A

we know so far...

structure constants here have a nice description!

certain family of elements in $\mathbb{C}[P_\infty] = \varprojlim \mathbb{C}[P_n]$

algebra Λ^* can be viewed as...

- (span Σ_π : π is a partition)

$\Sigma_\pi = \sum$ (rows are indexes, $\uparrow$)

support = entries of boxes

$x \in \mathbb{C}[P_\infty]$

this map looks best for "central" elements.

"Fourier transform"

- (span Ch_π : π is a partition)

certain family of functions on $\mathbb{Y}$

$Ch_\pi : \mathbb{Y} \to \mathbb{C}$

$f : \mathbb{Y} \to \mathbb{C}$

$Ch_\pi(\lambda) = \underbrace{n \cdots n-|\pi|+1}_{|\pi| \text{ factors}} \cdot \operatorname{tr} \rho^\lambda(\pi)$

$f(\lambda) = \operatorname{tr} \rho^\lambda(x)$

projection ~~restriction~~ of x to S_n, $n = |\lambda|$

$\{F$ is a shifted symmetric function$\}$

↓PAGE (2)↓

(B)

hint:
use filtration!

- $\mathrm{Alg}\,(\Sigma_1, \Sigma_2, \dots)$

Λ^* can be viewed as...

- $\mathrm{Alg}\,(T_2, T_3, T_4, \dots)$

$$T_{k+1} = k \sum_{1 \le i \le n} X_i^{k-1} \in \mathbb{C}[P_n]$$

Jucys-Murphy element

⚠ $T_2 = \cancel{k}1 \cdot \sum_{1 \le i \le n} X_i^0 \overset{⚠}{=} \sum_i (\mathrm{id}, \{i\}) = \Sigma_1 \in \mathbb{C}[P_1]$

≈ "symmetric polynomials in JM-elements"

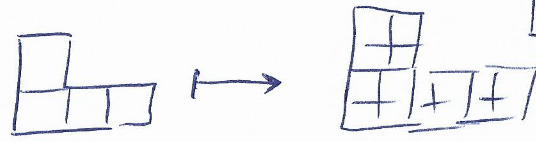
⚠ T_2 does not fit into this picture

PAGE C

- $\mathrm{Alg}\,(Ch_1, Ch_2, \dots)$

- $\mathrm{Alg}\,(T_2, T_3, \dots)$

$$T_{k+1}(\lambda) = k \cdot \sum_{\square \in \lambda} [c(\square)]^{k-1}$$



$\boxed{\Lambda^* \text{ is a GRADED algebra}}$ (why?)

- $\mathrm{Alg}\,(S_2, S_3, \dots)$

$$S_{k+1}(\lambda) = k \cdot \iint_{(x,y) \in \lambda} (x-y)^{k-1} \, dx \, dy$$

homogeneous, of degree $k+1$.

$$S_{k+1}(s\lambda) = s^{k+1} S_{k+1}(\lambda)$$

C

GRADATION

homogeneous elements of degree d =

$$= \left\{ F \in \Lambda^* : \quad F(s\lambda) = s^d F(\lambda) \right\}$$

why gradation?

① why homogeneous elements SPAN?
$\rightarrow$ basis $S_2, S_3, \ldots$

FILTRATION

inhomogeneous elements of degree $\le d$

$$= \left\{ F \in \Lambda^* : \quad s \longmapsto F(s\lambda) \text{ is a polynomial of degree } \le d \right\}$$

② why linearly independent? *

NEW!

↑ PAGE (B) ↑

↓ PAGE (D) ↓

- $Alg(R_2, R_3, \ldots \quad)$

~~R_k~~

R_{k+1} = homogeneous part of of degree $k+1$

Ch_k

$R_{k+1} \in \Lambda^*$

(formula for R_{k+1} via stanley

Ch_k is of degree $k+1$

$\rightarrow$ Stanley formula.

$s \mapsto Ch_k(s\lambda)$ is a polynomial of degree $k+1$.

$$R_{k+1}(\lambda) = [s^{k+1}] \quad Ch_k(s\lambda) =$$

$$= \lim_{s \to \infty} \frac{1}{s^{k+1}} \quad Ch_k(s\lambda)$$

* $f_0 + f_1 + \cdots + f_k = 0$

f_d - homogeneous of degree d.

$\Rightarrow \forall_\lambda \ f_d(\lambda) = 0 \qquad f_d \equiv 0$

D

why $R_2, R_3, \dots$ generate Λ^* ?

$\boxed{\forall_\pi \quad Ch_\pi = \text{Polynomial}\,(R_2, R_3, \dots, \dots)}$ use induction over degree $Ch_\pi = |\pi| + \ell(\pi)$

- $Ch_\pi = Ch_{\pi_1} \cdot Ch_{\pi_2} \cdots Ch_{\pi_\ell} + (\text{lower-degree term})$.

- enough to prove for $\pi = (k)$ having one part.

- $Ch_k = R_{k+1} + (\text{lower-degree term})$.

↑ PAGE (C) ↑

$\boxed{\text{Kerov polynomials}}$

$\longrightarrow$ arXiv: 1409.7533

$$Ch_1 = R_2$$

$$Ch_2 = R_3$$

$$Ch_3 = R_4 + R_2$$

$$Ch_4 = R_5 + 5 R_3$$

$$Ch_5 = R_6 + 15 R_4 + 5 R_2^2 + 8 R_2$$

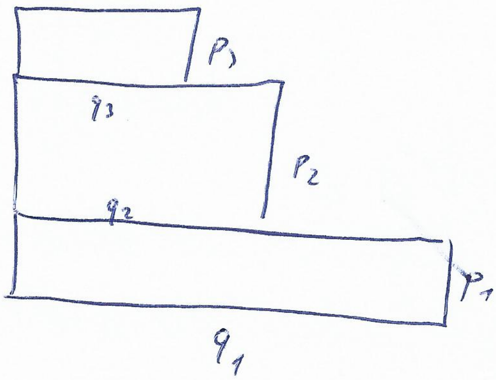
Kerov conjecture 2000

coefficients are <u>non-negative</u> integers.

interpretation?

Stanley coordinates.

$\lambda = p \times q =$



interesting concept from philosophical point of view

For any polynomial function $F \in \Lambda^*$

$F(p \times q)$ is a polynomial in $p_1, \dots, p_l$
$q_1, \dots, q_l$

Exercise.

$$\boxed{\frac{\partial}{\partial s_{k_1}} \cdots \frac{\partial}{\partial s_{k_i}} F \Big|_{s_2 = s_3 = \cdots = 0} = \left[p_1 q_1^{k_1 - 1} \, p_2 q_2^{k_2 - 1} \cdots \right] F(p \times q).}$$

PROOF...

Stanley coordinates

- Each Young diagram is of the form $p \times q$.

  (Hint: $p = (1, 1, \dots, 1)$).

  Bad idea!

- View as a polynomial in $p_1, p_2, \dots, q_1, q_2, \dots$

- Exotic shapes $p \times q$!

- ANALYSIS: infinitesimal changes of shape of λ!

- Important involution

  $$\lambda \longleftrightarrow \lambda'$$

  $$\rho^{\lambda'} = \mathrm{sign} \otimes \rho^{\lambda}$$

  is NOT reflected here.

- more democratic choice of coordinates is not helpful!

Polynomial function on $\mathbb{Y}$

$F \in \Lambda^*$

$\big\downarrow$

polynomial

each Young diagram is of this form!!!

why faithful? $\longrightarrow$ $\frac{\partial}{\partial s_{k_n}} \cdots \frac{\partial}{\partial s_{k_1}} F \Big|_{s_2, \ldots = 0} =$

$= \left[p_1 q_1^{k_1 - 1} \cdots \right] F(p \times q).$

$F(p \times q) = \text{polynomial}(p_1, \ldots, p_\ell, q_1, \ldots, q_\ell)$

Formally: element of the inverse limit

$\varprojlim \mathbb{C}[p_1, \ldots, p_\ell, q_1, \ldots, q_\ell]$

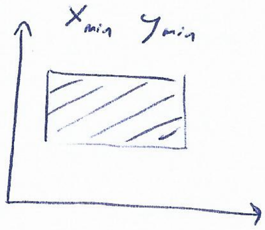
$$\boxed{\text{why Stanley polynomial (if exists) is unique?}}$$

idea: view it as polynomial.

square-free terms in $p_1, p_2, \ldots$

like the limit $p_1, p_2, \ldots \to 0$

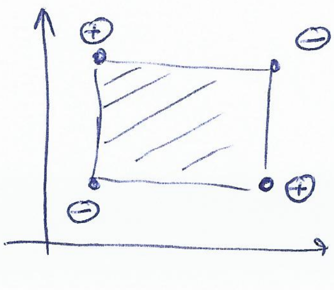
$$\int_{x_{min}}^{x_{max}}\int_{y_{min}}^{y_{max}} (k-1)\,(x-y)^{k-2}\,dx\,dy =$$



$$= \int_{x_{min}}^{x_{max}} (-1)\,(x-y)^{k-1}\Big|_{y_{min}}^{y_{max}}\,dx =$$

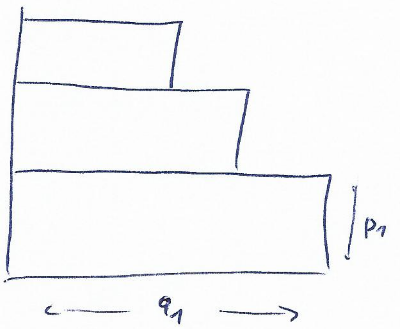
$$= \int_{x_{min}}^{x_{max}} \underbrace{-(x-y_{max})^{k-1}} + \underbrace{(x-y_{min})^{k-1}}\,dx =$$

$$= \frac{1}{k}\Big[\overline{(x_{max}-y_{min})^{k} - (x_{min}-y_{min})^{k}} + \overline{-(x_{max}-y_{max})^{k} + (x_{min}-y_{max})^{k}}\Big] =$$

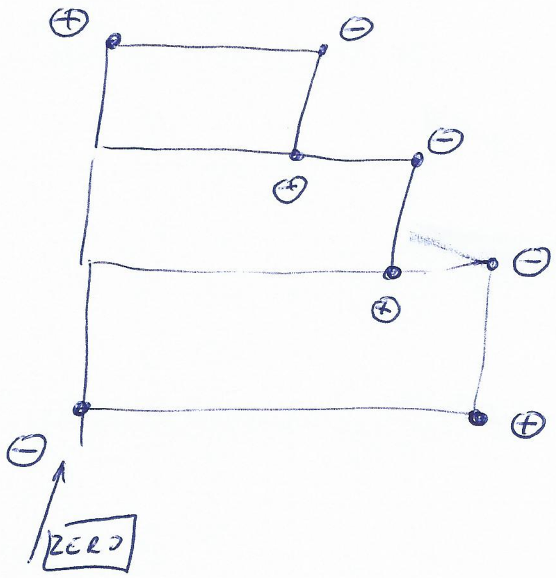
$$=$$ 

$\boxed{k \geq 2}$

$$S_k(\cancel{q}\, p \times q) = \iint (k-1)\,(x-y)^{k-2}\, dx\, dy =$$



$=$



$=$

$$= \frac{1}{k}\Big[q_1^k - (q_1 - p_1)^k + (q_2 - p_1)^k - (q_2 - p_1 - p_2)^k$$

$$+ (q_3 - p_1 - p_2)^k - (q_3 - p_1 - p_2 - p_3)^k$$

$$+ (-p_1 - p_2 - p_3)^k \Big]$$

$$\frac{\partial^l}{\partial p_{i_1}\partial p_{i_2}\cdots\partial p_{i_l}} S_k(p\times q)\Big|_{p_1=p_2=\cdots=0} =$$

$i_1 < i_2 < \ldots < i_l$

$\boxed{\text{Empty diagram!}}$

variables $p_1, \ldots$ over which we do not take derivative can be set to 0.

$\to$ case $(i_1, i_2, \ldots) = (1, 2, \ldots, l)$

$$= \frac{\partial^l}{\partial p_{i_1}\cdots\partial p_{i_l}} \frac{1}{k}\sum_j (q_j - p_1 - \cdots - p_{j-1})^k - (q_j - p_1 - \cdots - p_j)^k + (\text{ostatní dílečků}) = \ldots$$

gives p-square-free term of S_k.

~~not zero~~ only for ~~j = i_l~~

- for $j < i_l$ this does not depend on p_{i_l}
- for $j > i_l$ summands cancel.
- only $j = i_l$!

⚠ last summand $(k-1)\cdots(k-l+1)\, 0^{k-l}\, (-1)^{k-l}$

⚠ for $\boxed{k=l}$.

$$\ldots = \begin{cases} (k-1)(k-2)\cdots(k-l+1)\, q_{i_l}^{k-l} \cdot (-1)^{l-1} & \text{for } l<k \\ 0 & \text{for } l \geq k \end{cases}$$

≥ 1

⚠ for $l=k$ extra cancellation!

$\boxed{\text{Consequence}}$

[p-square free term] $\quad S_k(p \times q) =$

$$= \sum_{\substack{i_1 < i_2 < \dots < i_\ell \\ \ell < k}} p_{i_1} p_{i_2} \cdots p_{i_\ell} \cdot \underset{(k-1)\cdots(k-\ell+1)}{} q_{i_\ell}^{k-\ell} \cdot (-1)^{\ell-1}$$

[p-square-free term] $\quad S_{k_1} \cdots S_{k_m}(p \times q) =$

$$= \sum_{A,B} \prod_{i \in A} p_i \cdot \prod_{i \in B} q_i \cdot (\text{something}).$$

⚠ $\boxed{B \subseteq A}$ "variables q_i" $\subseteq$ "variables p_i"

for $k_1, k_2, \ldots \geq 2$! ⚠

$$[p_1 p_2 \cdots q_1^{k_1-1} q_2^{k_2-1} \cdots] \; S_{k_1} \cdot S_{k_2} \cdots =$$

$$= \frac{\partial}{\partial S_{k_1}} \cdots \frac{\partial}{\partial S_{k_2}} \cdots S_{k_1} \cdot S_{k_2} \cdots \Bigg|_{\substack{\lambda = \emptyset \\ S_2 = S_3 = \dots = 0}}.$$

Corollary.

For each polynomial function $F \in \Lambda^*$

$$F = \sum_{\alpha_2, \alpha_3, \dots \geq 0} \boxed{\frac{1}{\alpha_2!\, \alpha_3! \cdots} \left. \frac{\partial^{\alpha_2 + \cdots}}{(\partial S_2)^{\alpha_2} (\partial S_3)^{\alpha_3} \cdots} F \right|_{\substack{S_2 = S_3 = \dots = 0 \\ \lambda = \emptyset}}} \cdot S_2^{\alpha_2} S_3^{\alpha_3} \cdots$$

via Stanley polynomial.

Fantastic!

$S_2, S_3, \dots$ are easy to compute!

Why p-square-free terms of Stanley polynomials are enough?

① polynomial

$$F(p_1, p_2, \ldots, q_1, q_2, \ldots) \text{ is NOT symmetric.}$$

② ~~nevertheless~~,

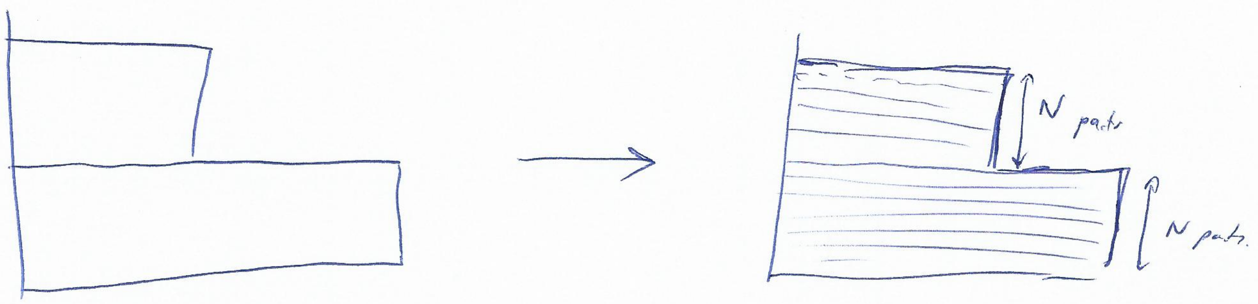
$$\left[p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots q_{i_1}^{\beta_1} q_{i_2}^{\beta_2} \cdots\right] F =$$

$$= \left[p_{f(i_1)}^{\alpha_1} p_{f(i_2)}^{\alpha_2} \cdots q_{f(i_1)}^{\beta_1} \cdots\right] F$$

for increasing f.

lot of symmetry!

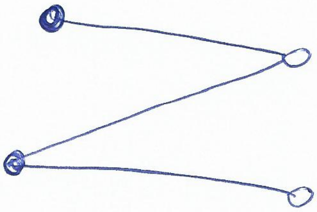
③

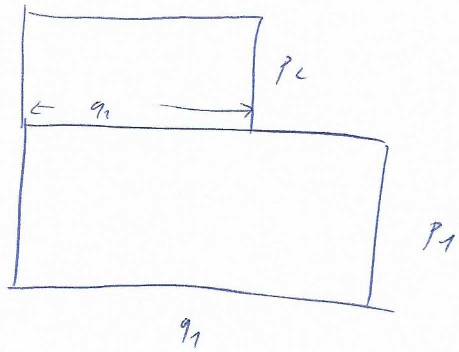


subdivide!

only p-square free terms give contribution for $N \to \infty$.

$\boxed{\text{Embeddings in Stanley coordinates}}$





$\mathcal{G}$ graph

$$N_{\mathcal{G}}(p\times q) = \sum_{f : V_\circ \to \{1,2,\dots\}} \quad ,$$

"number of the box"

$$\prod_{v\in V_\bullet} p_{f(v)} \quad .$$

$$\prod_{w\in V_\circ} q_{\substack{\max f(v) \\ v\in V_\bullet \\ v\sim w}}$$

$$[p_1 q_1^{k_1-1}\, p_2 q_2^{k_2-1} \cdots]\, N_{\mathcal{G}}(p\times q) = \dots\dots$$

<u>bijective</u> labelings of $V_\bullet$ st ....

Stanley formula in Stanley coordinates.

$\pi \in S_n$

$$\mathrm{Ch}_\pi = \sum_{\substack{\sigma_1,\sigma_2 \in S_n \\ \sigma_1\sigma_2=\pi}} (-1)^{\sigma_1} \, N_{\sigma_1,\sigma_2}$$

$$\mathrm{Ch}_\pi(p \times q) = \sum_{\substack{\sigma_1,\sigma_2 \in S_n \\ \sigma_1\sigma_2=\pi}} \; \sum_{\substack{f: C(\sigma_2) \to \\ \to \{1,2,\dots\}}} (-1)^{\sigma_1} \prod_{c \in C(\sigma_2)} p_{f(c)} \prod_{d \in C(\sigma_1)} q_{\max_{c \cap d \neq \emptyset} f(c)}$$

Corollary:

expansion of Ch_π in terms of

$S_2, S_3, \dots$

Free cumulants in terms of $S_1, S_2, \dots$

$$R_k = \left[\text{top-degree part of } Ch_{k-1}\right] =$$

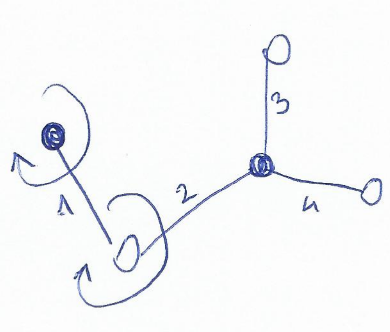
$$= \sum_{\substack{\sigma_1 \sigma_2 = (1,2,\dots,k-1) \\ \sigma_1, \sigma_2 \in S_{k-1} \\ \#C(\sigma_1) + \#C(\sigma_2) = k}} (-1)^{\sigma_1} N_{\sigma_1, \sigma_2}$$

$\Updownarrow$

$\|\sigma_1\| + \|\sigma_2\| = \|(1,2,\dots,k-1)\|$

MINIMAL FACTORIZATIONS =

= TREES.



$\circ = \sigma_1 = (3)(12)(3)(4)$

$\bullet = \sigma_2 = (1)(234)$

$\sigma_1 \sigma_2 = (1234)$

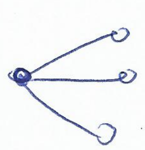
Rooted bicolored tree!!!

Yes, we can!

$$R_k = ? \; S_k + \sum_{a+b=k} \boxed{?} \; S_a \cdot S_b + \sum_{a+b+c} ? \; S_a S_b S_c + \dots$$

exercise. | closed, explicit formula.

$$[p_1 q_1^{k-1}] \, R_k(p \times q) =$$

$$= \Big[$$  $$\Big] = 1$$

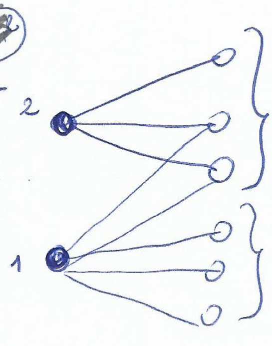
if $a \neq b$ if $a \neq b$ also for $a = b$

$$\frac{1}{2} \cdot \frac{\partial}{\partial S_a} \cdot \frac{\partial}{\partial S_b} R_k = \frac{1}{2} \left[p_1 p_2 \, q_1^{a-1} \, q_2^{b-1} \right] R_k(p \times q) =$$

(a,b) and (b,a) !

for $a=b$ this vanishes!

[redacted]

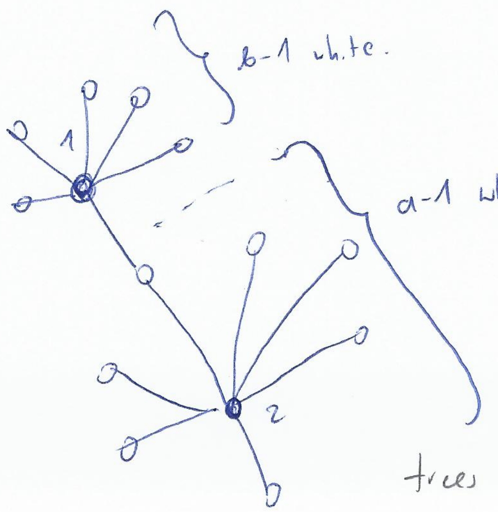
$$= \frac{1}{2} \Big[$$  $$\Big] =$$

it's a tree!

$b-1$ white.

$a-1$ white

[redacted]

$$\frac{1}{2} \Big[$$  $$\Big] = \frac{1}{2}(-1) \quad (k-1)$$

trees WITH choice of labels for ●

$$R_k = S_k - \frac{1}{2!}(k-1)\sum_{\substack{a+b=k\\ a,b\geq 2}} S_a S_b \;+$$

$$\underbrace{+\frac{1}{3!}(k-1)^2 \sum_{a+b+c=k} S_a S_b S_c}_{\text{"exercise"}} + \dots$$

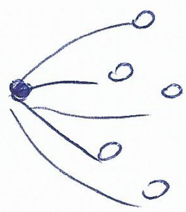
$$S_k = R_k + \frac{1}{2}(k-1)\sum_{a+b=k} R_a \cdot R_b \;+\; ?$$

Kerov polynomials.

"Una expression of Ch_n in terms of $S_2, S_3, \dots$ $\Rightarrow$ in terms of $R_2, R_3, \dots$"

$$[R_a]\, Ch_k = [S_a]\, Ch_k \cdot \underbrace{[R_a]\, S_a}_{1}$$

linear terms

$$= \Big[$$  $$\Big] =$$

a-1 vertices

$$= \# \quad \sigma_1, \sigma_2 \in S(k)$$

σ_2 has 1 cycle

σ_1 has a-1 cycles.

$\sigma_1 \sigma_2 = (1,2,\dots,k)$

σ_2 has 1 cycle

$\|\sigma_2\| = (k-1)$

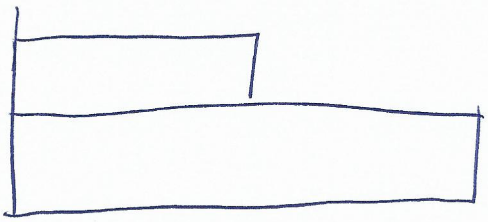
$\|(1,2,\dots,k)\| = (k-1)$

$(-1)^{\sigma_1} = 1$

"easy".

$\longrightarrow$ Stanley, Biane

Free cumulant, two rectangles



[square-free term] $R_k =$

$$= p_1 q_1^{k-1} \cdot 1 + p_2 q_2^{k-1} \cdot 1$$

$$+ p_1 p_2 \sum_{\substack{a+b=k-2 \\ a \geq 1 \\ b \geq 2}} q_1^a q_2^b \cdot (k-1) \, (-1)$$

$a \neq b$, $a, b \geq 2$

$$[R_a R_b]\, F = [p_1 p_2 q_1^{a-1} q_2^{b-1}]\, F - [p_1 p_2 q_2^{a+b-2}]\, F$$

Proof. a) $F = R_{a'} \cdot R_{b'}$

$L = 1 \iff \{a, b\} = \{a', b'\}$

$R = [\{a,b\} = \{a', b'\}] - 0$ ✓

b) $F = R_{a+b}$

$L = 0$

$R = - (a+b-1) - [-(a+b-1)] = 0$ ✓

c) $F = R \cdot R \cdot R$

$$\boxed{\text{if } \quad a, b \geq 2}$$

$$[p_1 p_2 \; q_1^{a-1} \; q_2^{b-1}]\, F = [p_1 p_2 \; q_1^{b-1} \; q_2^{a-1}]$$

Proof

a) $F = R_? \cdot R_?$ b) $F = R\,?$ c) $?\, R = R \cdot R$.