$\boxed{1}$

Normalized characters of symmetric groups.

Young diagram with n boxes

(usual) character : $\chi^\lambda(\pi) = \mathrm{Tr}\, \underbrace{\rho^\lambda}(\pi)$

permutation in S_n

irreducible representation corresponding to λ.

Typically, character is viewed as a function on the group.

makes sense because $\pi \in S_k \subset S_n$

<u>normalized</u> character :

Young diagram with n boxes

$$\Sigma_\pi(\lambda) = \underbrace{\underbrace{n \cdot (n-1) \cdots (n-k+1)}_{k \text{ factors}}}_{\text{"in how many ways } S_k \text{ can be embedded into } S_n\text{?"}} \frac{\overbrace{\operatorname{Tr} \rho^\lambda(\pi)}^{\text{makes sense because } \pi \in S_k \subset S_n}}{\underbrace{\operatorname{Tr} \rho^\lambda(e)}_{\text{dimension of the representation}}}$$

permutation in S_k

normalized character: we fix the conjugacy class π and we vary the Young diagram λ.

very smart idea! → Kerov

If $n < k$ we just define $\Sigma_\pi(\lambda) = 0$

2 Stanley formula

Stanley formula will be our favorite tool!

For $\pi \in S_k$

$$\Sigma_\pi(\lambda) = \sum_{\substack{\sigma_1, \sigma_2 \in S_k \\ \sigma_1 \sigma_2 = \pi}} (-1)^{\sigma_1} N_{\sigma_1, \sigma_2}(\lambda)$$

~~later~~ formulated by Stanley in a completely different way

proved by Féray

this formulation: Féray, Śniady

where...

$$N_{\sigma_1, \sigma_2}(\lambda) := N_{\mathcal{G}}(\lambda)$$

bipartite graph associated to σ_1, σ_2

$\mathcal{G}$: vertices : $\mathcal{V}_1 \sqcup \mathcal{V}_2$

empty vertices

solid vertices

empty ○ vertices correspond to cycles of σ_1

solid ● vertices correspond to cycles of σ_2

(vertices from $\mathcal{V}_i$ correspond to cycles of σ_i)

Vertices c_1 and c_2 are connected by an edge iff cycles c_1 and c_2 have nontrivial intersection.

$N_G(\lambda)$ is the number of pairs (h_1, h_2) s.t.:

- $h_i : \mathcal{V}_i \longrightarrow \mathbb{N}$

  we think that

  $h_1 : \mathcal{V}_1 \longrightarrow \mathbb{N}$ „columns"

  $h_2 : \mathcal{V}_2 \longrightarrow \mathbb{N}$ „rows"

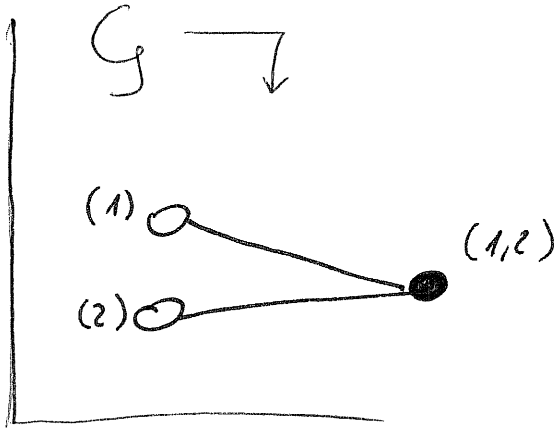
- if vertices $v_1 \in \mathcal{V}_1$, $v_2 \in \mathcal{V}_2$ are connected by an edge, we require that

$$\underbrace{\big(h_1(v_1),\ h_2(v_2)\big)}_{\substack{\text{box in } h_1(v_1) \text{ column,} \\ h_2(v_2) \text{ row}}} \in \lambda$$

Example

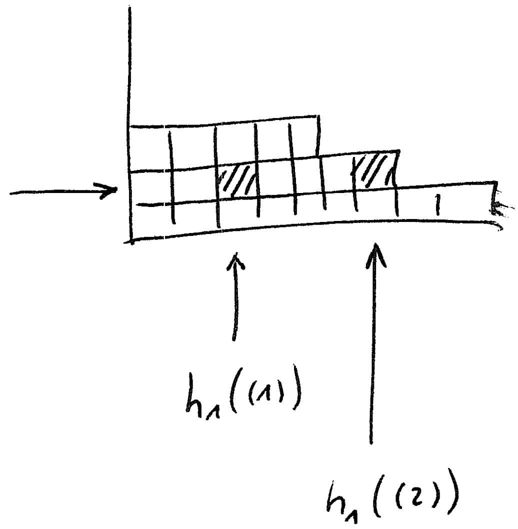
$(12) \in S_2$ transposition.

- $(12) = \underbrace{(1)(2)}_{\gamma_1} \cdot \underbrace{(12)}_{\gamma_2}$



$h_2((12))$

h_2 - we chose one of the rows of λ

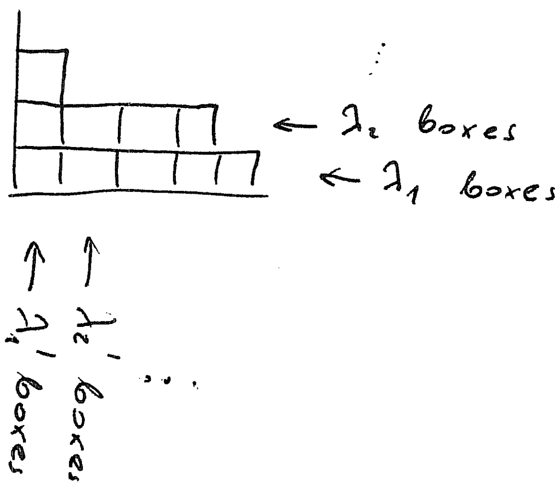


$h_1((1))$, $h_1((2))$ -

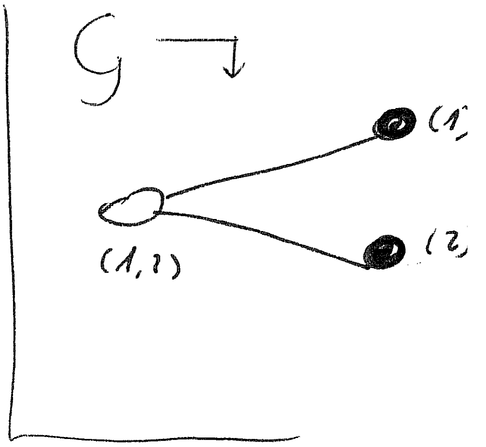
- we chose two columns of λ
- there are λ_j^2 choices!

$$N_{\gamma_1,\gamma_2}(\lambda) = \sum_j (\lambda_j)^2$$

We use french convention:



$$\bullet \quad (12) = \underbrace{(12)}_{g_1} \cdot \underbrace{(1)(2)}_{g_2}$$

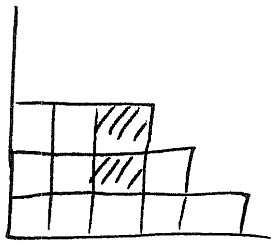


analogous:

$$N_{g_1, g_2}(\lambda) = \sum_j (\lambda'_j)^2$$

$h_2((2)) \rightarrow$

$h_2((1)) \rightarrow$



$\uparrow$

$j = h_1((12))$

Therefore:

$$\underbrace{\Sigma_{(1,2)}(\lambda)}_{= n(n-1)\frac{\mathrm{Tr}\, \rho^{\lambda}((12))}{\mathrm{Tr}\, \rho^{\lambda}(e)}} = + \sum_j \lambda_j^2 - \sum_j {\lambda'_j}^2$$

where $n = |\lambda|$

Lecture 1, Erratum

The result which I attributed to Anthony Wassermann was proved independently in the same year by Vershik and Kerov.

4 | Wassermann estimate

$$\beta_i = \frac{\lambda'_i}{n}$$

$$\alpha_i = \frac{\lambda_i}{n}$$

$$n = |\lambda|$$

Claim: For any permutation π

$$\frac{\operatorname{tr} \rho^\lambda(\pi)}{\operatorname{tr} \rho^\lambda(e)} = \prod_{\substack{c \mid \pi \\ \underbrace{|c|}\geq 2}} \left[\sum_i \alpha_i^{|c|} - \sum_i (-\beta_i)^{|c|} \right] + O\left(\frac{1}{n}\right)$$

number of elements in the cycle

for example: $|(1,2,\dots,k)| = k$

In the limit as $|\lambda| = n \longrightarrow \infty$.

Proof For simplicity we concentrate on the case when $\pi = (1,2,\dots,k)$ has only one non-trivial cycle.

$$\underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{tr} \rho^\lambda((1\,2 \dots k))}{\operatorname{tr} \rho^\lambda(e)} = \sum_{(1,2,\dots,k)}^{\lambda} =$$

$$= + N_{e,\,(1,2,\dots,k)}(\lambda) + \Bigg|\begin{array}{l} \delta_1 = e \\ \delta_2 = (1,2,\dots,k) \end{array}$$

$$+ (-1)^{k-1} N_{(1,2,\dots,k),\,e}(\lambda) + \Bigg|\begin{array}{l} \delta_1 = (1,2,\dots,k) \\ \delta_2 = e \end{array}$$

$$+ (\text{other summands}) =$$

$$= \sum_j (\lambda_j)^k - (-1)^k \sum_j (\lambda_j')^k +$$

$$+ (\text{other terms})$$

~~Therefore~~

$$\cancel{\frac{\operatorname{tr} \rho^\lambda((1,2,\dots,k))}{\operatorname{tr} \rho^\lambda(e)} = \frac{n^k}{n(n-1)\cdots(n-k+1)}}$$

Therefore

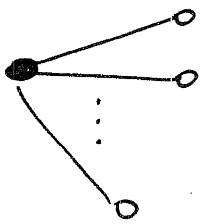
$$\frac{\operatorname{tr} \rho^{\lambda}((1,2,\dots,k))}{\operatorname{tr} \rho^{\lambda}(e)} = \underbrace{\frac{n^{k}}{n(n-1)\cdots(n-k+1)}}_{1+o(1)} \times$$

$$\times \left[\sum_{j} \left(\frac{\lambda_j}{n}\right)^{k} - \sum_{j} \left(-\frac{\lambda_j'}{n}\right)^{k} + \underbrace{+ \frac{(\text{other terms})}{n^{k}}}_{\text{why is it small?}} \right]$$

Step 1

~~Lemma.~~ Suppose that $\mathcal{G}$ is a bipartite graph without isolated vertices.

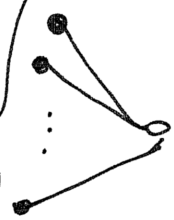
It is possible to remove some edges from $\mathcal{G}$ in such a way that the new graph $\mathcal{G}'$ is a disjoint union of



at least one empty vertex

and

at least one solid vertex.

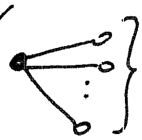


Hint: remove any edge which connects two vertices of degree at lest two. rinse, repeat.

Step 2. $$0 \leq N_{\mathcal{G}}(\lambda) \leq N_{\mathcal{G}'}(\lambda)$$

Step 3.

function $x \mapsto x^{i-1}$ is convex

N  i-1 vertices $= \sum_j (\lambda_j)^{i-1} \leq n^{i-1} =$

#vertices − #connected components

$= n$

Step 4.

$$N_{\mathcal{G}'}(\lambda) \leqslant n^{\underbrace{\#\text{vertices of } \mathcal{G}' - \#\text{connected components}}_{\text{when is it equal to } n^k \text{ or more?}}}$$

(otherwise $\mathcal{G}$ would not contribute!)

• #vertices of $\mathcal{G}'$ = #vertices of $\mathcal{G}$ = #cycles of γ_1 + #cycles of γ_2

$$= k - \|\gamma_1\| + k - \|\gamma_2\| \underset{\text{triangle inequality}}{\leqslant}$$

length on the S_k:

$\|\pi\|$ = minimal number of transpositions to write π as product of transpositions = $k - (\#\text{cycles of } \pi)$.

$$\leqslant 2k - \|\underbrace{\gamma_1 \cdot \gamma_2}_{=(1,2,\dots,k)}\| = 2k - (k-1) =$$

$$= k+1.$$

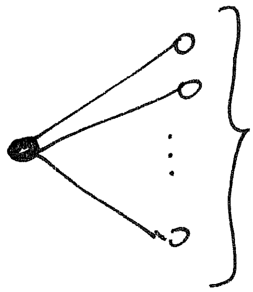
• # connected components of $\mathcal{G}' \geqslant 1$.

when equality holds?

Therefore...

#vertices of $\mathcal{G}'$ − # connected components of $\mathcal{G}' \leqslant k$

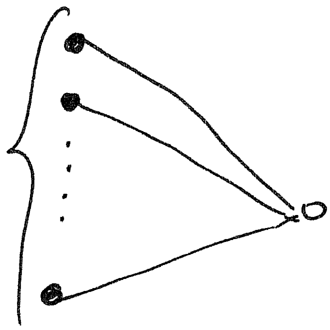
To have a non-zero limit, $\mathcal{G}'$ must look like this:



k empty vertices

or

k solid vertices.



$\mathcal{G}$ must also look like this. Therefore

$\sigma_1 \in S_k$ has k cycles so

$\sigma_1 = e$

$\sigma_2 = (1, 2, \dots, k)$

or

$\sigma_2 \in S_k$ has k cycles so

$\sigma_2 = e$.

$\sigma_1 = (1, 2, \dots, k)$

these two summands we already considered.

$\square$

5* Bonus material / Science fiction.

S_∞ is defined as the group of permutations with finite support:

$$S_\infty = \Big\{ \pi : \mathbb{N} \longrightarrow \mathbb{N} : \underbrace{\#\{ i \in \mathbb{N} : \pi(i) \neq i \}}_{\text{support of permutation.}} < \infty \Big\}$$

We are interested in ~~irreducible~~ factorial

bad idea!

the notion of irreducible representation of an infinite group sometimes behaves very badly!

representations :

$$\varrho : S_\infty \longrightarrow \underbrace{B(\mathcal{H})}_{\substack{\text{bounded operators on} \\ \text{a Hilbert space}}}$$

factorial :

$$\underbrace{\{ \varrho(\pi) : \pi \in S_\infty \}''}_{\text{is a von Neumann algebra which is a}} \subseteq B(\mathcal{H})$$

II_1 factor equipped with a canonical trace φ

Quick overview

von Neumann algebra =

*-algebra of bounded operators on a Hilbert space, closed in appropriate topology.

factor = von Neumann algebra with a property that its center is trivial ($= \mathbb{C} \cdot I$) (only multiples of the identity operator)

II_1-factor = a particularly nice factor; has a unique tracial state ("trace")

linear map $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$

$\mathcal{A}$: v. Neumann algebra

φ: trace

- $\varphi(1) = 1$
- $\varphi(ab) = \varphi(ba)$ "tracial property"
- state = some technicalities...

factorial representation ϱ of S_∞

$\updownarrow$ bijective correspondence

character of S_∞:

$$\chi(\pi) := \underbrace{\varphi}_{\text{the unique tracial state of the von Neumann algebra}} \left(\varrho(\pi) \right)$$

can be characterized as a function φ which
- $\varphi(e) = 1$
- is positive definite *
- and is extremal with these properties...

$\updownarrow$ Elmar Thoma (1964)

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0$$
$$\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant 0$$

such that

$$\sum_j \alpha_j + \sum_j \beta_j \leq 1$$

Positive definite:
for any $z_i \in \mathbb{C}$ $g_i \in G$

$$\sum_{i,j} z_i \bar{z}_j \, \varphi(g_i g_j^{-1}) \geq 0$$

$\Updownarrow$

φ can be used to define a scalar product on functions on G: $(\ell^2(G))$

$$\langle \delta_g, \delta_h \rangle = \varphi(g h^{-1})$$

$$\chi(\pi) = \prod_{\substack{c | \pi \\ |c| \geq 2}} \left[\sum_j \alpha_j^{|c|} - \sum_j (-\beta_j)^{|c|} \right]$$

Wassermann, Vershik, Kerov:

If a sequence of Young diagrams $\lambda^{(n)}$ is given such that

- $|\lambda^{(n)}| = n$
- $\dfrac{\lambda_i^{(n)}}{n} \longrightarrow \alpha_i \qquad \dfrac{\lambda_i^{(n)'}}{n} \longrightarrow \beta_i$

Then

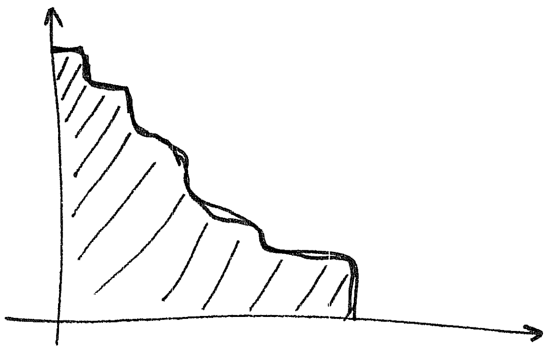
$$\underbrace{\frac{\operatorname{tr} \varrho^{\lambda^{(n)}}(\pi)}{\operatorname{tr} \varrho^{\lambda^{(n)}}(e)}}_{\text{irreducible character of } S_n} \longrightarrow \chi(\pi)$$

Thoma character of S_∞.

$\boxed{\text{Deep exlanations why this happens.}}$

Generalized Young diagrams

(in French notation)



profile: any bounded, non-increasing function

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$

with a compact support.

generalized Young diagram:

$$\lambda = \{ (x,y) : 0 \leq x \quad 0 \leq y \leq f(x) \}$$

new definition of N_G

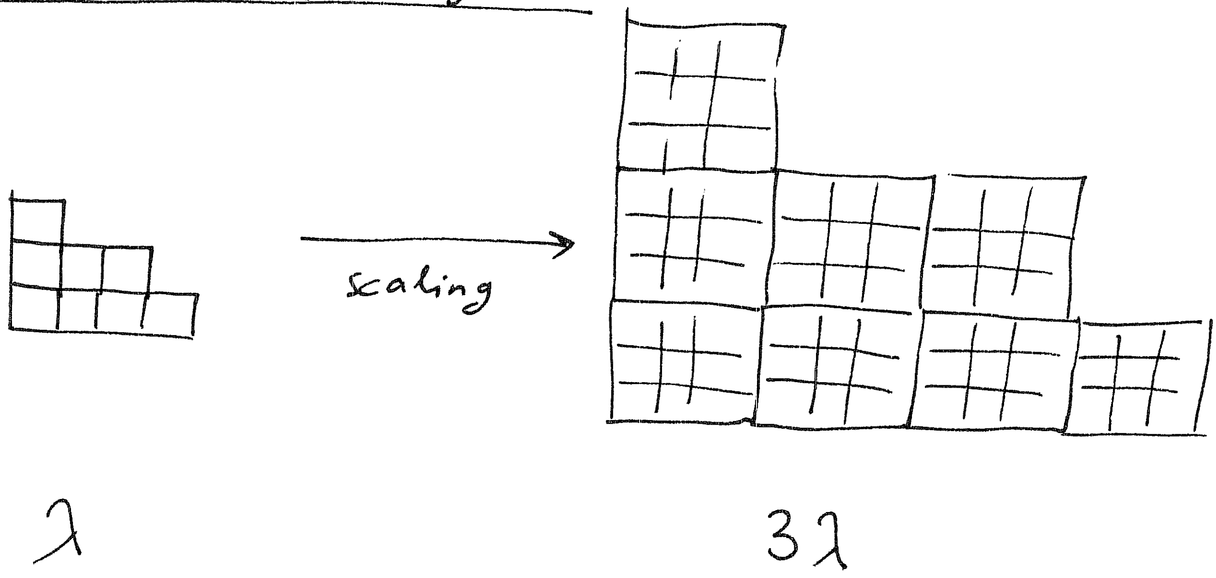
$N_G(\lambda)$ is the ~~number~~ volume of pairs (h_1, h_2) s.t.

- $h_i : \mathcal{V}_i \longrightarrow \cancel{\mathbb{N}}\ \mathbb{R}_+$
- if vertices $v_1 \in \mathcal{V}_1$, $v_2 \in \mathcal{V}_2$ are connected by an edge, we require that $(h_1(v_1), h_2(v_2)) \in \lambda$

$\mathcal{G}$ is a bipartite graph with vertices $\mathcal{V}_1 \cup \mathcal{V}_2$

$(h_1, h_2) \in \mathbb{R}_+^{|\mathcal{V}|}$

Scaling of Young diagrams.



If $s>0$ is not an integer then $s\lambda$ might be not a Young diagram, but a generalized Young diagram.

$\boxed{N_G \text{ is a homogeneous function!}}$ of degree #vertices.

$$N_G(s\lambda) = s^{\#\text{vertices of } G} \cdot N_G(\lambda)$$

→ We like homogeneous functions very much!

Stanley formula $\quad \pi \in S_k$

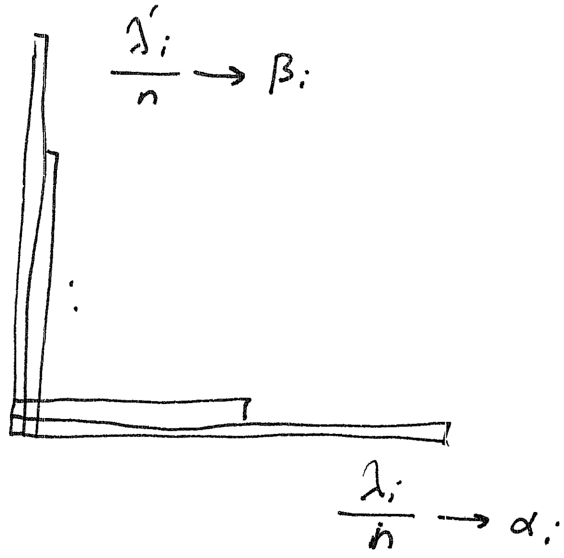
$$\Sigma_\pi(\lambda) = \sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_1 \sigma_2 = \pi}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}(\lambda)$$

makes sense even if λ is a generalized Young diagram! $\boxed{\text{Big surprise!}}$

if λ is a true (not generalized) Young diagram $|\lambda| = n$

$$= n(n-1)\ldots(n-k+1) \frac{\operatorname{tr} \rho^\lambda(\pi)}{\operatorname{tr} \rho^\lambda(e)}$$

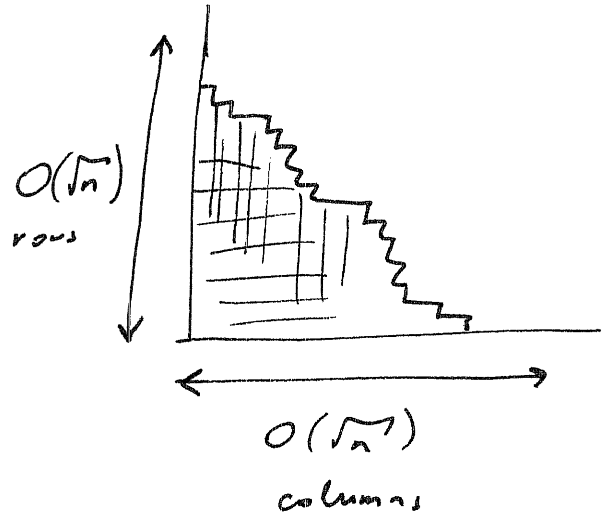
Balanced Young diagrams



"Thoma scaling":

long rows and columns

($\to$ Wassermann estimate)



Young diagram is called <u>C-balanced</u> if it has at most $C\sqrt{n}$ rows and at most $C\sqrt{n}$ columns, where n = number of boxes of λ

If we study asymptotic properties of balanced Young diagrams it makes sense to have a look at

$$\frac{1}{\sqrt{n}} \lambda^{(n)} \quad , \text{ if } \lambda^{(n)} \text{ has } n \text{ boxes.}$$

<u>For example:</u> a sequence of Young diagrams $\lambda^{(n)}$ is given, $|\lambda^{(n)}| = n$;

$$\frac{1}{\sqrt{n}} \lambda^{(n)} \xrightarrow[\text{topology}]{\text{some}} (\text{some generalized Young diagram}).$$

What can we say about characters of $\lambda^{(n)}$? for $n \to \infty$?

3 Maps

$$\sigma_1 \sigma_2 = \pi$$

$$\sigma_1, \sigma_2, \pi \in S_k$$

$\longleftrightarrow$ bipartite graph drawn on an oriented surface, edges labeled $1, 2, \dots, k$

σ_1 – structure of $\cancel{\bullet}$ ○ vertices

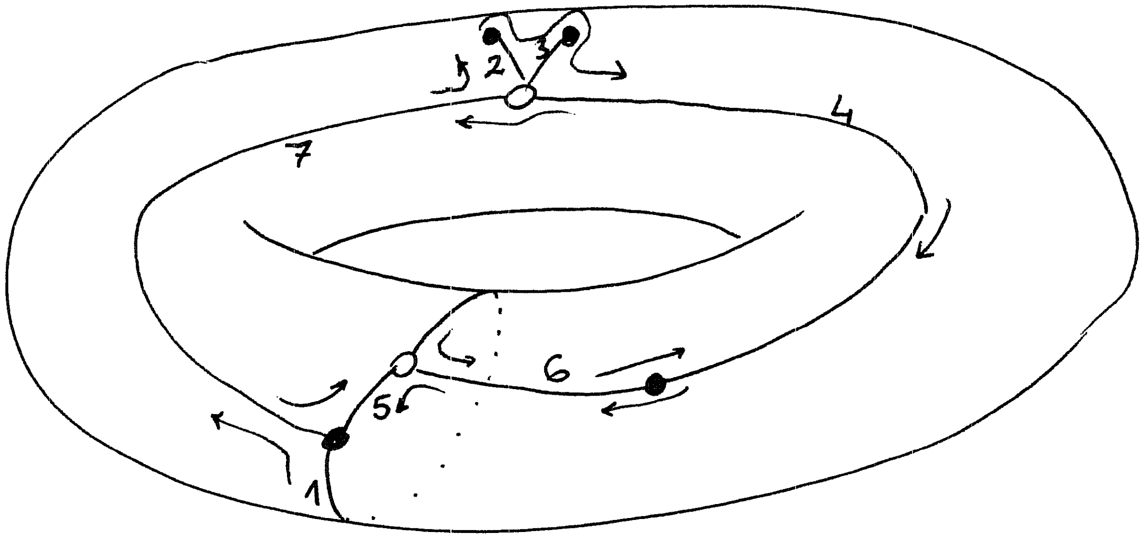
σ_2 – structure of ● vertices

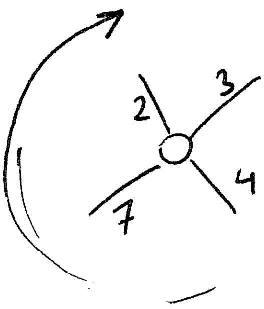
π – structure of faces

Example:

$\sigma_1 = (165)(2347)$

$\sigma_2 = (175)(2)(3)(46)$

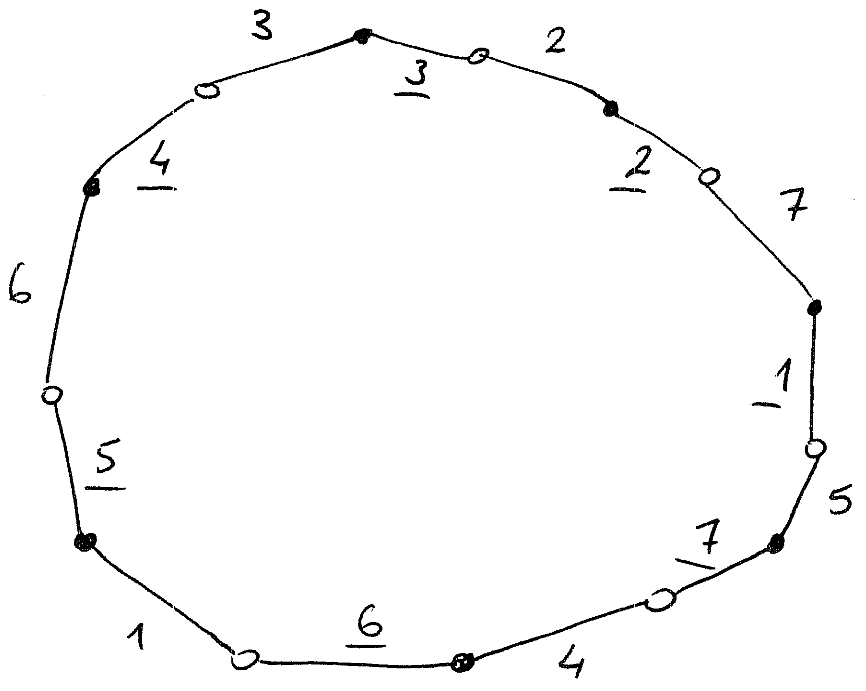




going around ○ vertices we read cycle structure of σ_1

going around ● vertices we read cycle structure of σ_2

$\Pi = (1\,2\,3\,4\,5\,6\,7)$



If we follow the arrows, we go along the boundary of the faces of the map. Reading every second label of the edge we recover the cycle structure of Π.

It is convenient to ~~treat~~ think that summands in Stanley formula correspons to maps. Therefore intead of $N_{\sigma_1,\sigma_2}(\lambda)$ we often write $N_M(\lambda)$ or even

$\uparrow$ map

$N_G(\lambda)$

$\uparrow$ bipartite graph

Characters on balanced Young diagrams

$\sigma_1, \sigma_2 \in S_k$

$\sigma_1 \cdot \sigma_2 = \pi$

Euler characteristic

$$\chi(M) = V - E + F$$

F: number of faces, equal to the number of cycles of $\sigma_1 \sigma_2 = \pi$

E: number of edges, equal to k

V: number of edges, equal to k.

$V = \chi(M) + k -$
$- \#\text{cycles of } \pi$

V depends only on topology of the map!

$$\chi(M) = \sum_{\substack{\text{connected}\\ \text{components of}\\ M}} 2 - 2\,\text{genus} \leq \underbrace{2 \cdot \#\text{connected components of } M}_{\substack{\text{equal to the number of orbits}\\ \text{in the action of the group}\\ \langle \sigma_1, \sigma_2 \rangle \text{ on } \{1, 2 \ldots, k\}}} \leq$$

$$\leq 2 \cdot \#\text{cycles of } \sigma_1 \sigma_2 = \pi$$

And it is easy to see when the equality holds true!

So...

$$V \leq k + \#\text{cycles of } \pi$$

Assume λ is C-balanced. Then

$$0 \leq N_{\sigma_1,\sigma_2}(\lambda) \leqslant (\#\text{ columns of } \lambda)^{\#\text{cycles of } \sigma_1} \cdot (\#\text{ rows of } \lambda)^{\#\text{cycles of } \sigma_2} \leqslant$$

$$\leqslant (C\sqrt{n})^{\#\text{vertices of } M} = (C\sqrt{n})^{\chi(M)+k-\text{cycles of } \pi} \leq O\left(n^{\frac{k+\#\text{cycles of } \pi}{2}}\right).$$

⚠ The contribution of σ_1,σ_2 asymptotically depends only on the topology of the map.

$\boxed{\text{Corollary from Stanley formula: for } \pi \in S_k,\ |\lambda|=n}$

$\boxed{\lambda \text{ is } C\text{-balanced}}$

$$\Sigma_\pi(\lambda) = \sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_1\sigma_2=\pi}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}(\lambda) = O\left(n^{\frac{k+\#\text{cycles of } \pi}{2}}\right)$$

$$\frac{\operatorname{tr}\rho^\lambda(\pi)}{\operatorname{tr}\rho^\lambda(e)} = \underbrace{\frac{1}{n(n-1)\cdots(n-k+1)}}_{\approx n^{-k}} \cdot \Sigma_\pi(\lambda) =$$

$$= O\left(\frac{1}{n^{\frac{k-\#\text{cycles of } \pi}{2}}}\right) = O\left(\frac{1}{\sqrt{n}^{\|\pi\|}}\right)$$

where $\|\pi\|$ = length of π = $k-\#$cycles of π is the minimal number of factors to write π as a product of transpositions

Comment:

The contribution of g_1, g_2 asymptotically depends only on the topology (Euler characteristic) of the corresponding map. Maps with big contribution correspond to spheres; the more holes (handles), the smaller contribution.

This phenomenon occurs as well in the random matrix theory, it is called "genus expansion".

Recomended reading:

S. K. Lando, A. K. Zvonkin, "Graphs on Surfaces and their Applications" Springer-Verlag 2004

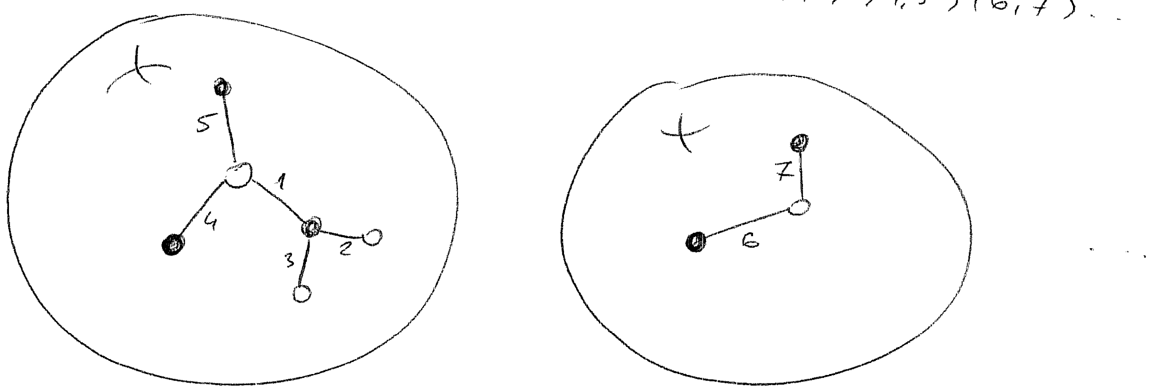
Which summands are really of the maximal degree?

- \# connected components of M must be equal to the ~~...~~ \# cycles of π, so

  each cycle of γ_1 and each cycle of γ_2 must be contained in one of the cycles of π

- each connected component of M must be a sphere with one face:

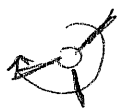
$\pi = (1,2,3,4,5)(6,7)\ldots$



in other words: each connected component of M is a bipartite, plane rooted tree

Vocabulary:

<u>plane tree:</u> tree drawn on a plane so that we know what



it means to go clockwise around a vertex

<u>rooted tree</u> : tree with a special vertex called <u>root</u>.

In our case it is not difficult to find some convention for defining the root since the edges of our map are labeled!

$\boxed{\text{Alternatively:}}$

$\sigma_1, \sigma_2 \in S_k$
$\sigma_1 \sigma_2 = \pi$

$$\|\pi\| = k - \#\text{cycles of } \pi$$

$\wedge$ triangle inequality

$$\|\sigma_1\| + \|\sigma_2\| = (k - \#\text{cycles of } \sigma_1) + (k - \#\text{cycles of } \sigma_2) =$$

$$= 2k - (\#\text{white vertices} + \#\text{black vertices}) =$$

$$= 2k - V$$

If $\|\sigma_1\| + \|\sigma_2\| > \|\pi\|$

then

$\|\sigma_1\| + \|\sigma_2\| \geq \|\pi\| + \underline{\underline{2}}$!

It is impossible that $\|\sigma_1\| + \|\sigma_2\| = \|\pi\| + 1$

because $(-1)^{\sigma} = (-1)^{\|\sigma\|}$;

so...

$$V \leqslant k + \#\text{cycles of } \pi$$

and the equality holds true iff

$$\|\sigma_1\| + \|\sigma_2\| = \|\sigma_1 \sigma_2\| = \|\pi\|$$

„minimal factorizations of π".

Free cumulants

„free cumulant of λ"

Good news! R_{k+1} is a homogeneous function of degree $k+1$!

$$R_{k+1}(\lambda) := \sum_{\substack{\sigma_1, \sigma_2 \in S_k \\ \sigma_1 \sigma_2 = (1,2,\dots,k) \\ \|\sigma_1\| + \|\sigma_2\| = \|(1,2,\dots,k)\|}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}(\lambda)$$

minimal factorizations

„dominant part of Σ_k".

$\boxed{\text{Corollary:}}$ for C-balanced Young diagram λ, $|\lambda| = n$

$$\Sigma_{(1,2,\dots,k)}(\lambda) = \underbrace{R_{k+1}(\lambda)}_{\text{this part is } O\left(\sqrt{n}^{\frac{k+1}{2}}\right)} + O\left(\sqrt{n}^{\frac{k-1}{2}}\right)$$

at first sight you may think that it should be $O\left(\sqrt{n}^{\frac{k}{2}}\right)$, but the parity argument shows it is not possible.

$\boxed{\text{Corollary 2}}$ for C-balanced Young diagram λ $|\lambda| = n$

$$\Sigma_\pi(\lambda) = \prod_{c | \pi} R_{|c|+1}(\lambda) + O\left(\sqrt{n}^{\frac{k + \#\text{cycles of } \pi - 2}{2}}\right).$$

Alternative formulation

If $\lambda^{(n)}$ is a sequence of C-balanced Young diagrams, ~~and~~ $\cancel{\lambda^{(n)}}$ $|\lambda^{(n)}| = n$ and

$$\frac{1}{\sqrt{n}} \lambda^{(n)} \longrightarrow \lambda$$

(generalized Young diagram)

(in some nice topology to be specified)

then

$$\frac{\operatorname{tr} \rho^{\lambda^{(n)}}(\pi)}{\operatorname{tr} \rho^{\lambda^{(n)}}(e)} \cdot \sqrt{n}^{\|\pi\|}$$

$$\prod_{c | \pi} R_{|c|+1}(\lambda)$$

$$\Sigma_{\pi}(\lambda) \cdot \frac{1}{\sqrt{n}^{\,k + \#\text{cycles of } \pi}}$$

$(\pi \in S_k)$

Summary

Asymptotics of characters on balanced Young diagrams is given by free cumulants.

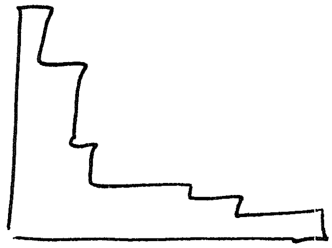
Free cumulants are (almost) scale-invariant (= homogeneous), can be seen as functions of the shape of λ.

"Shape of λ determines the characters".

Questions for future lectures:

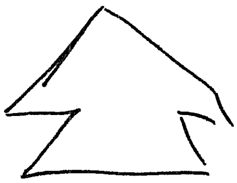
- how to compute free cumulants efficiently?
- how to determine a Young diagram if we know its characters / free cumulants?

General idea to Lectures 1 – 4 (approx.)



¿ what is the relationship between ... ? $\chi^\lambda(\pi)$

"shape" of a Young diagram $\longleftrightarrow$ characters



Lecture 1-2:

"shape" = encoded in "the number of colorings" N_{θ_1,θ_2}

Lecture 3:

"shape" = contents of the boxes

Lecture 4:

"shape" = ...

Further reading.

- Valentin Féray, Piotr Śniady,
  Asymptotics of characters ... related to Stanley formula
  Ann. Math. 173 ~~([illegible])~~ (2011), p. 887–906

  (proof of Stanley's formula and overview of some of its implications)

- S.V. Kerov, Asymptotic representation theory of symmetric groups ...
  Translations of Mathematical Monographs, 219, AMS 2003

  (remarkable book! It gives a good overall picture of what asymptotic representation theory is)