Lecture 5

$\boxed{\text{Outlook}}$

JM-elements

functionals of shape

free cumulants

$$\boxed{\text{algebra} \quad \Lambda^* = \operatorname{span}\left(Ch_\pi : \pi\text{-partition} \right)}$$

$= \operatorname{span} \left(\Sigma_\pi : \pi\text{-partition} \right)$

- shifted symmetric function.

jej charakteryzacje

jej wiele baz algebraicznie $\to$ $Ch_\pi, Ch_2, \ldots$

$\to S$

$\to R$

$\to T$

$\to$ polynomials in JM elements.

filtracje

gradacje

$\boxed{\text{algebra of polynomial functions}}$

todo: filtrations / gradations

$\boxed{\text{Symmetric functions}}$

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each symmetric function is a polynomial in

- elementary symmetric functions $e_1, e_2, \dots$
- power-sum symmetric functions

Jucys-Murphy elements.

$$X_1, \dots, X_n \in \mathbb{C}[S_n]$$

$$X_k = (1,k) + (2,k) + \dots + (k-1,k)$$

$X_1, \dots, X_n$ commute

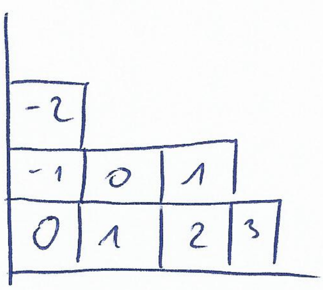
$\longrightarrow$ ideas of proof in the last lecture.

Fact

For any Young diagram λ with n boxes and any <u>symmetric</u> polynomial $P(z_1, \dots, z_n)$

$$\rho^\lambda\big(P(X_1, \dots, X_n)\big) = P(c_1, \dots, c_n) \cdot I$$

where $c_1, \dots, c_n$ is the multiset of <u>contents</u> of the boxes of λ



content of a box = (x-coordinate) - (y-coordinate).

"shape of $\lambda \longleftrightarrow$ information about characters"

Surprising

if $P(z_1, \dots, z_n)$ is a symmetric polynomial

then

$$P(X_1, \dots, X_n) \in Z\,\mathbb{C}[S_n]$$

belongs to the center of $\mathbb{C}[S_n]$.

explanation:

each symmetric polynomial can be expressed as a polynomial in elementary symmetric polynomials:

$$P = f(e_1, e_2, \dots, e_n)$$

VERTÉ →

What do we need to know about JM elements, (cont.)?

elementary symmetric polynomial ~~function~~

$$e_k(z_1, \dots, z_n) = \sum_{i_1 < i_2 < \dots < i_k} z_{i_1} z_{i_2} \cdots z_{i_k}$$

Reading on symmetric polynomials:

I. G. Macdonald
„Symmetric functions and Hall polynomials"
Oxford University Press.

- $$e_k(X_1, \dots, X_n) = \cancel{\sum_{|\nu|=k} C_\nu^{(n)}} = \underbrace{\sum_{\substack{\pi \in S_n \\ \|\pi\| = k}} \pi}_{\text{belongs to the center } Z\mathbb{C}[S_n]}$$

to large extent this equality is n-independent. Which is nice!

$\boxed{\text{Proof}}$ (sketch!)

$$\longrightarrow (1+tX_1)(1+tX_2)\cdots(1+tX_n) = \sum_{k \geq 0} t^k \, e_k(X_1, \dots, X_n)$$

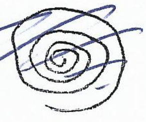
if we multiply out the brackets, and $X_i = (1,i) + \dots + (i-1,i)$ every permutation occurs exactly once

$$\longrightarrow (1+tX_1)(1+tX_2)\cdots(1+tX_n) = \sum_{\pi \in S_n} t^{\|\pi\|} \cdot \pi$$

$\|\pi\|$ = minimal number of factors necessary to write π as product of transpositions.

$\square$

~~Top-secret page which is necessary for some argument in Lecture 91 to work.~~

~~CORRECTION: we do not really need it! But it is nice!~~ $\longrightarrow$ see 

If we view $X_i = \underbrace{(1,i)} + \dots + \underbrace{(i-1,i)}$

partial permutations with 2-element support

then the proof from the previous page show that

$$e_k(X_1, \dots, X_n) = \sum_{\substack{\pi \in S_n \\ \|\pi\| = k}} \underbrace{(\pi, \text{support of } \pi)}_{\text{partial permutation}} =$$

$$= \text{linear combination of conjugacy classes } \left(\Sigma_\pi\right)$$

Corollary. If P is a symmetric function,

$$P(X_1, \dots, X_n) = \text{linear combination of conjugacy classes } \left(\Sigma_\pi\right).$$

Example

$$e_1(X_1,\dots,X_n) = \frac{1}{2}\Sigma_{2}$$

$$e_2(X_1,\dots,X_n) = \frac{1}{3}\Sigma_{3} + \frac{1}{2!\,2^2}\Sigma_{2,2}$$

$$\vdots$$

We abuse notation and write

$$P(X_1,X_2,\dots) = \lim P(X_1,\dots,X_n)$$ } element of the inverse limit.

Moral: for each symmetric function P,

$$P(X_1,X_2,\dots) \in \Lambda^*$$

span $(\Sigma_\pi : \pi$ is a partition$)$

$$\mathrm{span}\left(P(X_1,X_2,X_3,\dots) : P \text{ symmetric function}\right) \subsetneq \Lambda^*$$

?

equality?

Yes, but requi...

Filtrations

Filtrations

Consider the algebra spanned by (Σ_μ)

("algebra of conjugacy classes").

Suppose that for any μ we have specified

$$\deg \Sigma_\mu \in \{0,1,2,\dots\}$$

„degree of Σ_μ".

in such a way that whenever

$$\star \qquad \Sigma_{\mu_1} \cdot \Sigma_{\mu_2} = \sum_\nu \underbrace{n_\nu}_{\text{for some numbers } n_\nu} \Sigma_\nu$$

then

$$n_\nu \neq 0 \implies \deg \Sigma_\nu \leq \deg \Sigma_{\mu_1} + \deg \Sigma_{\mu_2}$$

if this is the case,
we can use deg to define
filtration on $\operatorname{span}(\Sigma_\mu)$,
but this is just a formalization
which does not give too
much insight.

In typical applications,
we can use the degree
to say that some equalities
hold true, up to some
garbage which is of small
degree.

Example 1

We declare

$$\deg \Sigma_\mu = |\mu|$$

(the size of support).

this degree is well-suited
for studying of "Thoma scaling"

Thoma filtration

Very easy exercise

1. this degree fulfills condition (✩).

2. $$\Sigma_{\mu_1} \cdot \Sigma_{\mu_2} = \Sigma_{\mu_1 \cup \mu_2} + \text{(terms of degree at most } |\mu_1|+|\mu_2|-1\text{)}$$

This result can be translated to a result about reduced cycle classes...

if we declare

$$\deg k^{\alpha} \cdot C_{\nu} = \alpha + |\nu| + \underbrace{\ell(\nu)}_{\text{the number of parts of } \nu}$$

Alternatively: declare that
$\deg k = 1$
$\deg C_{\nu} = |\nu| + \ell(\nu)$

then the isomorphism $(\clubsuit)$ preserves this degree ~~exercise~~ (easy exercise).

$\boxed{\text{Corollary:}}$

$$C_{\nu_1} \cdot C_{\nu_2} = C_{\nu_1 \cup \nu_2} + \left(\text{terms of degree at most } |\nu_1| + \ell(\mu_1) + |\nu_2| + \ell(\nu_2) - 1 \right)$$

With respect to this degree

$$\sum_i X_i^{\,k} = \underbrace{\frac{1}{k+1}\,\Sigma_{k+1}}_{= C_k} + \text{(conjugacy classes of degree at most } k\text{)}$$

$X_i = (1,i) + \dots + (i-1, i)$ is the Jucys-Murphy element.

Proof

we consider the filtration on the algebra of partial permutations given by

$$\deg(\underset{\substack{\uparrow \\ \text{support}}}{\pi, A}) = |A|$$

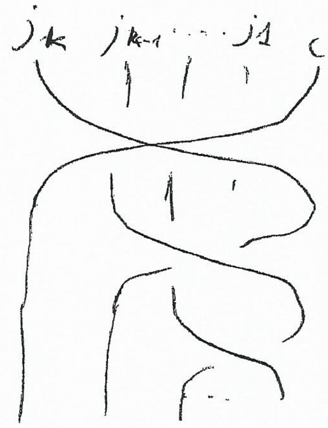
This degree is compatible with our filtration on the algebra of conjugacy classes.

$$\sum_i X_i^{\,k} = \sum_{j_1, j_2, \dots, j_k < i} \underline{(j_1, i)}\,\underline{(j_2, i)} \cdots \underline{(j_k, i)} =$$

View as partial permutations!

This is where we use the result from Lecture 3.

$$= \sum_{\substack{j_1,\dots,j_k < i \\ j_1,\dots,j_k \text{ are all different}}} \underbrace{(j_1,i)(j_2,i)\cdots(j_k,i)}_{= (i, j_k, j_{k-1}, \dots, j_1)} \; + \quad \Big]$$



$$C_k = \frac{1}{k+1} \Sigma_{k+1}$$

$$+ \sum_{\substack{j_1,\dots,j_k < i \\ j_1,\dots,j_k \text{ are NOT all different}}} \underbrace{(j_1,i)\cdots(j_k,i)}_{}$$

support $= \{j_1,\dots,j_k,i\}$, degree at most k

~~Lecture 3, top secret page~~

~~this is a linear combination of conjugacy classes (Σ_ν)~~

The whole sum is a linear combination of conjugacy classes (Σ_ν).

Def.

For $k+1 \geq 2$

$$T_{k+1} = k \cdot \sum_i X_i^{k-1} \in \mathbb{C}[S_n]$$

$$\boxed{⚠ \quad T_2 = \sum_i X_i^0 = \underbrace{n}_{=\Sigma_1} \in \mathbb{C}[S_n]}$$

$$T_{k+1} = \Sigma_k + \left(\text{terms of degree at most } k-1\right)$$

linear combination of Σ_π,

$$\Downarrow$$

$$T_{k_1+1} \cdot T_{k_2+1} \cdots T_{k_\ell+1} = \Sigma_{k_1, k_2, \dots, k_\ell} + \left(\text{term of degree at most } k_1 + \dots + k_\ell - 1\right)$$

$$\Downarrow$$

$$\Sigma_{k_1, \dots, k_\ell} = T_{k_1+1} \cdots T_{k_\ell+1} + \left(\text{term of degree at most } k_1 + \dots + k_\ell - 1\right)$$

For each Σ_π there exits polynomial f,

$$\Sigma_\pi = f(T_2, T_3, \dots)$$

Moral

$$\boxed{Alg(T_2, T_3, \dots) = \mathrm{Span}\left(\Sigma_\pi : \pi\text{-partition}\right)}$$

Comment:

thanks to the „degree" concept we

1) proved existence of something

2) we found the top-degree part of this something.

This is a quite typical situation in asymptotic representation theory.
Getting ~~xx~~ information about subdominant terms might be hard.

Explicit form of polynomial P_μ

$\longrightarrow$ M. Lassalle
„An explicit formula..."
Math. Ann. 340 (2008) no. 2, 383-405

$\longrightarrow$ Book of Ceccherini-Silberstein, ...

Exercise

$Ch_2, Ch_3, \ldots$ generate

the algebra of polynomial functions.

Algebra Λ^*

(partial) permutations

$\mathrm{span}\left(\Sigma_\pi\right)$

$k\sum x_i^{k-1}$

$=$

$\mathrm{Alg}\left(T_{k+1} : k+1 \geq 2\right)$

algebra of polynomial functions on $\mathbb{Y}$

functions on $\mathbb{Y}$

Λ^* (set of shifted-symmetric functions)

$\mathrm{span}\left(Ch_\pi\right)$

$=$

$\mathrm{Alg}\left(T_{k+1} : k+1 \geq 2\right)$

$\mathrm{Alg}\left(Ch_2, Ch_3, \ldots\right)$

$$T_{k+1} : \lambda \to k \cdot \sum_{\square \in \lambda} \mathrm{content}(\square)^{k-1}$$

What is a shifted-symmetric function?

$$P(z_1, z_2, \ldots)$$

"polynomial in infinitely many variables"

- $P = (P_1, P_2, \ldots)$ is a <u>sequence</u>

element of ~~a~~ the inverse limit!

- $P_i = P_i(z_1, \ldots, z_i)$ is a polynomial which is <u>shifted</u> symmetric

$\cancel{P_i(\ldots, a, b, \ldots)}$

$$P_i(\ldots, a, b, \ldots) =$$

$$= P_i(\ldots, b-1, a+1, \ldots)$$

inverse limit in the category of <u>filtered</u> rings

- $P_{i+1}(z_1, \ldots, z_i, 0) = P_i(z_1, \ldots, z_i)$

- $\sup_i (\text{degree of } P_i) < \infty$

subtle!

for each $F : \mathbb{Y} \to \mathbb{C}$, $F \in \Lambda^*$

we claim there exists a shifted-symmetric function P

s.t.

$$F(\lambda_1, \dots, \lambda_i) = P(\lambda_1, \dots, \lambda_i)$$

for all $\lambda_1 \geq \dots \geq \lambda_i \geq 0$

general phenomenon:

domain of functions for Λ^* can be

extended from the set of Young diagrams

to more general objects.

Why elements of Λ^* are shifted-symmetric?

$\longrightarrow$ enough to check on some generating set.

$$\frac{1}{i+1} T_{i+2}(\lambda) = \sum_{\square \in \lambda} \text{content}(\square)^i =$$

$$= \sum_{r \geq 1} \sum_{1 \leq c \leq \lambda_r} (c-r)^i =$$

$$= \sum_{r \geq 1} Z(\lambda_r - r) - Z(-r)$$

there exists a polynomial

$Z \in \mathbb{Q}[x]$ — degree $i+1$

s.t.

$$Z(x) - Z(x-1) = x^i$$

$$\Downarrow$$

$$\cancel{Z(\lambda_r) - Z(0)}$$

$$Z(\lambda_r - r) - Z(-r)$$

$$= (1-r)^i + (2-r)^i + \dots$$

$$\dots + (\lambda_r - r)^i$$

RHS defines a shifted-symmetric function

$$(z_1, \dots, z_i) \longmapsto \sum_{1 \leq r \leq i} Z(z_r - r) - Z(-r)$$

test

$$(\dots, a, b, \dots) \longmapsto \dots + [Z(a-j) - Z(j)] + [Z(b-j-1) - Z(j-1)] + \dots$$

$$(\dots, b-1, a+1, \dots) \longmapsto \dots + [Z(b-1-j) - Z(j)] + [Z(a+1-j-1) - \quad]$$

$\boxed{\text{why each shifted-symmetric functions is an element of } \cancel{A} \text{ ? span}(Ch_\pi : \pi \text{ is a partition})\text{?}}$

power-sum symmetric functions are a basis of the algebra of symmetric functions,

<u>shifted</u> power-sum symmetric functions are a basis of the algebra of <u>shifted</u> symmetric functions.

→ enough to prove for p_k^*.

$P = z^k$

$$p_k^*(z_1, z_2, \dots, z_\ell) =$$

$$= \sum_j (z_j - j)^k - (-j)^k = \sum_j P(z_j - j) - P(-j)$$

we know that $\forall i$

some polynomial of degree $i+1$

$$\underbrace{\frac{1}{i+1} T_{i+2}(\lambda)}_{\in \text{span}(Ch_\pi)} = \sum_{j \geq 1} Z(\lambda_j - j) - Z(-j)$$

$\boxed{\text{Exercise}}$ $\boxed{\text{Thoma filtration}}$

two ways to define filtration on Λ^*

- $\deg \mathrm{Ch}_\pi = |\pi|$

  "size of the support"

- $F : \mathbb{Y} \to \mathbb{C}$ is a ~~polynomial~~ shifted symmetric function

  it makes sense to speak about its degree

$\boxed{\text{Is it the same?}}$

Homogeneous functionals of shape

$$T_i(\lambda) = (i-1) \sum_{\square \in \lambda} (\text{x-coord} - \text{y-coord})^{i-2}$$

$$S_i(\lambda) = (i-1) \int_{(x,y)\in\lambda} (x-y)^{i-2} \, dx \, dy$$

Well defined even if λ has some exotic shape!

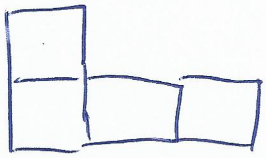
$$(i-1) \int_{x_0}^{x_0+1} \int_{y_0}^{y_0+1} (x-y)^{i-2} \, dx \, dy =$$

$$= \underbrace{\frac{1}{i}\left[-2(x_0-y_0)^{i} + (x_0-y_0+1)^{i} + (x_0-y_0-1)^{i}\right]}_{\text{Polynomial in } (x_0-y_0) \text{ of degree } i-2}$$

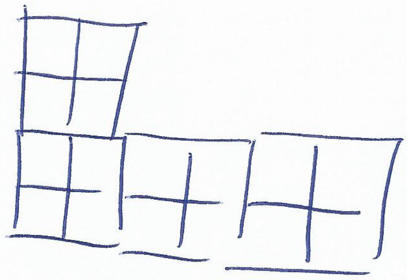
$\longrightarrow$ S_i is a linear combination of $T_2, T_3, \dots, T_i$

$\longrightarrow$ T_i is a linear combination of $S_2, S_3, \dots, S_i$

$Alg(S_2, S_3, \dots) = Alg(T_2, T_3, \dots) = \Lambda^*$



λ



$D_2\,\lambda$

$$S_i(D_s\lambda) = s^i\, S_i(\lambda)$$

homogeneous function in s of degree i.

$\Downarrow$

~~"Thoma VS~~

"Vershik-Kerov gradation"

Motivation? balanced Young diagram.

homogeneous elements of Λ^* of degree d are spanned by

$$S_2^{e_2}\, S_3^{e_3} \cdots \quad ; \quad 2e_2+3e_3+\dots = d.$$

Exercise:

deg Ch_π = ? Hint: $\longrightarrow$ stanley formula

$\boxed{\text{Kerov polynomials}}$

$\rightarrow$ arXiv: 1409.7533

$$Ch_1 = R_2$$

$$Ch_2 = R_3$$

$$Ch_3 = R_4 + R_2$$

$$Ch_4 = R_5 + 5R_3$$

$$Ch_6 = R_6 + 15R_4 + 5R_2^2 + 8R_2$$

Kerov conjecture 2000

coefficients are <u>non-negative</u> integers.

interpretation?

$\boxed{\text{Free cumulants}}$ $\boxed{i \geq 2}$

$$R_i = R_i(\lambda) =$$

$$= [\text{homogeneous part of degree } i] \; Ch_{i-1}$$

$R_i(\lambda) = \; ?$ $\longrightarrow$ Stanley formula

" $R_i \approx Ch_{i-1}$ "

$\boxed{R_2, R_3, R_4, \ldots \text{ generate } \Lambda^*}$

1. why $R_i \in \Lambda^*$?

2. why Ch_π is a polynomial in $R_2, R_3, \ldots$?

How to compute R_i efficiently?