_1?



$\{\square\} = \{$the box on jdt trajectory with the biggest number$\leq n\} =$

$$Q(w_1, w_2, \dots, w_n) \quad \setminus \quad Q(w_2, \dots, w_n)$$

why Θ exists and is a function of w_1?

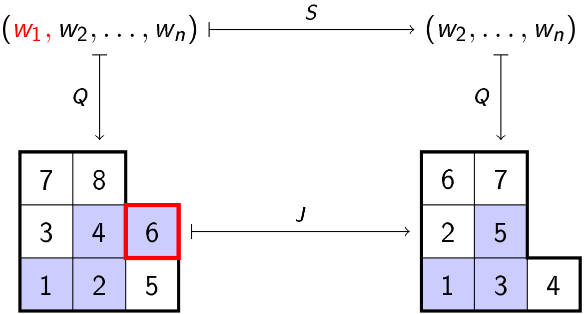


$$\{\square\} = \{\text{the box on jdt trajectory with the biggest number} \leq n\} =$$

$$Q(w_1, w_2, \dots, w_n) \setminus Q(w_2, \dots, w_n) =$$

$$Q(1 - w_n, \dots, 1 - w_2, 1 - w_1) \setminus Q(1 - w_n, \dots, 1 - w_2)$$

why Θ exists and is a function of w_1?



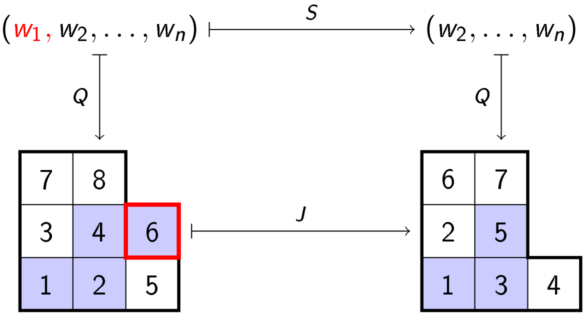
$\{\square\} =$ {the box on jdt trajectory with the biggest number $\leq n\} =$

$$Q(w_1, w_2, \dots, w_n) \setminus Q(w_2, \dots, w_n) =$$

$$Q(1-w_n, \dots, 1-w_2, 1-w_1) \setminus Q(1-w_n, \dots, 1-w_2) =$$

the box with the biggest number in $Q(1-w_n, \dots, 1-w_2, 1-w_1)$

why Θ exists and is a function of w_1?



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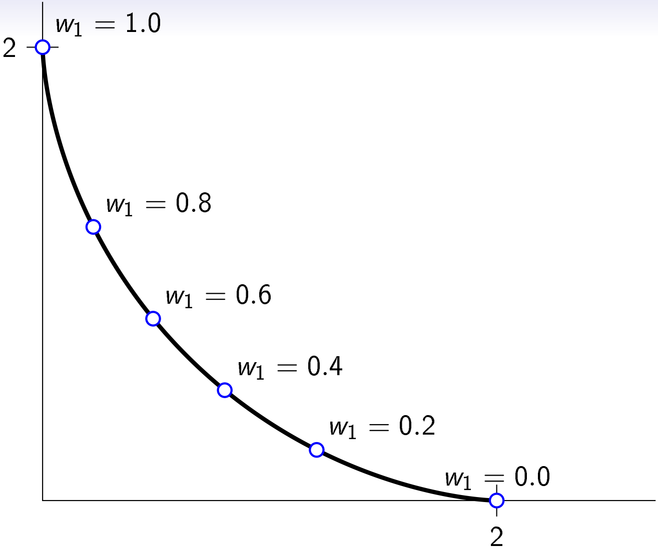
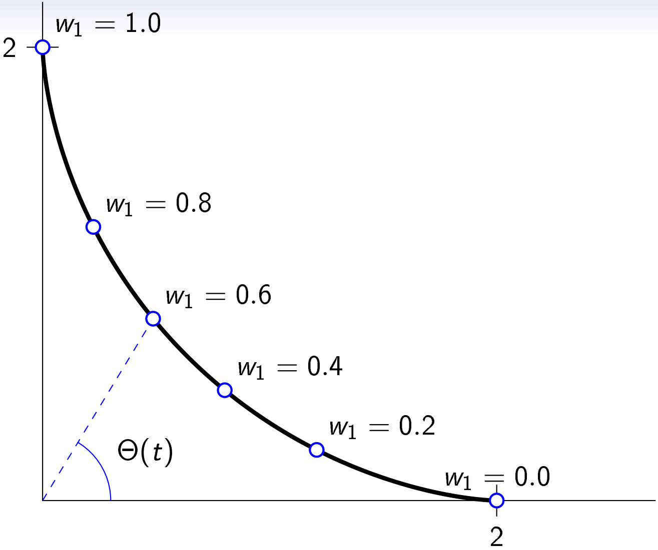
the box with the biggest number in $Q(1-w_n, \ldots, 1-w_2, 1-w_1)$

$\longrightarrow$ *asymptotic determinism of the last box insertion*



$w_1 = 1.0$
2
$w_1 = 0.8$
$w_1 = 0.6$
$w_1 = 0.4$
$w_1 = 0.2$
$w_1 = 0.0$
2



$w_1 = 1.0$
2
$w_1 = 0.8$
$w_1 = 0.6$
$w_1 = 0.4$
$w_1 = 0.2$
$w_1 = 0.0$
$\Theta(t)$
2

outlook: Lecture 2

- how to prove asymptotic determinism of the last box insertion?
- what can we learn about the characters of the infinite symmetric group $\mathfrak{S}_\infty$?

  $\longrightarrow$lectures of Cédric Lecouvey and Philippe Biane

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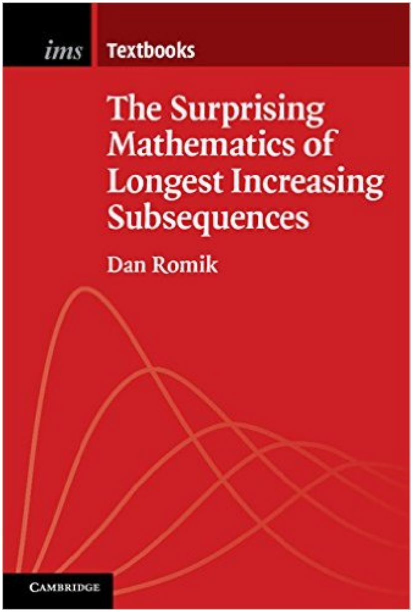
transparencies,
list of problems,
extras

are available on my personal website:
google$\longleftarrow$Śniady CIRM lectures


ims Textbooks
The Surprising Mathematics of Longest Increasing Subsequences
Dan Romik
CAMBRIDGE

Dan Romik, Piotr Śniady
Jeu de taquin dynamics on infinite Young tableaux and second class particles
Ann. Probab, Volume 43, Number 2 (2015), 682-737

- asymptotic determinism of the last box in RSK,
- trajectories of jeu de taquin,

Dan Romik, Piotr Śniady
Limit shapes of bumping routes in the Robinson–Schensted correspondence
Random Structures & Algorithms, Volume 48, Issue 1, January 2016, Pages 171–182

- bumping routes,