Suggested reading:

~~Cecch~~ Ceccherini-Silberstein, Scarabotti & Tolli, "Representation Theory of the Symmetric Group" Cambridge Univ. Press 2010

→ REMARKABLE BOOK! ←

Jucys-Murphy elements

Some ~~a~~ authors call them Young-Jucys-Murphy elements.

$$X_1 = 0$$
$$X_2 = (1,2)$$
$$X_3 = (1,3) + (2,3)$$
$$\vdots$$
$$X_i = (1,i) + (2,i) + \cdots + (i-1,i)$$
$$\vdots$$
$$X_n = (1,n) + \cdots + (n-1,n)$$

$$\in \mathbb{C}[S_n]$$

Jucys-Murphy elements play the key role in the modern approach to representation theory of S_n of Okounkov and Vershik!

→ see book of C-SST ←

What do we need to know about JM elements?

this is really a minimal list, they have many more nice properties!

- $X_1, \dots, X_n$ commute

  Hint:

  $$X_i = \underbrace{(\text{sum of all transpositions in } S_i)}_{\text{central element in } \mathbb{C}[S_i]} - \underbrace{(\text{sum of all transpositions in } S_{i-1})}_{\text{central element in } \mathbb{C}[S_{i-1}]}$$

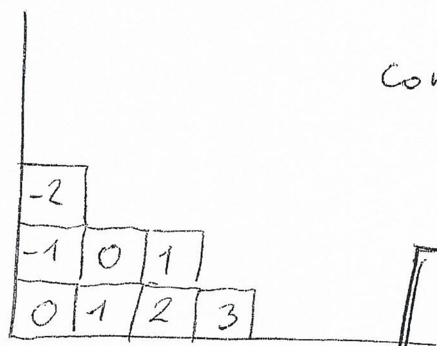
- for any Young diagram λ

and any symmetric polynomial $P(z_1, \dots, z_n)$

symmetric: $P(z_1, \dots, z_n) = P(z_{\pi(1)}, \dots, z_{\pi(n)})$ for any $\pi \in S_n$

$$\rho^\lambda\big(P(X_1, \dots, X_n)\big) = P(c_1, \dots, c_n) \cdot I,$$

where $c_1, \dots, c_n$ are the <u>contents</u> of the boxes of λ (listed in arbitrary order)



content of the box = (x-coordinate) − (y-coordinate)

The collection $(c_1, c_2, \dots, c_n)$ is a nice (but not perfect) way of describing shape of a Young diagram.

Example take

$$P(z_1, \dots, z_n) = z_1 + \dots + z_n.$$

then

$$P(X_1, X_2, \dots, X_n) = \sum_{i<j} (i,j) \qquad \text{sum of all transpositions}$$

$$\rho^\lambda \left(P(X_1, \dots, X_n) \right) = \rho^\lambda \left(\sum_{i<j} (i,j) \right) = (c_1 + c_2 + \dots + c_n) \, I$$

apply trace...

$$\binom{n}{2} \cdot \operatorname{tr} \rho^\lambda ((1,2)) = (c_1 + \dots + c_n) \operatorname{tr} \rho^\lambda (e)$$

Exercise:

compare this formula to the formula

$$\Sigma_2(\lambda) = \sum_i \lambda_i^2 - \sum_i (\lambda_i')^2$$

which we proved some time ago...

Comment: the contents $(c_1, \dots, c_n)$ can be ~~relati~~ easily extracted from the shape of λ. Therefore computation of $P(c_1, \dots, c_n)$ is "easy". The problem is to chose P in such a way that we calculate some useful quantity.

proof

We will need the following

Fact from representation theory of S_n:

Let V^λ be the irreducible representation of S_n corresponding to λ. There exists a basis (v_T) of the vector space V^λ indexed by T - standard Young tableaux of shape λ

~~in fact this~~ this is the key observation behind ~~the~~ Okounkov-Vershik approach to repr. of S_n!

called "Gelfand-Tsetlin basis"

Example

$$T = \begin{array}{|c|c|c|c|} \hline 6 & & & \\ \cline{1-1} 2 & 5 & 7 & \\ \cline{1-3} 1 & 3 & 4 & 8 \\ \hline \end{array}$$

standard Young tableau: numbers $1, 2, \dots, |\lambda|$, increase along rows and columns.

and such that

action of JM elements is diagonal in this basis.

$$X_i \, v_T = \underbrace{(\text{content of the box of } T \text{ which ~~has~~ contains number } i)}_{c_i^{(T)}} \cdot v_T$$

Example.

$$X_2 \, v_T = (-1) \, v_T$$

$$X_4 \, v_T = 2 \cdot v_T.$$

⚠ sequence $c_1^{(T)}, \dots, c_n^{(T)}$ depends on T!

Proof, continued.

$$X_i \, v_T = c_i \, v_T$$

$$\Downarrow$$

$$X_{i_1} \cdots X_{i_k} \, v_T = c_{i_1}^{(T)} \cdots c_{i_k}^{(T)} \, v_T$$

!
sequence $c_1^{(T)}, \ldots, c_n^{(T)}$ depends on T!

$$\Downarrow$$

$$P(X_1, \ldots, X_n) \, v_T = P(c_1^{(T)}, c_2^{(T)}, \ldots, c_n^{(T)}) \, v_T$$

for arbitrary polynomial!
not necessarily symmetric!

$$\underbrace{P(X_1, \ldots, X_n)}_{\text{if } P \text{ is symmetric.}} \, v_T = P\underbrace{(c_1^{(T)}, \ldots, c_n^{(T)})}_{} \, v_T$$

if P is symmetric.

multiset $(c_1^{(T)}, c_2^{(T)}, \ldots, c_n^{(T)})$ does not depend on T, it depends only on λ

In the base (v_T) the action of $P(X_1, \ldots, X_n)$ ~~is by~~ is a diagonal matrix

$$\begin{bmatrix} P(c_1, \ldots, c_n) & & \\ & P(c_1, \ldots, c_n) & \\ & & \ddots \end{bmatrix}.$$

$\square$

What do we need to know about JM elements, (cont.)?

elementary symmetric ~~function~~ polynomial

$$e_k(z_1,\dots,z_n) = \sum_{i_1<i_2<\dots<i_k} z_{i_1} z_{i_2} \cdots z_{i_k}$$

Reading on symmetric polynomials:

I. G. Macdonald
„Symmetric functions and Hall polynomials"
Oxford University Press.

- $$e_k(X_1,\dots,X_n) = \cancel{\sum_{|\nu|=k} C_\nu^{(n)}} = \sum_{\substack{\pi\in S_n \\ \|\pi\|=k}} \pi$$

to large extent this equality is n-independe. Which is nice!

$\boxed{\text{Proof}}$ (sketch!)

$$\longrightarrow (1+tX_1)(1+tX_2)\cdots(1+tX_n) = \sum_{k\geq 0} t^k \, e_k(X_1,\dots,X_n)$$

if we multiply out the brackets, and $X_i = (1,i)+\dots+(i-1,i)$ every permutation occurs exactly once

$$\longrightarrow (1+tX_1)(1+tX_2)\cdots(1+tX_n) = \sum_{\pi\in S_n} t^{\|\pi\|}\cdot\pi$$

$\|\pi\|$ = minimal number of factors necessary to write π as product of transpositions.

$\square$

~~Top-secret page which is necessary for some argument in Lecture 9.1 to work.~~

~~CORRECTION: we do not really need it! But it is nice!~~ → see

If we view $X_i = \underbrace{(1,i)} + \cdots + \underbrace{(i-1,i)}$

partial permutations with 2-element support

then the proof from the previous page show that

$$e_k(X_1, \dots, X_n) = \sum_{\substack{\pi \in S_n \\ \|\pi\| = k}} (\pi, \text{support of } \pi) =$$

partial permutation

$=$ linear combination of conjugacy classes $\left(\Sigma_\pi\right)$

Corollary. If P is a symmetric function,

$P(X_1, \dots, X_n) =$ linear combination of conjugacy classes (Σ_π).

Example

$$e_1(x_1, \dots, x_n) = \frac{1}{2} \sum_{2}$$

$$e_2(x_1, \dots, x_n) = \frac{1}{3} \sum_{3} + \frac{1}{2!\, 2^2} \sum_{2,2}$$

$$\vdots$$

Filtrations

Consider the algebra spanned by (Σ_μ)
("algebra of conjugacy classes").

Suppose that for any μ we have specified

$$\deg \Sigma_\mu \in \{0,1,2,\dots\}$$

"degree of Σ_μ".

☆ in such a way that whenever

$$\Sigma_{\mu_1} \cdot \Sigma_{\mu_2} = \sum_\nu \underbrace{n_\nu}_{\text{for some numbers } n_\nu} \Sigma_\nu$$

then

$$n_\nu \neq 0 \implies \deg \Sigma_\nu \leq \deg \Sigma_{\mu_1} + \deg \Sigma_{\mu_2}$$

if this is the case, we can use deg to define filtration on $\operatorname{span}(\Sigma_\mu)$, but this is just a formalizatio which does not give too much insight.

In typical applications, we can use the degree to say that some equalities hold true, up to some garbage which is of small degree.

$\boxed{\text{Example 1}}$

this degree is well-suited for studying of "Thoma scaling"

We declare

$$\deg \Sigma_\mu = |\mu| \qquad \text{(the size of support).}$$

$\boxed{\text{Very easy exercise}}$

1. this degree fulfills condition $(\star)$.
2. $$\Sigma_{\mu_1} \cdot \Sigma_{\mu_2} = \Sigma_{\mu_1 \cup \mu_2} + \text{(terms of degree at most } |\mu_1|+|\mu_2|-1\text{)}$$

This result can be translated to a result about reduced cycle classes...

if we declare

$$\deg \kappa^\alpha C_\nu = \alpha + |\nu| + \underbrace{\ell(\nu)}_{\text{the number of parts of } \nu}$$

Alternatively: declare that
$\deg \kappa = 1$
$\deg C_\nu = |\nu| + \ell(\nu)$

then the isomorphism $(\clubsuit)$ preserves this degree ~~exercise~~ (easy exercise).

$\boxed{\text{Corollary:}}$

$$C_{\nu_1} \cdot C_{\nu_2} = C_{\nu_1 \cup \nu_2} + \left(\begin{array}{l} \text{terms of degree} \\ \text{at most} \\ |\nu_1| + \ell(\cancel{\mu}_1) + \\ |\nu_2| + \ell(\nu_2) - 1 \end{array} \right)$$

With respect to this degree

$$\sum_i X_i^{\,k} = \underbrace{\frac{1}{k+1} \sum_{k+1}}_{= C_k} + \left(\text{conjugacy classes of degree at most } k\right)$$

$X_i = (1,i) + \dots + (i-1,i)$ is the Jucys-Murphy element.

<u>Proof</u>

we consider the filtration on the algebra of partial permutations given by

$$\deg(\underbrace{\pi, A}_{\text{support}}) = |A|$$

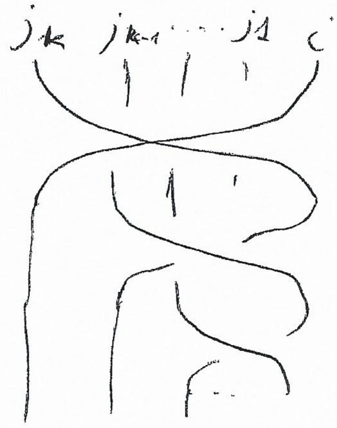
This degree is compatible with our filtration on the algebra of conjugacy classes.

$$\sum_i X_i^{\,k} = \sum_{j_1, j_2, \dots, j_k < i} \underline{(j_1, i)}\,\underline{(j_2, i)} \cdots \underline{(j_k, i)} =$$

View as partial permutations!

~~This is where we use the result from lecture~~ 3.

$$= \sum_{\substack{j_1,\dots,j_k<i \\ j_1,\dots,j_k \text{ are all different}}} \underbrace{(j_1,i)(j_2,i)\cdots(j_k,i)}_{= (i,j_k,j_{k-1},\dots,j_1)} \; + $$



$$C_k = \frac{1}{k+1}\Sigma_{k+1}$$

$$+ \sum_{\substack{j_1,\dots,j_k<i \\ j_1,\dots,j_k \text{ are NOT all different}}} \underbrace{(j_1,i)\cdots(j_k,i)}_{\substack{\text{support} = \{j_1,\dots,j_k,i\}, \\ \text{degree at most } k}}$$

~~Lecture 3, top secret page~~

~~this is a linear combination of conjugacy classes (Σ_ν)~~

The whole sum is a linear combination of conjugacy classes (Σ_ν).

size of support

Declare

$\deg \Sigma_\pi := k$ for $\pi \in S_k$

or

$\deg \Sigma_\pi := |\pi|$ for a partition π.

Filtration on the algebra of polynomial functions.

$\boxed{\text{Fact}}$ $\sum_i X_i^k = \frac{1}{k} \Sigma_k +$ (summands of smaller degree).

Comment:

thanks to the „degree" concept we

1) proved existence of something

2) we found the top-degree part of this something.

This is a quite typical situation in asymptotic representation theory.
Getting information about subdominant terms might be hard.

Explicit form of polynomial P_μ

→ M. Lassalle
„An explicit formula ..."
Math. Ann. 340 (2008) no. 2, 383–405

→ Book of Ceccherini-Silberstein, ...

$\boxed{\text{Example 2}}$

This scaling is well-suited for study of balanced Young diagrams scaling.

We declare

$$\deg \Sigma_\mu = |\mu| + \ell(\mu).$$

$\boxed{\text{difficult! exercise}}$

1. this degree fulfills condition ($\star$)

2. $\Sigma_{\mu_1} \cdot \Sigma_{\mu_2} = \Sigma_{\mu_1 \cup \mu_2}$ + (terms of degree at most $\deg \Sigma_{\mu_1} + \deg \Sigma_{\mu_2} - 2$)

If you know a <u>simple</u> proof,

I would be very interested in it!