Plan for today: guided ~~[illegible]~~ tour through the zoo.

If you like some animal, feel free to ask questions, we have plenty of time

One problem in many perspectives.

Suggested leisure reading:

arXiv: 1203.6509

IMPAN 1A.

→ psniady@faculty.wmi.amu.edu.pl

Combinatorics & interactions

representations of S_n, $n \to \infty$

representations of S_∞

representations of $U(d)$ $d \to \infty$

Combinatorics & interactions

IMPAN 1A.

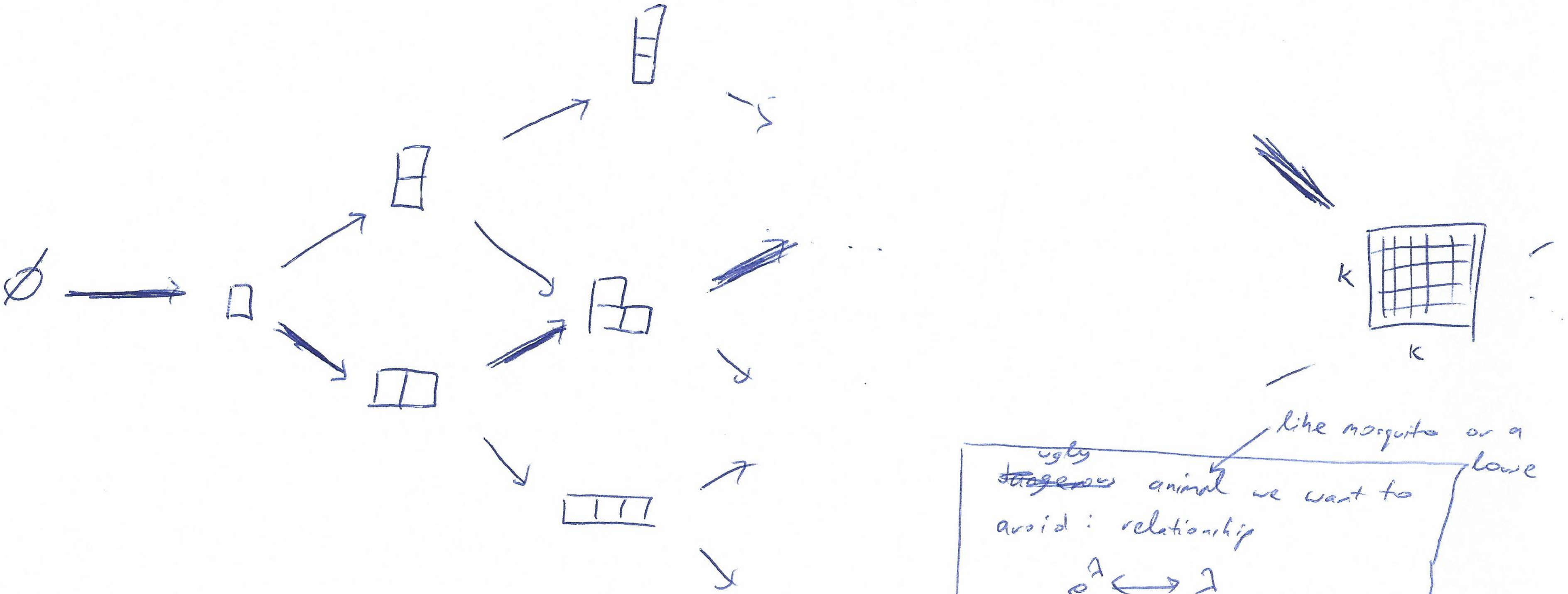
$\boxed{\text{psniady.faculty.wmi.amu.edu.pl}}$

Combinatorics & interactions

IHP Jan.-March 2017

~~asymptotic algeb~~

questions from the audience?



ugly ~~dangerous~~ animal we want to avoid: relationship

like mosquito or a louse

$$\varrho^\lambda \longleftrightarrow \lambda$$

Vertices = irreps of the symmetric groups S_n, $n \geq 0$.

$\longrightarrow$ infinity!

Young graph. $\mathbb{Y}$

Boundary of $\mathbb{Y}$?

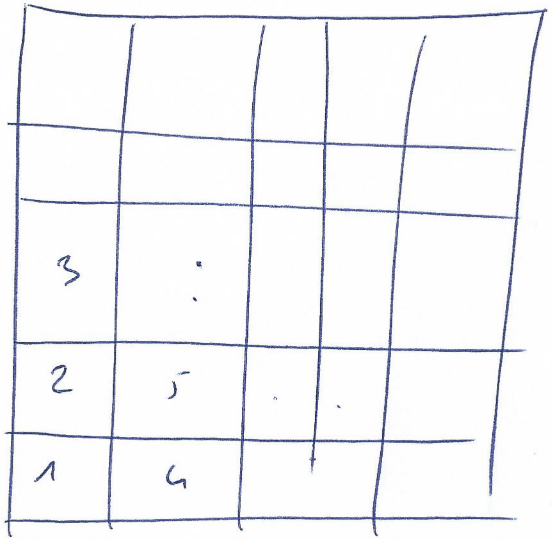
Cool problem not for today

Concrete problem for today

Q: Random path with specified endpoints $\varnothing$ and $\boxed{k \times k}$

What can we say about it?

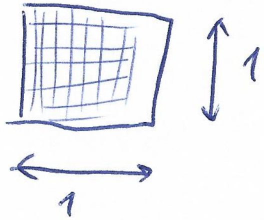
Young tableaux $\longleftrightarrow$ finite paths in $\mathbb{Y}$



isohypse curves?

In which language should we formulate the answer?

Hint:



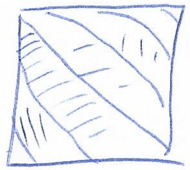
draw Young diagrams with small boxes

Law of large numbers? CLT?

PROBABILITY

Our hope:

there is some typical terrain



Q: random standard tableau filling a specified shape?

representation theory

Problem 1

$$\varrho^{\boxplus} \Big\downarrow^{S_{k^2}}_{S_{\frac{1}{2}k^2}} = \bigoplus_{|\mu| = \frac{1}{2}k^2} m_\mu \cdot \varrho^\mu$$

what can we say about a random irreducible component?

$$\mathbb{P}(\mu) = \frac{m_\mu \cdot \dim \varrho^\mu}{\dim \varrho^{\boxplus}}$$

in general:

what can we say about random irreducible component of a given interesting reducible representation?

another ~~attractive~~ cool problem: **Problem 2**

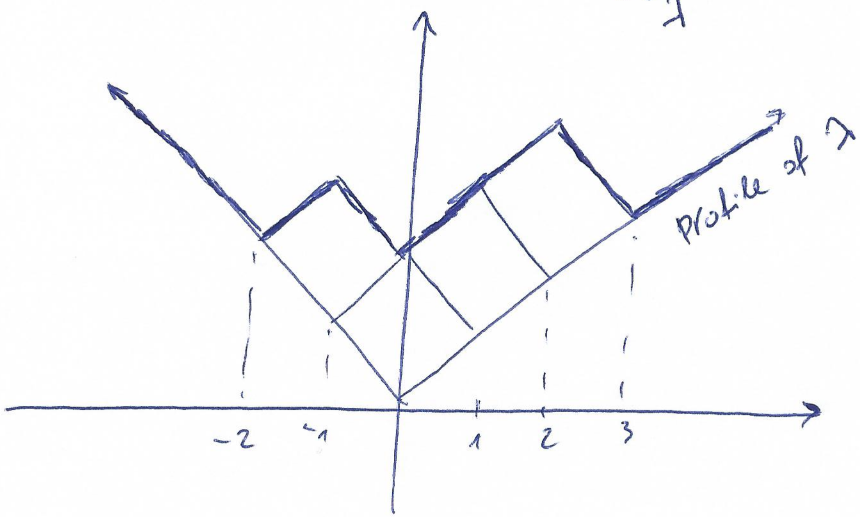
$$\varrho^\lambda \circ \varrho^\mu = \varrho^\lambda \otimes \varrho^\mu \Big\uparrow^{S_{|\lambda|+|\mu|}}_{S_{|\lambda|} \times S_{|\mu|}}$$

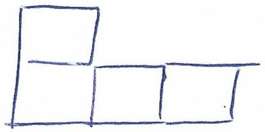
use the following blackboard layout

Problem 1	Problem 2

answers are in the form...

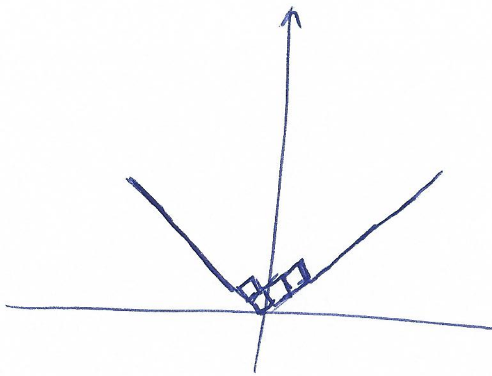
ω_λ





λ

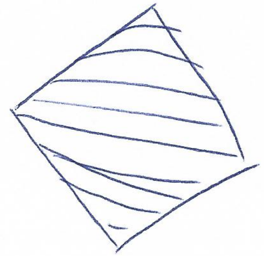
if $c > 0$



$\omega_{c\lambda}$

$c\lambda$

$\boxed{\text{Thm.}}$



for $\alpha \in (0,1)$ $\exists\, \omega : \mathbb{R} \to \mathbb{R}_+$ st.

if λ is a random irreducible component of $S^{\boxplus} \big\downarrow^{S_{k^2}}_{S_{\lfloor \alpha k^2 \rfloor}}$

then

$$\omega_{\frac{1}{k}\lambda} \longrightarrow \omega$$

in probability.

rescaled fluctuation of $\omega_{\frac{1}{k}\lambda}$ around ω are asymptotically gaussian.

$\boxed{\text{DETOUR}}$

before we understand our random Young diagrams we have to visit another part of the zoo.

random hermitian matrix with ~~a~~ fixed eigenvalues $x_1, \dots, x_n$.

$$A = \underbrace{U}_{\substack{\in U(n) \\ \text{random, Haar measure on } U(n)}} \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} U^{-1}$$

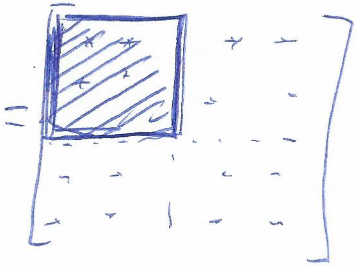
"fix eigenvalues, the eigenvectors are uniformly random"

$$B = V \begin{bmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{bmatrix} V^{-1}$$

U and V are independent.

Two cool problems

$$A = \Big[\, a_{ij} \,\Big]_{1 \le i,j \le n} =$$



A - random hermitian matrix with known eigenvalues.

$$A\Big\downarrow^{n}_{n/2} = \Big[\, a_{ij} \,\Big]_{1 \le i,j \le \frac{n}{2}}$$

$$\boxed{\text{eigenvalues of } A\Big\downarrow^{n}_{n/2} \;?}$$

(at least for $n \to \infty$)

Missing assumptions

what is the relationship between μ_A, μ_B and (random) μ_{A+B}?

A - independent hermitian matrices

B - with specified eigenvalues.

$$\boxed{\text{eigenvalues of } A+B \;?}$$

in which language we want the answer?

empirical eigenvalues distribution

$$\mu_X = \frac{1}{n}\left(\delta_{\lambda_1} + \dots + \delta_{\lambda_n}\right)$$

if X is a (random) matrix with eigenvalues $\lambda_1, \dots, \lambda_n$

random probab. measure on $\mathbb{R}$

can we have LLN for μ_{A+B}?

ITERATED DETOUR

$\left.\begin{array}{l} X \\ Y \end{array}\right\}$ independent random variables.

$\left.\begin{array}{l} \mu_X \\ \mu_Y \end{array}\right\}$ distributions.

$$\mu_{X+Y} = ?$$

$$\uparrow$$

$$\mu_X * \mu_Y$$

~~Fourier transform~~

$$\text{convolution of measures} \xleftrightarrow{\text{Fourier transform / Laplace}} \text{product of Laplace transform}$$

$$\text{convolution of measures} \xleftrightarrow{\text{log of Laplace transform}} \text{sum of log Laplace transform.}$$

$$\log \underbrace{\mathbb{E}\, e^{tX}}_{\text{Laplace transform}} = \sum_{m\geq 1} \underbrace{\kappa_m(X)}_{\substack{\text{m-ta kumulanta} \\ \text{zmiennej losowej } X.}} \frac{t^m}{m!}$$

$$\kappa_1(X) = \mathbb{E}X$$

$$\kappa_2(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \operatorname{Var} X$$

$$\kappa_3(X) = \dots\dots$$

$$\sum_{m\geq 1} \kappa_m(X+Y)\frac{t^m}{m!} = \log \mathbb{E}\, e^{t(X+Y)} \overset{\text{independence}}{=} \log\left(\mathbb{E}\, e^{tX} \cdot \mathbb{E}\, e^{tY}\right) = \sum_{m\geq 1} \left[\kappa_m(X) + \kappa_m(Y)\right]\frac{t^m}{m!}$$

$$\boxed{\kappa_m(X+Y) = \kappa_m(X) + \kappa_m(Y)}$$

"cumulants linearize convolution"

Profi version:

joint cumulant

$$\kappa(X_1, \dots, X_m) = [t_1 \cdots t_m] \log \mathbb{E}\, e^{t_1X_1 + \dots + t_mX_m}$$

$\boxed{\text{Back to random matrices.}}$

key assumption! A is unitarily invariant

Crazy idea:

spectral measure μ

information about $A = \left[a_{ij} \right]_{1 \le i,j \le n}$ $\longrightarrow$ information about the corner a_{11} $\longrightarrow$ cumulant of a_{11}

(lot of information!) (little information)

$$\underbrace{R_m(A)}_{\text{"free cumulant"}} = \underbrace{\frac{1}{m!}\, n^{m-1}}_{\substack{\text{we will not}\\ \text{discuss this}\\ \text{normalization today}}} K_m\left(a_{11} \right)$$

⚠ some normalization factor of type $m!$ might be missing

~~for $n \to \infty$~~

$\cancel{R_1(A) = \frac{1}{n}\sum \lambda_i = \int x \, d\mu_n = \mathbb{E} A}$

$R_2(A) = \mathbb{E} A^2 - (\mathbb{E} A)^2$

This idea does not work perfectly for finite values of n, but for $n \to \infty$ it works amazingly well!

$\boxed{\text{for } n \to \infty}$

$$\varphi(X) = \frac{1}{n} \mathbb{E} \operatorname{Tr} X$$

if X is a random $n \times n$ ~~m~~ matrix.

$$R_1 = \varphi(A)$$

"mean value"

$$R_2 = \varphi(A^2) - [\varphi(A)]^2$$

"covariance"

$$R_3 = \dots$$

$\boxed{\text{Two cool problems,}}$ $\boxed{\text{come back!}}$

$$R_m(A\downarrow^{n}_{\alpha n}) = \frac{1}{m!}(\alpha n)^{m-1} K_m(a_{11}) =$$

$$= \alpha^{m-1} \frac{1}{m!} n^{m-1} K_m(a_{11}) =$$

$$= \alpha^{m-1} R_m(A)$$

A, B } independent random matrices,

spectrum of $A+B$ = ?

$\boxed{\text{CHEATING AHEAD!}}$

cumulants
linearize
convolution

$$R_m(A+B) = \frac{1}{m!} n^{m-1} K_m(a_{11} + b_{11}) =$$

$$= \frac{1}{m!} n^{m-1} K_m(a_{11}) + \frac{1}{m!} n^{m-1} K_m(b_{11}) =$$

$$= R_m(A) + R_m(B)$$

"free cumulants linearize convolution of random matrices"

Come back to random Young diagrams.

$\boxed{\text{Problem 1}}$

$$\rho^{\lambda} \Big\downarrow^{S_n}_{S_{\lfloor \alpha n \rfloor}}$$

$\boxed{\text{Problem 2}}$

$$\rho^{\lambda} \circ \rho^{\mu} = \rho^{\lambda} \otimes \rho^{\mu} \Big\uparrow^{S_{|\lambda|+|\mu|}}_{S_{|\lambda|} \times S_{|\mu|}}$$

$\lambda \longrightarrow$ (quantum!) <u>random matrix</u> related to representation $\longrightarrow$ spectral measure, $\longrightarrow$ free cumulants

$R_1(\lambda) = 0$

$R_2(\lambda) = |\lambda|$

$R_3(\lambda) = ?$

ν - typical Young diagram

$$R_m(\nu) = \alpha^{m-1} R_m(\lambda)$$

ν - typical Young diagram

$$R_m(\nu) = R_m(\lambda) + R_m(\mu)$$

K10 ~~Combinatorics~~

Enumerative combinato...

Kerov polynomials

$$Ch_1 = R_2$$

$$Ch_2 = R_3$$

$$Ch_3 = R_4 + R_2$$

$$Ch_4 = R_5 + 5 R_3$$

$$Ch_5 = R_6 + 15 R_4 + 5 R_2^2 + 8 R_2$$

$$Ch_6 = R_7 + 35 R_5 + 35 R_3 R_2 + 84 R_3$$

"the principle of enumerative combinatorics"
if some numbers are positive integers for no obvious reasons, there is some hidden combinatorial structure behind it

it is a religious belief

Kerov conjecture: coefficients non-negative integers

coefficients are counted by maps

$R_k = R_k(\lambda) =$ "free cumulant of λ"

$Ch_k = Ch_k(\lambda) =$ "normalized character on λ"

$$= \underbrace{|\lambda| \cdot (|\lambda|-1) \cdots (|\lambda|-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda((1 2 \cdots k))}{\operatorname{Tr} \rho^\lambda(id)}$$

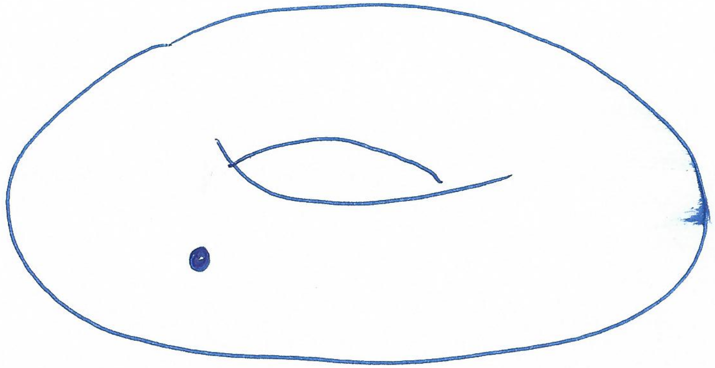
Maps

EDGE-LABELED,
BICOLORED

ORIENTED

___ map = ___ graph drawn on a ___ surface

(fill in the adjectives, such as: oriented, ~~bipartite~~, bicolored, labeled, rooted, painted, ...)



map = algebraic / combinatorial object + topological intuition