IMPAN lecture 04

Plan

Partial permutations

Normalized conjugacy classes
Characters and random Young diagrams.
Structure constants and Farahat-Higman

Normalized characters

Algebra of polynomial functions

~~JM elements~~

Partial permutations

Reading:
Ivanov, Kerov,
„The algebra of conjugacy classes ..."
J. Math. Sciences 2001, 175:5, 4212–4230

Partial permutation of set X is a pair

$$(\pi, A)$$

A - (finite) subset of X
A is called "support"

ALTERNATIVELY

$\pi : X \to X$
is a bijection s.t.
$\pi(x) = x$ if $x \notin A$

$\pi : A \to A$
is a bijection

π acts in a non-trivial way only on ~~the support~~ the support

http://goo.gl/N72B9

Reading: Philippe Biane,
„Characters of symmetric groups and free cumulants"
Springer Lecture Notes in Mathematics 1815 (2003)

$$(\pi_1, A_1) \cdot (\pi_2; A_2) = (\pi_1 \pi_2, A_1 \cup A_2)$$

Partial permutations form a semigroup with the unit $(id, \emptyset)$. P_n = semigroup of partial permutations of $\{1, \dots, n\}$.

Philosophical remark:

~~any permutation can b~~

if we want to turn a permutation into a partial permutation, we need to decide which of its fix-points should belong to the support. This might seem silly but it is a source of major headache if we work with the usual permutations. Partial permutations remember which of its fix-points are „true fix-points" and which are „cycles of length one".

In particular, a product of two permutations may contain a lot more fixpoints than the original factors.

We shall see very soon that partial permutations are much better than permutations!

Inverse system:

$$\mathbb{C}[P_1] \xleftarrow{r_1} \mathbb{C}[P_2] \xleftarrow{r_2} \mathbb{C}[P_3] \longleftarrow \cdots$$

P_n = semigroup of partial permutations of $\{1,2,\cdots,n\}$

$\mathbb{C}[P_n]$ = partial ~~pe~~ permutations algebra

$r_k : \mathbb{C}[P_{k+1}] \longrightarrow \mathbb{C}[P_k]$ is the <u>restriction</u>, r_k is an <u>algebra homomorphism</u>!

$$r_k(\pi, A) = \begin{cases} (\pi, A) & \text{if } A \subseteq \{1,2,\cdots,k\} \\ 0 & \text{otherwise} \end{cases}$$

Exercise: why if we work with symmetric group algebras $\mathbb{C}[S_n]$ this does not work

$$\mathbb{C}[S_1] \longleftarrow \mathbb{C}[S_2] \longleftarrow \cdots \quad ?$$

This inverse system is very interesting, but too big for our purposes. We will need only some special elements in it.

$\varprojlim \mathbb{C}[P_n]$ is an algebra where we can study convolution of conjugacy classes in all symmetric groups at the same time

(Normalized) conjugacy classes

For $\pi \in S_k$ we define

usually we will not write this!

$$\Sigma_\pi^{(n)} \in \mathbb{C}[P_n]$$

as

$$\Sigma_\pi^{(n)} = \sum_{\substack{f:\{1,\dots,k\}\to\{1,\dots,n\} \\ \text{injection}}} \Big(\underbrace{f\circ\pi\circ f^{-1}}_{f\circ\pi\circ f^{-1}\,:\, A \to A} \;,\; \underbrace{\text{Image of } f}_{\substack{A := \\ (\text{support})}} \Big)$$

$$\begin{tikzcd}
\mathbb{C}[P_1] & \mathbb{C}[P_2] \arrow[l, "r_1"'] & \cdots \arrow[l] \\
\Sigma_\pi^{(1)} \arrow[u, phantom, sloped, "\in"] & \Sigma_\pi^{(2)} \arrow[u, phantom, sloped, "\in"] \arrow[l, mapsto, "r_1"'] & \cdots \arrow[l, mapsto]
\end{tikzcd}$$

$$\Sigma_\pi = \varprojlim \Sigma_\pi^{(n)}$$ is a "Platonic object", a heavenly conjugacy class described by π.

Philosophical remark:

this "inverse system" business is just an abstract-nonsense way of stating results ~~which~~ on conjugacy classes in S_n in a way which is n-independent

Notation

Since Σ_π depends only on the conjugacy class of π, we can also define Σ_μ if μ is a partition:

$$\Sigma_\mu := \Sigma_\pi \quad \text{where } \pi \in S_{|\mu|} \text{ is}$$

any permutation with the conjugacy class specified by μ.

Most important:

$$\Sigma_k = \Sigma_{(1,2,\dots,k)}.$$

Alternative description: (but equivalent)

$$\Sigma_\mu = \sum_{\substack{\text{all tableaux of} \\ \text{shape } \mu}} \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \cline{1-1} 11 & 1 & 7 & \\ \cline{1-3} 9 & 3 & 5 & 4 \\ \hline \end{array} \in \mathbb{C}[P_n]$$

all tableaux of shape μ,
we fill boxes of μ with $\{1, 2, \dots, n\}$
each number should occur <u>at most</u> once

we interpret each row as a cycle of a permutation.
support = set of the labels.

Multiplication of conjugacy classes - Example.

$$\Sigma_{2} \cdot \Sigma_{2} = \quad ?$$

$$= \sum_{\substack{f_1 : \{1,2\} \to \\ \to \{1,\dots,n\}}} \cdots \sum_{\substack{f_2 : \{3,4\} \\ \to \{1,\dots,n\}}} \cdots =$$

there are several possibilities on how ~~the~~ image of f_1 intersects image of f_2

$$= \sum_{\substack{f_1 : \{1,2\} \to \dots \\ f_2 : \{3,4\} \to \dots}} (\cdots)(\cdots)$$

the following possibilities might occur:

A) Image f_1 is disjoint with image of f_2

B) $f_1(1) = f_2(3)$, $f_1(2) \neq f_2(4)$

C) $f_1(1) = f_2(4)$, $f_1(2) \neq f_2(3)$

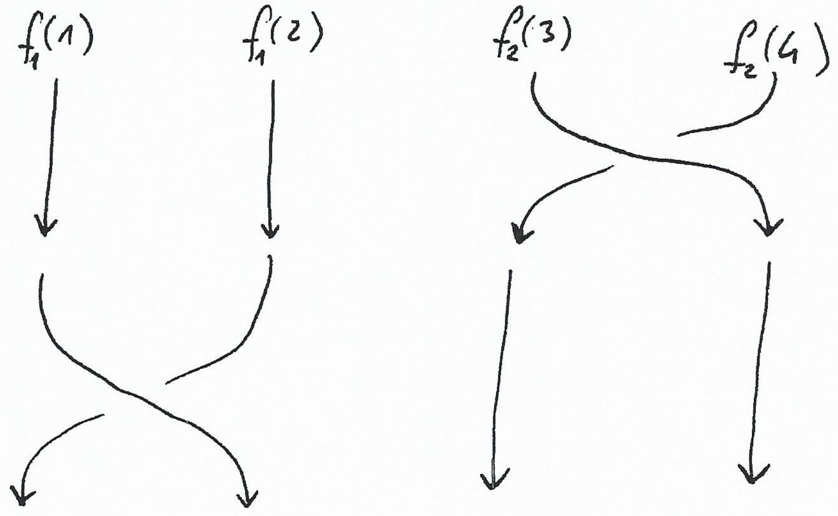
D) $\vdots$

E)

F) $f_1(1) = f_2(3)$, $f_1(2) = f_2(4)$

G) $f_1(1) = f_2(4)$, $f_1(2) = f_2(3)$

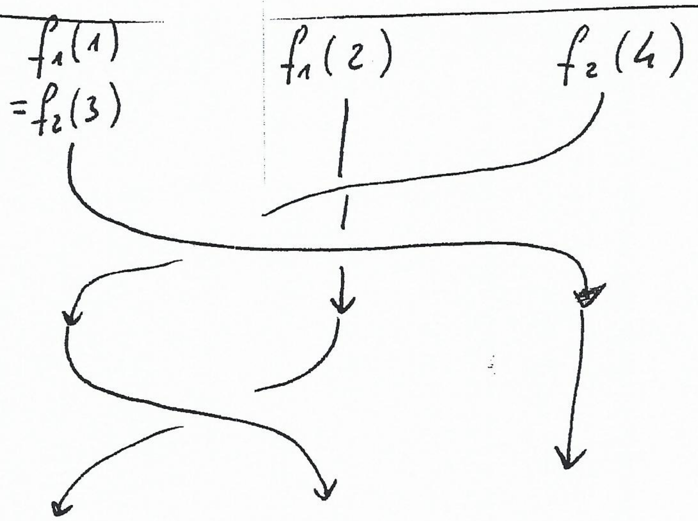
A



$$= \sum_{f:\{1,2,3,4\} \to \dots} \Big(\big(f(1), f(2)\big) \cdot \big(f(3), f(4)\big), \ \text{Image of } f \Big)$$

$$= \sum\nolimits_{2,2}$$

B



$$= \sum_{f:\{1,2,4\}} \Big(\big(f(1), f(4), f(2)\big), \ \text{Image of } f \Big) =$$

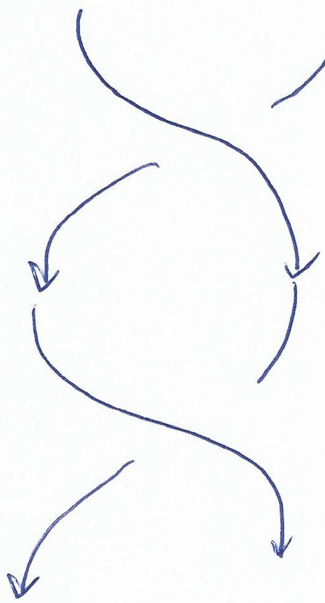
$$= \sum\nolimits_{3}$$

$\vdots$

etc.

F

$$f_1(1) = f_2(3) \qquad f_4(2) = f_2(4)$$



$$= \sum_{f:\{1,2\}} (\,\mathrm{id}\,,\ \mathrm{Image}\ f\,) = \Sigma_2$$

finally:

$$\Sigma_2 \cdot \Sigma_2 = \Sigma_{2,2} + 4 \cdot \Sigma_3 + 2 \cdot \Sigma_{1,1}$$

Fact / Exercise. For any partitions μ_1, μ_2 there exist (non-negative) integer numbers n_ν s.t.

$$\Sigma_{\mu_1} \cdot \Sigma_{\mu_2} = \sum_\nu n_\nu \Sigma_\nu$$

- these numbers are unique (*)
- only finitely many of them are non-zero.
- we can think that n_ν are „connection coefficients" since they express a product of two conjugacy classes as a combination of conjugacy classes.

In other words, (Σ_μ) span an <u>algebra</u> and (Σ_μ) is a linear basis of this algebra.

(*)

if we want to have uniqueness of these numbers, we should treat

$$\Sigma_\mu = \lim_{\longleftarrow} \Sigma_\mu^{(n)}$$ as the element of the inverse system.

~~If we work with~~

If we treat $\Sigma_\mu = \Sigma_\mu^{(n)}$ as an element of P_n then we are in trouble since

$$\Sigma_\mu^{(n)} = 0 \quad \text{if} \quad |\mu| > n,$$

so <u>no</u> uniqueness.

philosophical remark:

algebra ~~gen~~ spanned by Σ_π is commutative

so it might seem trivial. Not true!

a large part of the asymptotic representation

theory is to understand some subtle

questions related to the structure constants.

Partial permutations and normalized characters.

Let λ be a Young diagram with n boxes,

$\pi \in S_k$.

$$\rho^\lambda \big(\underbrace{\Sigma_\pi}_{} \big) = \; ?$$

formally, element of $\mathbb{C}[P_n]$, but if we forget about support, element of $\mathbb{C}[S_n]$ which is central.

$$\rho^\lambda (\Sigma_\pi) = \mathrm{Id} \cdot \underbrace{(\text{constant})}_{= \, ?}$$

$$\Sigma_\pi = \underbrace{\sum_{f : \{1, \dots, k\} \to \{1, 2, \dots, n\}}}_{} (\text{something conjugate to } \pi, \; \dots\dots)$$

$n(n-1) \dots (n-k+1)$ summands

$\vdots$

$$\boxed{\text{o! this looks exactly like something which we called the normalized character } \Sigma_\pi(\lambda)\text{!}}$$

$$\rho^\lambda \big(\Sigma_\pi \big) = n(n-1) \dots (n-k+1) \cdot \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\operatorname{Tr} \rho^\lambda(e)}$$

finite value of n

$$\begin{tikzcd}
Z\mathbb{C}[P_n] \arrow[d] \arrow[dd, dashed, bend right, "\text{not injective!}"'] \\
Z\mathbb{C}[S_n] \arrow[d, "\text{isomorphism}"] \\
\{f : \mathbb{Y}_n \to \mathbb{C}\}
\end{tikzcd}$$

linear span of $\Sigma_\pi^{(n)}$

center of the symmetric group algebra

functions on Young diagrams with n boxes

with pointwise product

$$\Sigma_\pi^{(n)} \longmapsto Ch_\pi$$

$$\sum_{\pi \in S_n} g_\pi \cdot \pi \overset{\text{"Fourier transform"}}{\longmapsto} f$$

$$f(\lambda) := \sum_{\pi \in S_n} g_\pi \operatorname{tr} \rho^\lambda(\pi)$$

ALGEBRA HOMOMORHISMS

$\overbrace{Z\mathbb{C}[P_n]}^{\text{algebra of random variables}}$, $\chi : Z\mathbb{C}[P_n] \longrightarrow \mathbb{C}$

expected value

character of S_n

$Z\mathbb{C}[S_n]$

ρ - reducible representation of S_n

$\underbrace{\{ f : \mathbb{Y}_n \longrightarrow \mathbb{C} \}}_{\text{algebra of random variables}}$, $\mathbb{P}$ = probability measure on $\mathbb{Y}_n$

'random irrep of ρ'

$$\boxed{\begin{array}{l} \text{if } \quad x \in Z\mathbb{C}[P_n] \\ \qquad\quad \downarrow \\ \qquad f : \mathbb{Y}_n \to \mathbb{C} \\ \text{then} \\ \chi(x) = \mathbb{E}_{\mathbb{P}} f \end{array}}$$

difficult result of Kerov!

Toy problem

SCIENCE FICTION

(Version A)

Let λ be a random irreducible component of the left-regular representation $\mathbb{C}[S_n]$.

Show that for each k

$$\sqrt{n}^{\,k}\, \operatorname{tr} \rho^{\lambda}\big((1,2,\dots,k)\big) \xrightarrow{d} \underbrace{N(0,1)}_{\text{Gaussian distribution.}}$$

(~~after some rescaling~~)

$\Uparrow$

IMPORTANT TODAY

(Version B).

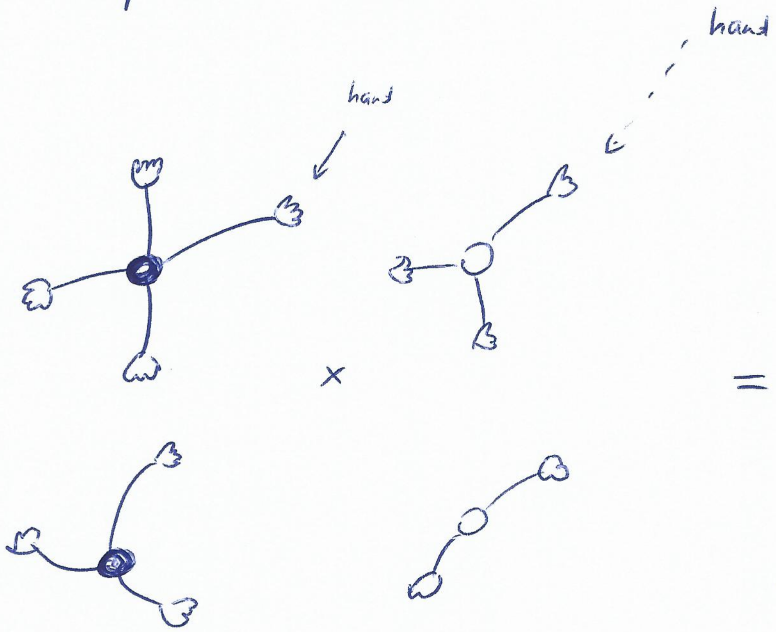
$$\underbrace{\chi^{(n)}}_{\substack{\text{character of}\\ \mathbb{C}[S_n]}}\left(\Sigma_\mu\right) := \begin{cases} n^{\underline{a}} = n(n-1)\cdots(n-a+1) & \text{if } \mu = (\underbrace{1,1,\dots,1}_{a \text{ times}}) \\ 0 & \text{otherwise.} \end{cases}$$

show that

$$\frac{\chi^{(n)}\left(\Sigma_k^{\,s}\right)}{n^{\frac{k}{2}}} \longrightarrow \begin{cases} 0 & \text{if } s \text{ odd} \\ (s-1)!! & \text{if } s \text{ even.} \end{cases}$$

Combinatorics of the structure constants

$$\Sigma_{\mu_1} \cdot \Sigma_{\mu_2} = \; ?$$

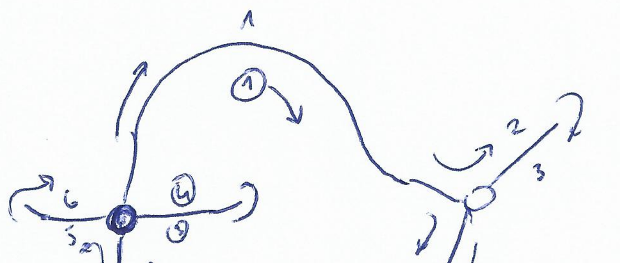


$$= \sum$$

vertices of degrees given by partition μ_1

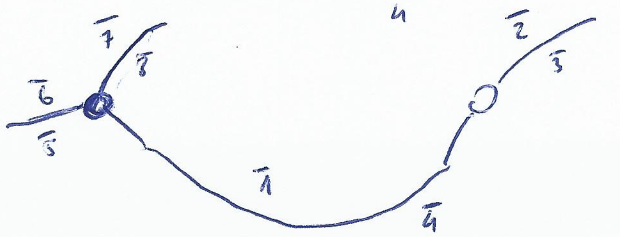
vertices of degrees given by partition μ_2

any hand of ● can be connected to a hand of ○



count the lengths of the faces / 2 :

$$\frac{6}{2}, \frac{4}{2}, \frac{8}{2} = 6, 2, 4$$



$$\leftarrow \Sigma_{6,2,4}$$

in the inverse limit, $n \to \infty$

$$
\begin{tikzcd}
\varprojlim_{n} Z\mathbb{C}[P_n] \arrow[d, "\text{injective}"] & = & \text{linear span of } \Sigma_\pi \arrow[d, "\text{isomorphism!}"] \\
\{ f : \mathbb{Y} \longrightarrow \mathbb{C} \} & \supseteq & \text{linear span of } Ch_\pi
\end{tikzcd}
$$

"algebra of polynomial functions"

? how to characterise functions on $\mathbb{Y}$ which belong to the linear span of Ch_π?

Reduced cycle structure.

For a permutation $\pi \in S_k$ let

$\mu_1 \geq \mu_2 \geq \dots \geq \mu_\ell$ be the lengths of the cycles of π. Notice that $|\mu| = \mu_1 + \mu_2 + \dots = k$.

The <u>reduced cycle structure</u> of π is defined as

$$\nu = (\mu_1 - 1 \geq \mu_2 - 1 \geq \dots \geq \mu_\ell - 1).$$

Notice that $|\nu| = \|\pi\|$ = minimal number of factors necessary to write π as product of transpositions.

sometimes we skip it.

Define

$$C_\nu^{(k)} = \sum_{\substack{\text{reduced cycle type} \\ \text{of } \pi \text{ is equal to} \\ \nu}} \pi \in \mathbb{C}[S_k].$$

"conjugacy class".

algebra of partial permutations $\qquad$ algebra of (usual) permutations

$$\mathbb{C}[P_n] \longrightarrow \mathbb{C}[S_k]$$

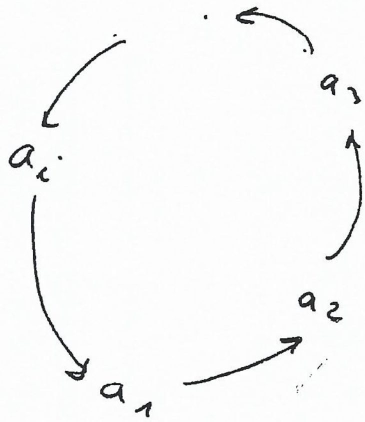
forgetting support

$$\Sigma_\mu \longmapsto (\text{some number})\ C^{(k)}_\nu$$

for example:

- for $i \geq 2$ $\qquad \Sigma_i \longmapsto i\ C_{i-1}$

$\{1, 2, \dots i\}$



there are i ways to produce each cycle

- for $i \geq 2$ $\qquad \Sigma_{i,i} \longmapsto i^2\ 2!\ C_{i-1,\,i-1}$

not difficult to guess the general formula!

- $$\Sigma_1 \longmapsto k \cdot C_\emptyset$$
- $$\Sigma_{1,1} \longmapsto \underbrace{k(k-1)}_{\substack{\text{polynomial} \\ \text{function of } k}} \cdot C_\emptyset$$

not difficult to guess the general formula!

Corollary: (example, continued)

$$\Sigma_2 \cdot \Sigma_2 = \Sigma_{2,2} + 4\,\Sigma_3 + 2 \cdot \Sigma_{1,1}$$

$$\big\downarrow$$

$$2\,C_1 \cdot 2\,C_1 = 8\,C_{1,1} + 12\,C_2 + 2 \cdot k(k-1)\,C_\emptyset$$

so...

$$C_1 \cdot C_1 = 2\,C_{1,1} + 3\,C_2 + \frac{k(k-1)}{2}\,C_\emptyset$$

polynomials in k

Thm For any partitions ν_1 and ν_2 there exist unique polynomials $f_\mu \in \mathbb{C}[k]$ s.t.

$$C^{(k)}_{\nu_1} \cdot C^{(k)}_{\nu_2} = \sum_{\mu} f_\mu \; C^{(k)}_\mu$$

Original proof (in some sense more complicated!):

Farahat, Higman

"The centers of symmetric group rings"

Proc. Roy. Soc. London Ser. A.

250 (1959) 212-221

In other words:

linear span of $(k^\alpha C_\nu)_{\alpha=0,1,2,\dots}$ forms an algebra. ~~The linear~~

$(k^\alpha C_\nu)$ forms a linear basis of it.

This algebra is called Farahat-Higman algebra.

Therefore . . .

$\mathrm{span}\left(\Sigma_\nu\right)$ $\xleftrightarrow{\text{isomorphism } (\;)}$ Farahat-Higman algebra $=$ $\mathrm{span}\left(k^\alpha C_\nu\right)$

calculations are usually easier

for some people this object is more intuitive

this isomorphism allows us to translate results from one language to another