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Lecture 2. Stanley character formula.

global erratum : replace globally

$\hat{S}_n$ by $\tilde{S}_n$

and

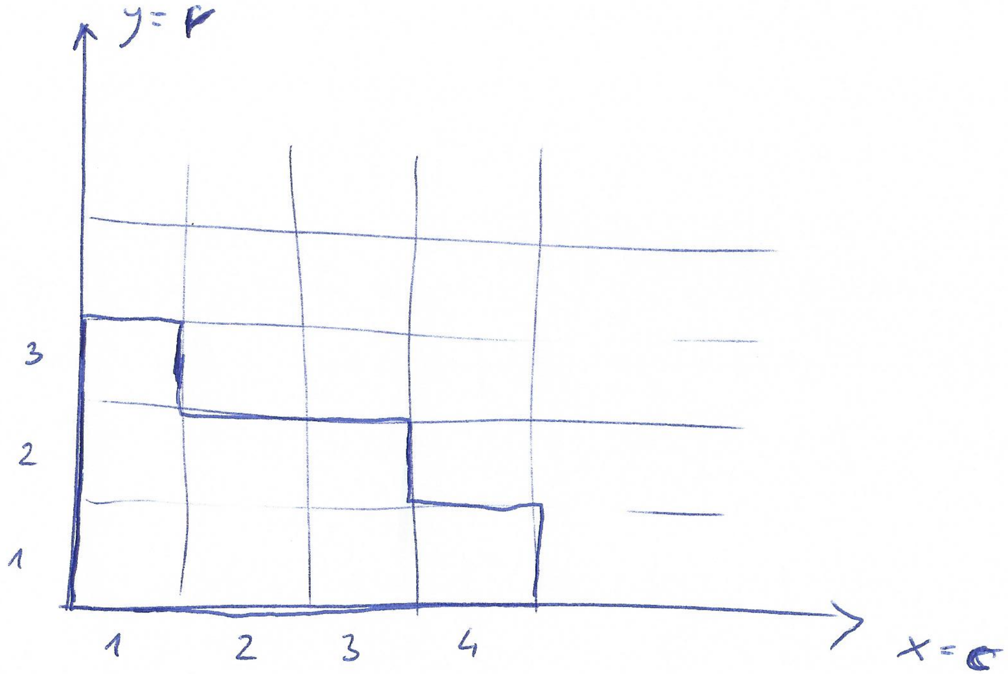
$\hat{\pi}$ by $\tilde{\pi}$

etc.

notations

"abstract" permutation group S_n permutes $\{1, 2, \dots, n\}$

"concrete" permutation group $\hat{S}_n$ permutes boxes of λ, $|\lambda| = n$



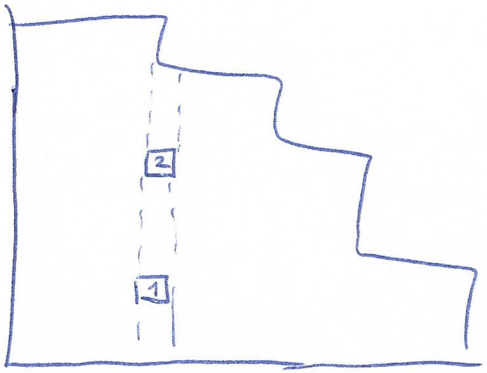
$r(\square)$ = row of $\square \in \{1, 2, 3, \dots\}$

$c(\square)$ = column of $\square \in \{1, 2, 3 \dots\}$

$\square = (c, r)$ Cartesian coordinates: first x then y!

column inversions

$$(-1)^{\mathrm{cinv}\,\hat{\pi}} = \prod_{\substack{\square_1,\,\square_2\in\lambda\\ c(\square_1)=c(\square_2)\\ r(\square_1)<r(\square_2)}} \times \begin{cases} 1 & r\,\hat{\pi}(\square_1) < r\,\hat{\pi}(\square_2) \\ -1 & r\,\hat{\pi}(\square_1) > r\,\hat{\pi}(\square_2) \\ 0 & r\,\hat{\pi}(\square_1) = r\,\hat{\pi}(\square_2) \end{cases}$$



"if two boxes are in the same column, their images are not in the same row"!

readers of Ann. Math 173, 887–906!

the roles played by π and π^{-1} are reversed in this note compared to [Ann. Math.]!

$\boxed{\text{This does not create any problems!}}$

$\boxed{\text{Corollary}}$

it is a ~~tot~~ corollary from a result which we prove later!

$$\chi^\lambda(\pi) := \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\dim V_\lambda} = \frac{1}{n!} \sum_{\mu \in \hat{S}_n} (-1)^{\operatorname{cinv} \hat{\mu}^{-1} \hat{\pi} \hat{\mu}} =$$

$$= \mathbb{E} \, (-1)^{\operatorname{cinv} \hat{\hat{\pi}}}$$

$\hat{\hat{\pi}}$ - random permutation of boxes ~~with~~ from ~~cycle~~ conjugacy class of π.

$\boxed{\text{Proof follows} \ldots}$ $\longrightarrow$

Specht module

left-regular representation

$$\mathbb{C}\hat{S}_n \, c_\lambda$$

projection, up to a scalar multiple.

→ book of Fulton & Harris

$$c_\lambda = a_\lambda \cdot b_\lambda$$

$$a_\lambda = \sum_{\hat{\sigma}_\lambda \in \hat{S}_n} \hat{\sigma}_\lambda$$

$\hat{\sigma}_\lambda$ preserves each row of λ

P_λ = group of permutations which preserve each row

(~~to really write~~ $\hat{\sigma}_\lambda$)

$$b_\lambda = \sum_{\hat{\sigma}_\lambda \in \hat{S}_n} (-1)^{\hat{\sigma}_\lambda} \hat{\sigma}_\lambda$$

$\hat{\sigma}_\lambda$ preserves each column of λ

Q_λ = group of permutations which preserve each column

$$P_\lambda = \frac{c_\lambda}{\alpha_\lambda}$$ is a minimal projection

$$\alpha_\lambda = \frac{n!}{\dim V_\lambda}$$

Warning: for some strange reason [Ann. Math 173, 887-906] uses

$$c_\lambda = b_\lambda a_\lambda.$$

Can we justify their choice?

$c_\lambda = a_\lambda \cdot b_\lambda$ is the standard notation (~~Yes, we can justify that~~

Convention which I will try to use:

cycles of $\underline{\underline{\sigma_1}}$ should define **the first** coordinate so

σ_1 ~~permutes~~ preserves each **column**. $\sigma_1 \in Q_\lambda$

cycles of $\underline{\underline{\sigma_2}}$ should define **the second** coordinate so

σ_2 preserves each **row**. $\sigma_2 \in P_\lambda$

Young's lemma

~~Conceptions which I try to~~ $\hat{\delta}_1$

for a given $\hat{\pi}$, a permutation of boxes of a Young diagram λ . . .

$\iff (-1)^{\text{cinv}\,\hat{\pi}} \neq 0$

$\exists\, \hat{\delta}_1, \hat{\delta}_2$ $\quad \hat{\pi} = \hat{\delta}_2\ \hat{\delta}_1$ $\quad \iff \quad \forall\ \square_1 \neq \square_2 \in \lambda$

$\hat{\delta}_1$ preserves each column, $\hat{\delta}_1 \in Q_\lambda$

$\hat{\delta}_2$ preserves each row $\hat{\delta}_2 \in P_\lambda$.

$\square_1, \square_2$ are in the same column

$\Downarrow$

Such $\hat{\delta}_1, \hat{\delta}_2$ (if exist) are determined uniquely; $(-1)^{\hat{\delta}_1} = (-1)^{\text{cinv}\,\hat{\pi}}$

$\hat{\pi}\,\square_1$, $\hat{\pi}\,\square_2$ are <u>not</u> in the same row.

($\Rightarrow$) is easy.

($\Leftarrow$) is NOT that easy. We use the fact that λ is a (NOT skew!) Young diagram.

$\frac{n!}{\dim V_\lambda} \operatorname{Tr} \varrho^\lambda(\pi) =$

trace in $V_\lambda = \mathbb{C} S_n p_\lambda$ trace in $\mathbb{C} S_n$

$$\frac{n!}{\dim V_\lambda} \operatorname{Tr} \varrho^\lambda(\pi^{-1}) = \frac{1}{\alpha_\lambda} \underbrace{\operatorname{Tr} \varrho^\lambda(\pi^{-1})}_{\text{trace of } \pi^{-1} \text{ acting on the left on the space } \mathbb{C} S_n p_\lambda} = \frac{1}{\alpha_\lambda} \sum_{\hat\mu \in \hat S_n} \langle \delta_{\hat\mu}, \hat\pi^{-1} \delta_{\hat\mu} p_\lambda \rangle$$

see the separate page

$$= \sum_{\hat\mu \in S_n} \sum_{\hat\delta_1 \in Q_\lambda} \sum_{\hat\delta_2 \in P_\lambda} (-1)^{\hat\delta_1} \cdot [\hat\mu = \hat\pi^{-1} \hat\mu \, \hat\delta_2 \, \hat\delta_1] =$$

$\hat\delta_1 \in Q_\lambda$ preserves each column

$\hat\delta_2 \in P_\lambda$ preserves each row

$$[\hat\mu^{-1} \hat\pi \hat\mu = \hat\delta_2 \hat\delta_1]$$

$$[\hat\mu^{-1} \hat\pi^{-1} \hat\mu = \hat\delta_1^{-1} \hat\delta_2^{-1}]$$

we will use this later.

For each μ $\hat\delta_1$ and $\hat\delta_2$ (if exist) are determined uniquely

$\to$ Young's lemma.

~~This is one of many occasions when it does not matter if we write δ_i or δ_i^{-1}.~~

$$= \sum_{\hat\mu \in S_n} (-1)^{\text{cinv}\, \hat\mu^{-1} \hat\pi \hat\mu}$$

$\boxed{\square \text{ end of proof}}$

⚠ $\boxed{\text{explanation: 1.}}$

define for $x \in \mathbb{C}S_n$

$$R\,x := x\,p_\lambda$$

we act by p_λ from the right

$$\underbrace{\mathrm{Tr}\ \rho^\lambda(\pi^{-1})}_{\text{trace in } V_\lambda = R\,\mathbb{C}S_n} = \mathrm{Tr}\ \underbrace{R\,\pi^{-1}}_{\text{viewed as a linear map on } \mathbb{C}S_n} =$$

$$= \sum_{\tilde\mu \in \tilde S_n} \langle \underbrace{\delta_{\tilde\mu}}_{\text{basis of } \mathbb{C}S_n}, \pi^{-1} R\, \delta_{\tilde\mu} \rangle = \sum_{\tilde\mu \in \tilde S_n} \langle \delta_{\tilde\mu}, \pi^{-1} \delta_{\tilde\mu}\, p_\lambda \rangle$$

$\boxed{\text{explanation 2}}$

This is one of many occasions when it does not matter

if we write δ_i or δ_i^{-1}.

Also, it often does not matter if we write

$$\pi = \delta_1 \delta_2 \qquad \text{or} \qquad \pi = \delta_2 \delta_1$$

because...

$$\Updownarrow$$

$$\pi^{-1} = \delta_2^{-1} \delta_1^{-1}$$

replace by δ_1.

replace by δ_2

often we can replace π by π^{-1}

normalized characters of S_n

philosophy: ....

$S_1 \subset S_2 \subset S_3 \subset \dots$

how to study all symmetric groups at the same time?

for a permutation $\pi \in S_\ell$

and a Young diagram λ, $|\lambda| = n$

$$\mathrm{Ch}_\pi(\lambda) = \begin{cases} \underbrace{n(n-1)\cdots(n-\ell+1)}_{\ell \text{ factors}} \dfrac{\mathrm{Tr}\, \rho_\lambda(\pi)}{\dim V_\lambda} & \text{if } \ell \leq n. \\ 0 & \text{if } \ell > n \end{cases}$$

The usual viewpoint: fix ~~an~~ irreducible representation λ,

character $\pi \longmapsto \mathrm{Tr}\, \rho^\lambda(\pi)$

is a function on a group

Dual viewpoint: fix conjugacy class π

character

$\lambda \longmapsto \mathrm{Ch}_\pi(\lambda)$

is a function on Young diagrams.

Big goal for today: for $\pi \in S_\ell$

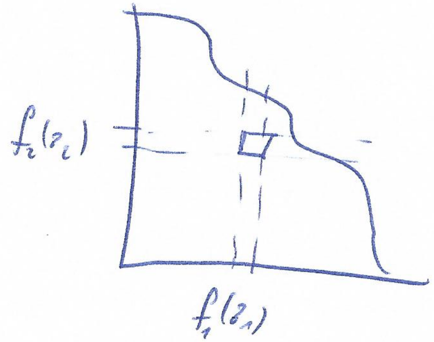
, $$\mathrm{Ch}_\pi(\lambda) = \sum_{\substack{\sigma_1, \sigma_2 \in S_\ell \\ \sigma_1 \sigma_2 = \pi}} (-1)^{\sigma} N_{\sigma_1, \sigma_2}(\lambda)$$

proof of this result will take several steps!

where $N_{\sigma_1, \sigma_2} = \{ (f_1, f_2)$

$$\underbrace{f_1 : C(\sigma_1)}_{\text{cycles of } \sigma_1} \longrightarrow \underbrace{\mathbb{N}}_{\text{set of columns}}, \qquad f_2 : C(\sigma_2) \longrightarrow \underbrace{\mathbb{N}}_{\text{set of rows}}$$

such that $\forall c_1 \in C(\sigma_1), c_2 \in C(\sigma_2) \quad c_1 \cap c_2 \neq \emptyset \Rightarrow (f_1(c_1), f_2(c_2)) \in \lambda \}$



For $\sigma_1, \sigma_2 \in S_\ell$

$\widetilde{N}_{\sigma_1,\sigma_2}(\lambda) := \#\Big\{ f : \{1, \dots, \ell\} \longrightarrow \lambda$, injection ,

$r \circ f$ is constant on each cycle of σ_1

$c \circ f$ is constant on each cycle of $\sigma_2 \Big\}$

"f maps cycles of σ_1 to columns, cycles of σ_2 to rows"

$\boxed{\text{Prop.}}$ for $\pi \in S_\ell$

$$Ch_\pi(\lambda) = \sum_{\substack{\sigma_1, \sigma_2 \in S_\ell \\ \sigma_1 \sigma_2 = \pi}} (-1)^{\sigma_1} \widetilde{N}^{\lambda}_{\sigma_1, \sigma_2}$$

in the big goal we use non-injective versions

until now, we viewed all permutations as permutations of boxes of λ.

Now we view permutations as permutations of abstract elements.

$\boxed{\text{case } l = n}$

$$\frac{n!}{\dim V_\lambda} \operatorname{Tr} \rho^\lambda(\pi) = \sum_{\mu \in S_n} \sum_{\hat\sigma_1 \in Q_\lambda} \sum_{\hat\sigma_2 \in P_\lambda} (-1)^{\hat\sigma_1} \left[\hat\mu^{-1} \hat\pi \hat\mu = \hat\sigma_1 \hat\sigma_2 \right] =$$

$$\| \qquad Ch_\pi(\lambda) =$$

at no cost we can replace $\hat\pi \leftrightarrow \hat\pi^{-1}$, $\hat\sigma_1 \leftrightarrow \hat\sigma_1^{-1}$, $\hat\sigma_2 \leftrightarrow \hat\sigma_2^{-1}$.

multiset over $\hat\mu$ $\left\{ \hat\mu^{-1} \hat\pi \hat\mu \right\}$ = multiset over f $\left\{ f \pi f^{-1} \right\}$

~~f bijection~~

$f : \{1, 2, \dots, l\} \to \lambda$ is a bijection

$$\sigma_1 := f^{-1} \hat\sigma_1 f$$

$$\sigma_2 := f^{-1} \hat\sigma_2 f$$

~~$\sigma_1 \leftrightarrow \hat\sigma_1$ is a bijection between S_l and S_λ.~~

$$\pi = \sigma_1 \sigma_2 \iff \pi = f^{-1} \hat\sigma_1 \hat\sigma_2 f \iff$$

$$\underbrace{f \pi f^{-1}}_{\hat\mu^{-1} \hat\pi \mu} = \hat\sigma_1 \hat\sigma_2$$

$\sigma_1 := f^{-1}\hat{\sigma}_1 f$

$\hat{\sigma}_1 \longleftrightarrow \sigma_1$ is a bijection between $\hat{S}_n$ and S_n

~~P_λ card~~

$\hat{\sigma}_1 \in Q_\lambda \iff c \circ f$ is constant on each cycle of σ_1

$\hat{\sigma}_2 \in P_\lambda \iff r \circ f$ is constant on each cycle of σ_2

$$\ldots = \sum_{f \text{ : bijection}} \; \sum_{\sigma_1, \sigma_2 \in S_\ell} (-1)^{\sigma_1} \, [\pi = \sigma_1 \sigma_2] =$$

$c \circ f$ is constant on each cycle of σ_1

$r \circ f$ is constant on each cycle of σ_2

$$= \sum_{\substack{\sigma_1, \sigma_2 \in S_\ell \\ \sigma_1 \sigma_2 = \pi}} (-1)^{\sigma_1} \, \tilde{N}_{\sigma_1, \sigma_2}(\lambda)$$

[end of proof for $n = \ell$].

$\boxed{\ell = n \text{, revisited}}$

support 1

$$\frac{n!}{\dim V_\lambda} \operatorname{Tr} \rho^\lambda(\pi) = \sum_{\substack{\sigma_1, \sigma_2 \in S_\ell \\ \sigma_1 \sigma_2 = \pi}} (-1)^{\sigma_1} \tilde{N}_{\sigma_1, \sigma_2}(\lambda)$$

Claim We can restrict the above sum to σ_1, σ_2 s.t.

$$\operatorname{supp} \sigma_1, \operatorname{supp} \sigma_2 \subseteq \operatorname{supp} \pi.$$

Proof. if $m \in \operatorname{supp} \sigma_1 \setminus \operatorname{supp} \pi = \operatorname{supp} \sigma_2 \setminus \operatorname{supp} \pi$

then for each $f : \{1, \dots, n\} \to \lambda$ bijection ...

~~$f(m) \neq f(\sigma_2(m))$~~

OR

Support 2

$$f(m) \neq f(\sigma_2(m))$$

OR

$$r \circ f(m) \neq r \circ f \circ \sigma_2(m)$$

$r \circ f$ is NOT constant on cycles of σ_2

$$c \circ f(m) \neq c \circ f \circ \sigma_2(m)$$

$$= c \circ f \circ \underbrace{\sigma_1 \circ \sigma_2}_{\pi}(m)$$

$c \circ f$ is NOT constant on cycles of σ_1

$l \leq n$ $\qquad \pi \in S_l$

$$\frac{n!}{\dim V_\lambda} \operatorname{Tr} \varrho^\lambda(\pi) = \sum_{\substack{\sigma_1,\sigma_2 \in S_l \\ \sigma_1\sigma_2 = \pi}} (-1)^{\sigma_1} \underbrace{\tilde{N}^{S_n}_{\sigma_1,\sigma_2}(\lambda)} =$$

⚠ for this we view σ_1, σ_2 as permutations from S_n, which have $n-k$ extra fixpoints

$$= \sum_{\substack{\sigma_1,\sigma_2 \in S_l \\ \sigma_1\sigma_2 = \pi}} (-1)^{\sigma_1} \underbrace{\tilde{N}^{S_l}_{\sigma_1,\sigma_2}(\lambda)}_{\text{now } \sigma_1,\sigma_2 \in S_l} \cdot (n-k)!$$

Corollary

$$Ch_\pi(\lambda) = \sum_{\substack{\sigma_1,\sigma_2 \in S_l \\ \sigma_1\sigma_2 = \pi}} (-1)^{\sigma_1} \tilde{N}_{\sigma_1,\sigma_2}(\lambda).$$

forgetting injectivity.

$$Ch_\pi(\lambda) = \sum_{\substack{\sigma_1,\sigma_2\in S_\ell \\ \sigma_1\sigma_2=\pi}} (-1)^{\sigma_1} \hat{N}_{\sigma_1,\sigma_2}(\lambda) \overset{?}{=} \sum_{\substack{\sigma_1,\sigma_2\in S_\ell \\ \sigma_1\sigma_2=\pi}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}(\lambda)$$

injective functions $f:\{1,\dots,\ell\}\to\lambda$

s.t. $c\circ f$ constant on cycles of σ_1

$r\circ f$ — " — " σ_2.

functions $f:\{1,\dots,\ell\}\to\lambda$

s.t. $c\circ f$ const. on cycles of σ_1

$r\circ f$ const on cycles of σ_2

$\boxed{\text{Proof}}$ Let $f:\{1,\dots,\ell\}\to\lambda$, such as on rhs be non-injective;

exist $a,b\in\{1,\dots,\ell\}$ s.t. $f\circ(a,b) = f$.

Consider the set of pairs (σ_1,σ_2) for which f contributes; there is an involution on this set $(\sigma_1,\sigma_2) \longleftrightarrow (\underbrace{\sigma_1(a,b)}_{\sigma_1'}, \underbrace{(a,b)\sigma_2}_{\sigma_2'})$

$$c\circ f = c\circ f\circ\sigma_1^{-1} \iff c\circ f = c\circ f\circ \underbrace{(a,b)\circ\sigma_1^{-1}}_{\sigma_1'^{-1}}$$

$$r\circ f = r\circ f\circ\sigma_2 \iff r\circ f = r\circ f\circ \underbrace{(a,b)\sigma_2}_{\sigma_2'}$$

Contribution of f:

$$\sum_{\substack{\sigma_1,\sigma_2 \in S_e \\ \sigma_1\sigma_2 = \pi}} (-1)^{\sigma_1} = (-1) \sum_{\sigma_1', \sigma_2' \in S_e} (-1)^{\sigma_1'}$$

$\vdots$

c of cont. on cycles of σ_1

r of cont on cycles of σ_2

$$X = -X$$

$$\Downarrow$$

$$X = 0.$$