JEU DE TAQUIN
AND ASYMPTOTIC REPRESENTATION THEORY

PROBLEM LIST 1

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One of the main ideas in the first lecture was investigation of the following commutative diagram:

$$\begin{array}{ccc} \big([0,1],\text{Lebesgue}\big)^{\mathbb{N}} & \xrightarrow{S} & \big([0,1],\text{Lebesgue}\big)^{\mathbb{N}} \\ \big\downarrow Q & & \big\downarrow Q \\ (\mathcal{T},\text{Plancherel}) & \xrightarrow{J} & (\mathcal{T},\text{Plancherel}) \end{array} \tag{1}$$

in which all arrows are measure-preserving maps between probability spaces, where $Q(\mathbf{w})$ is the recording tableau associated to an infinite word $\mathbf{w}$; furthermore $S(w_1,w_2,\dots)=(w_2,w_3,\dots)$ is the left shift and J is the jeu de taquin transformation. We will use the convention that $\mathbb{N}=\{1,2,\dots\}$.

The key result was that the vertical arrows are invertible maps (in some special sense!).

The goal of this problem list is to address the following quite general question.

Problem 1. *Is it possible to extend the commutative diagram* (1) *by adding some additional* interesting *rows (maybe between the two existing rows; maybe below the bottom row; maybe above the top row?) in such a way that the diagram remains commutative and the vertical arrows remain invertible?*

The order in which the problems are presented here is intended to make them as difficult as possible. After solving Problem x, for some $x\geq 2$, you may revisit Problem y, for all choices of $y\in\{1,\dots,x-1\}$.

1. Arrangements on $\mathbb{N}$.

We shall view a permutation $\sigma_k\in\mathfrak{S}_k$ as a word in the alphabet $\{1,\dots,k\}$. For a permutation $\sigma_k=(i_1,\dots,i_k)$ we denote by $\sigma_k^{-1}(m)$ the m-th entry of the inverse permutation, i.e. the unique index $j\in$

$\{1,\dots,k\}$ such that $i_j = m$. For a permutation $\sigma_k = (i_1,\dots,i_k)$ and $k \geq 2$ we define a permutation

$$\sigma\!\downarrow := (i_1,\dots,\not{k},\dots,i_k) \in \mathfrak{S}_{k-1}$$

obtained by removing the symbol k from the word σ.

An *arrangement of* $\mathbb{N}$ is a sequence $\sigma = (\sigma_1,\sigma_2,\dots)$ such that $\sigma_k \in \mathfrak{S}_k$ and $\sigma_{k-1} = \sigma_k\!\downarrow$ (in other words, the set of arrangements is the inverse limit $\varprojlim \mathfrak{S}_k$ of the sets of permutations). The set of arrangements of $\mathbb{N}$ will be denoted by $\mathfrak{A}$.

There is a simple bijective correspondence between arrangements of $\mathbb{N}$ and linear orders on $\mathbb{N}$ given as follows: to an arrangement $\sigma \in \mathfrak{A}$. we associate a linear order $\prec$ such that for any $i \neq j$ and any $k \geq i, j$

$$i \prec j \iff \text{the symbol } i \text{ precedes the symbol } j \text{ in the permutation } \sigma_k.$$

Problem 2.

(1) *How to define the* uniform *measure on* $\mathfrak{A}$*?*
(2) *Let* $\sigma = (\sigma_1,\sigma_2,\dots)$ *be a random arrangement with the uniform distribution on* $\mathfrak{A}$*. Prove that for each* $i \in \mathbb{N}$ *the limit*

$$X_i := \lim_{k\to\infty} \frac{\sigma_k^{-1}(i)}{k}$$

exists almost surely.
(3) *Find explicitly the joint distribution of the random variables* $X_1, X_2, \dots$.

For a permutation $\sigma_k = (i_1,\dots,i_k) \in \mathfrak{S}_k$, $k \geq 2$ we denote by $T(\sigma_k) = (i_1 - 1,\dots,\not{0},\dots,i_k - 1) \in \mathfrak{S}_{k-1}$ the permutation which is obtained by (a) subtracting 1 from each entry of the permutation, and then (b) removing the entry equal to zero. For an arrangement $\sigma = (\sigma_1,\sigma_2,\dots) \in \mathfrak{A}$ we define $T\sigma := (T\sigma_2, T\sigma_3,\dots) \in \mathfrak{A}$.

Problem 3. *We equip the set* $\mathfrak{A}$ *with the uniform probability measure. Prove that* $T : \mathfrak{A} \to \mathfrak{A}$ *is measure-preserving and ergodic.*

If $x = (x_1, x_2, \dots) \in \mathbb{R}$ is a sequence of distinct elements, we define $\Phi x \in \mathfrak{A}$ to be the arrangement which corresponds to the following linear order on $\mathbb{N}$: for any $i, j \in \mathbb{N}$

$$i \prec j \iff x_i < x_j.$$

Problem 4. *Check that the following diagram commutes:*

$$\begin{array}{ccc} [0,1]^{\mathbb{N}} & \xrightarrow{S} & [0,1]^{\mathbb{N}} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathfrak{A} & \xrightarrow{T} & \mathfrak{A} \end{array}$$

Problem 5. *Let* $X = (X_1, X_2, \dots)$ *be a sequence of iid random variables, with the uniform distribution on the interval* $[0, 1]$.

(1) *Prove that* $(\sigma_1, \sigma_2, \dots) := \Phi X$ *is a random arrangement with the uniform distribution on* $\mathfrak{A}$.
(2) *Prove that* Φ *is a homomorphism (version for enthousiasts: isomorphism) between the following two dynamial systems:*
   - *the product probability space* $\big([0, 1], \text{Lebesgue}\big)^{\mathbb{N}}$ *equipped with the dynamical transformation given by the left shift* S, *and*
   - *the probability space* $\mathfrak{A}$ *with the uniform measure, equipped with the transformation* T.
(3) *Prove that*
$$X_i = \lim_{k\to\infty} \frac{\sigma_k^{-1}(i)}{k}$$
holds true almost surely.
(4) *Come back to previous problems.*

2. Zigzag tableaux

Olshanski and Gnedin [Coherent permutations with descent statistic and the boundary problem for the graph of zigzag diagrams. International Mathematics Research Notices, 2006:51968, 2006] show how to a given arrangement associate something called *zigzag tableau*, see also the work of Tarrago [Zigzag diagrams and Martin boundary, arXiv:1501.07087].

Problem 6. *Show how to a zigzag tableau associate an infinite standard Young tableau in such a way that the commutative diagram* (1) *has an additinal row between the two existing ones.*

For the following problems you might need some information from Lecture 2.

Problem 7. *How is the result of Tarrago related to jeu de taquin? Write a paper about it.*