Piotr Śniady
IMPAN

Lectures on random matrices and free probability theory

Lecture 1.
What is random matrix about?

October 15, 2019

psniady.impan.pl

Suggested reading 0.

Persi Diaconis

"What is... a random matrix?"

Notices of the AMS, vol. 52, number 11, pp. 1348–1349.

KAWAI

Suggested reading 1

Hint: PDF files with all books are legally available on authors' websites.

$\boxed{\text{MS}}$ = James Mingo, Roland Speicher,
"Free probability and random matrices"

recommended!

$\boxed{\text{NS}}$ = Alexandru Nica, Roland Speicher
"Lectures on the combinatorics of free probability"

Terence Tao
"Topics in random matrix theory"

Tao offers interesting philosophical insight.
→ Section 2.5.

Suggested reading 2.

Anderson, Guionnet, Zeitouni

"An introduction to random matrices".

N. C. Snaith P. J. Forrester J. J. M. Verbaarschot

"Developments in random matrix theory"

J. Phys. A 36 (2003) R1

arXiv: cond-mat/0303207

good survey of topics which WILL NOT

be covered by this series of lectures.

general plan of Lecture 1.

LECTURE 1.

"what this series of Lectures is NOT about"

→ general picture of the theory

- universality, Riemann ξ, chaos,
- determinantal point processes, Baik-Deift-Johansson,

"what this series IS about"

→ more specific picture of the series

- Voiculescu's free probability
- free cumulants
- combinatorics
- Weingarten calculus.
- Brown measure, non-hermitian random matrices

What is a random matrix?

random matrix is a random variable with values in $M_N(\mathbb{C})$

$$X : \Omega \longrightarrow M_N(\mathbb{C})$$

entries of $X = (X_{ij})_{1 \le i,j \le N}$ are random variables

$$X_{ij} : \Omega \longrightarrow \mathbb{C}$$

---

toy example: random permutation matrix

$\Omega = S_n$, the set of permutations of $\{1, .., N\}$

$\mathbb{P}$ = the uniform measure on S_n

$$X(\sigma) = \begin{bmatrix} & 1 & & & \\ 1 & & & & \\ & & & 1 & \\ & & 1 & & \\ & & & & 1 \end{bmatrix}$$

lots of ZEROS

one 1 in $\sigma(2)$ row

one 1 in $\sigma(1)$-row

$$\boxed{\text{random matrix is something that your computer can generate.}}$$

typical problem: distribution of eigenvalues

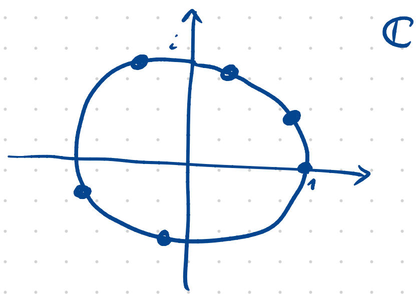
random matrix

$$X : \Omega \longrightarrow M_N(\mathbb{C})$$

for each $\omega \in \Omega$ we look on the multiset of eigenvalues of X] „random set of eigenvalues"

toy example: random permutation matrix

eigenvalues are on the unit circle



if you prefer to get eigenvalues which are real numbers, you better consider symmetric / / hermitian random matrices

$$X : \Omega \longrightarrow M_n^{sa}(\mathbb{C}) = \{ M \in M_n(\mathbb{C}) : M = M^* \}$$

[i.e. $M_{ij} = \overline{M_{ji}}$]

$$M_n^{sym}(\mathbb{R}) = \{ M \in M_n(\mathbb{R}) : M = M^T \}$$

[i.e. $M_{ij} = M_{ji}$]

Ginibre ensemble.

X is a $N\times N$ matrix with entries (f_{ij})

$(\mathrm{Re}\, f_{ij}, \mathrm{Im}\, f_{ij})_{ij}$ – family of independent Gaussian random variables $\sim N(0, \sigma^2)$

Exercise. Write a computer program in some fancy language which → generates a Ginibre random matrix, → calculates its (complex!) eigenvalues, and → plots them on the plane. Play with the program for large N

Hint: use SageMath

(the distribution of) Ginibre ensemble can be uniquely determined by saying that ① the joint distribution of $(\mathrm{Re}\, f_{ij}, \mathrm{Im}\, f_{ij})$ is centered Gaussian, ② specifying the covariance:

$$\mathbb{E}\, f_{ij}\, f_{kl} = 0$$

$$\mathbb{E}\, \overline{f_{ij}}\, \overline{f_{kl}} = 0 \qquad\qquad C = 2\sigma^2$$

$$\mathbb{E}\, f_{ij}\, \overline{f_{kl}} = [i=k]\,[j=l]\, C$$

GUE

suppose X is a random matrix from Ginibre ensemble.

we say that $Y := X + X^*$ is a

GUE (Gaussian Unitary Ensemble) random matrix.

The distribution of a GUE random matrix can be uniquely determined by saying that the entries

(g_{ij}) of Y form a complex centered Gaussian random vector with

① $g_{ij} = \overline{g_{ji}}$

② $\mathbb{E}\, g_{ij}\, g_{kl} = \mathbb{E}\left(f_{ij} + \overline{f_{ji}}\right)\left(f_{kl} + \overline{f_{lk}}\right) =$

$= [i = l]\,[j = k]\; 2c$

$\boxed{2c = \frac{1}{N}}$ prefered normalization used by Mingo and Speicher.

$\boxed{2c = 1}$ another normalization used by M&S

GOE

don't like complex numbers?
Ok!

$$X = (X_{ij})$$

$\uparrow$ independent real Gaussian $\sim N(0, \sigma^2)$

we say that

$$Y := X + X^T$$

is a Gaussian Orthogonal Ensemble (GOE) random matrix.

Energy level in a quantum system

$\mathcal{H}$ - Hilbert space (usually infinite-dimensional)

$H : \mathcal{H} \longrightarrow \mathcal{H}$ (unbounded) linear operator "Hamiltonian"

self-adjoint: $H = H^*$

toy examples: • $\mathcal{H} = \mathbb{C}^n$

H is a hermitian matrix

$H_{ij} = \overline{(H_{ji})}$

• $\mathcal{H} = L^2(\mathbb{R})$

$Hf = -\frac{d^2}{dx^2} f + V \cdot f$

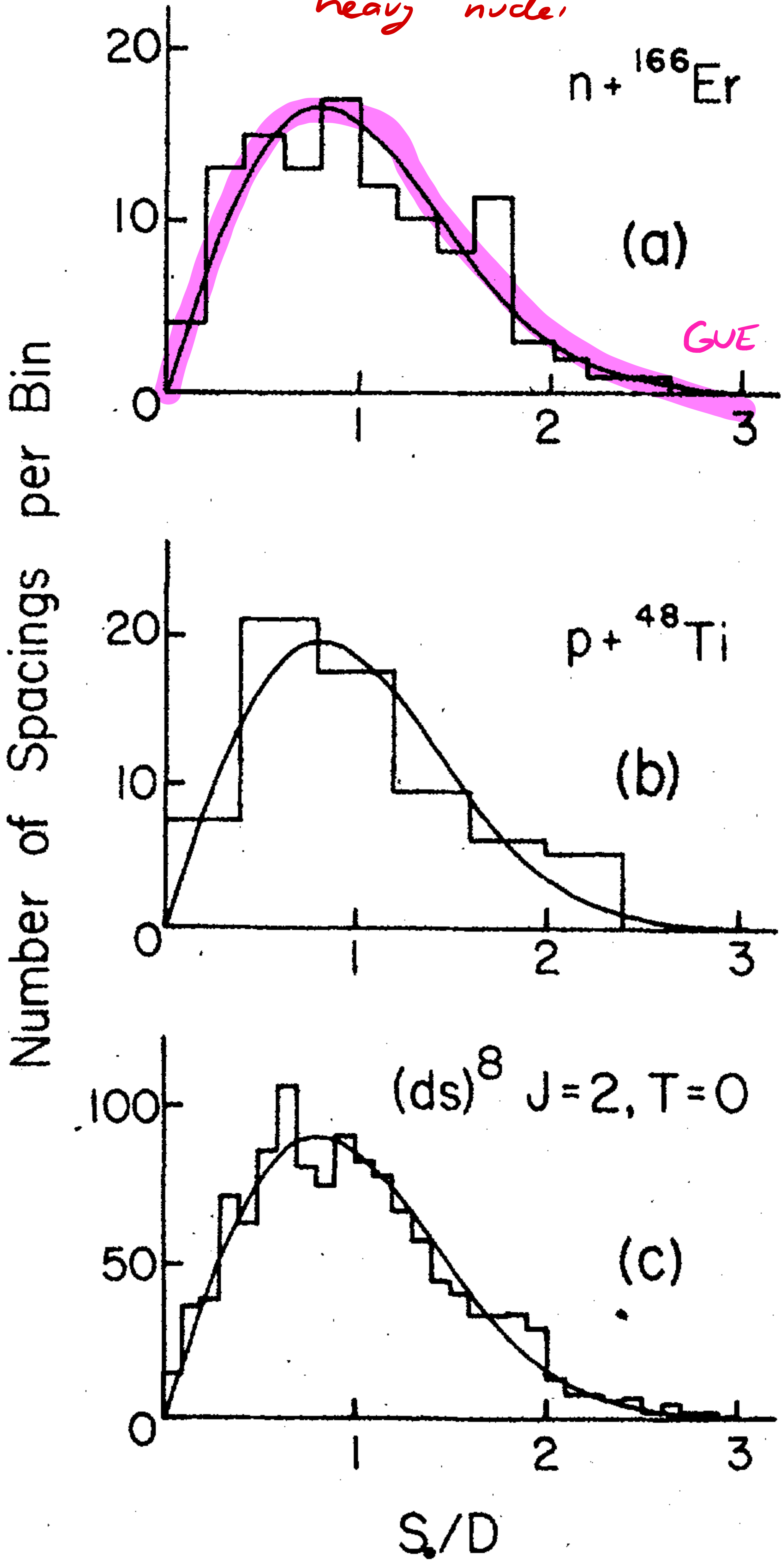
"potential", nice function on $\mathbb{R}$

dictionary:

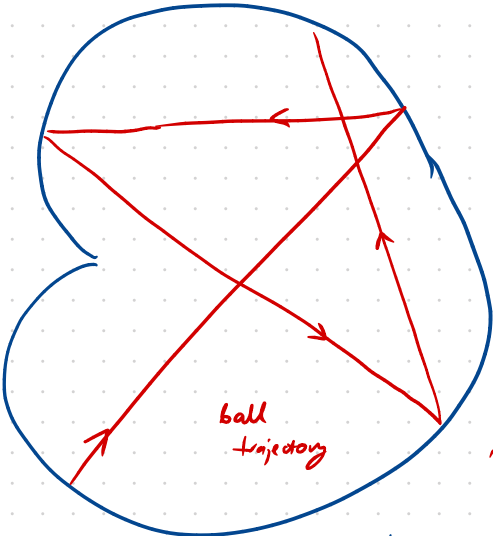
physics	mathematics
energy levels of a quantum system	eigenvalues of H.


energy levels for heavy nuclei
Poisson noise
(a)
(b)
(c)
(d)
collisions of energy levels are rare
often collisions of random points
"energy levels are evenly distributed"
a lot of random fluctuations.
n+166Er
p+48Ti
(ds)8 2+0
Poisson

Spacings between energy levels in heavy nuclei



Classical billiard



cardioid

"classical billiard"

"is a typical trajectory chaotic?"

Quantum billiard

$$\mathcal{H} = \text{functions in } \mathcal{L}^2(\;\cdot\;)$$

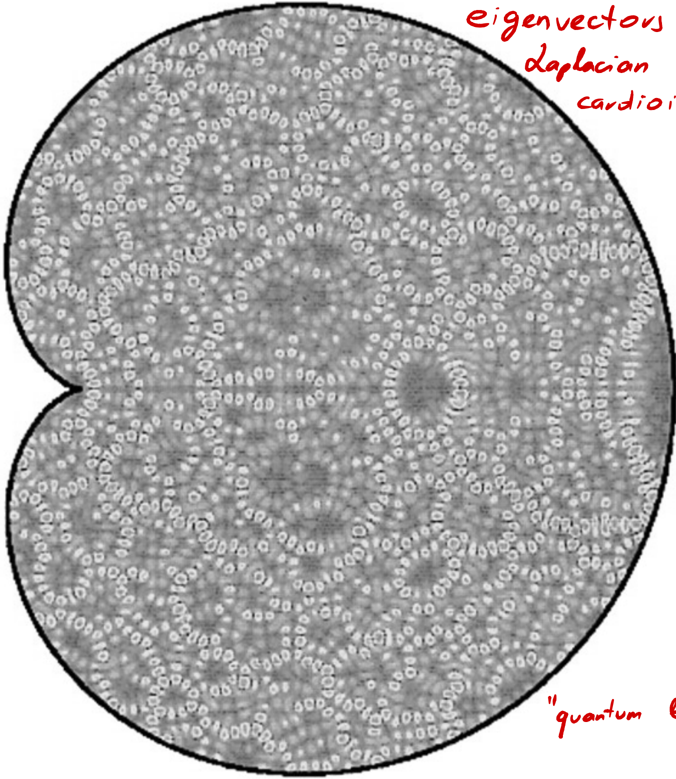
which are $\equiv 0$ on the boundary

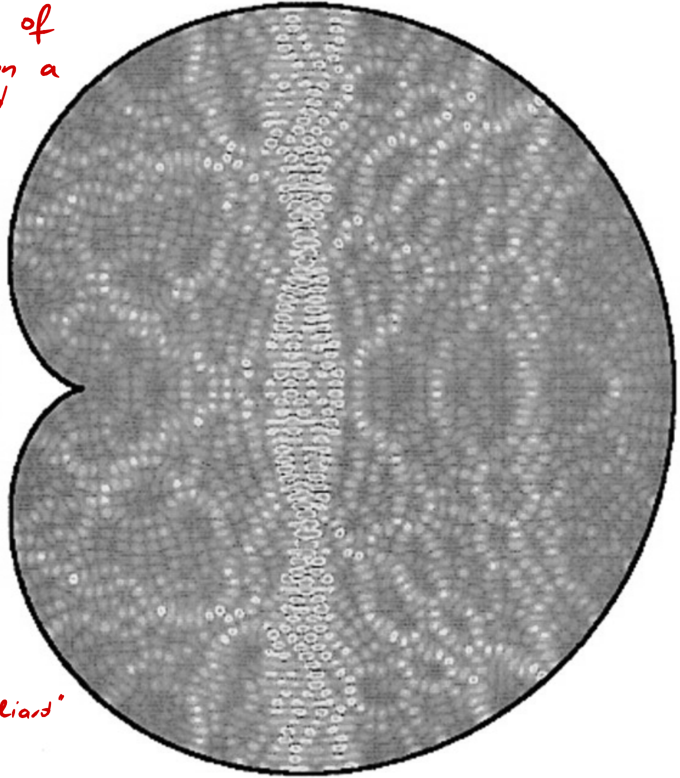
$$H = -\nabla^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$ Laplacian

$\rightarrow$ Can one hear the shape of a drum?
Mark Kac

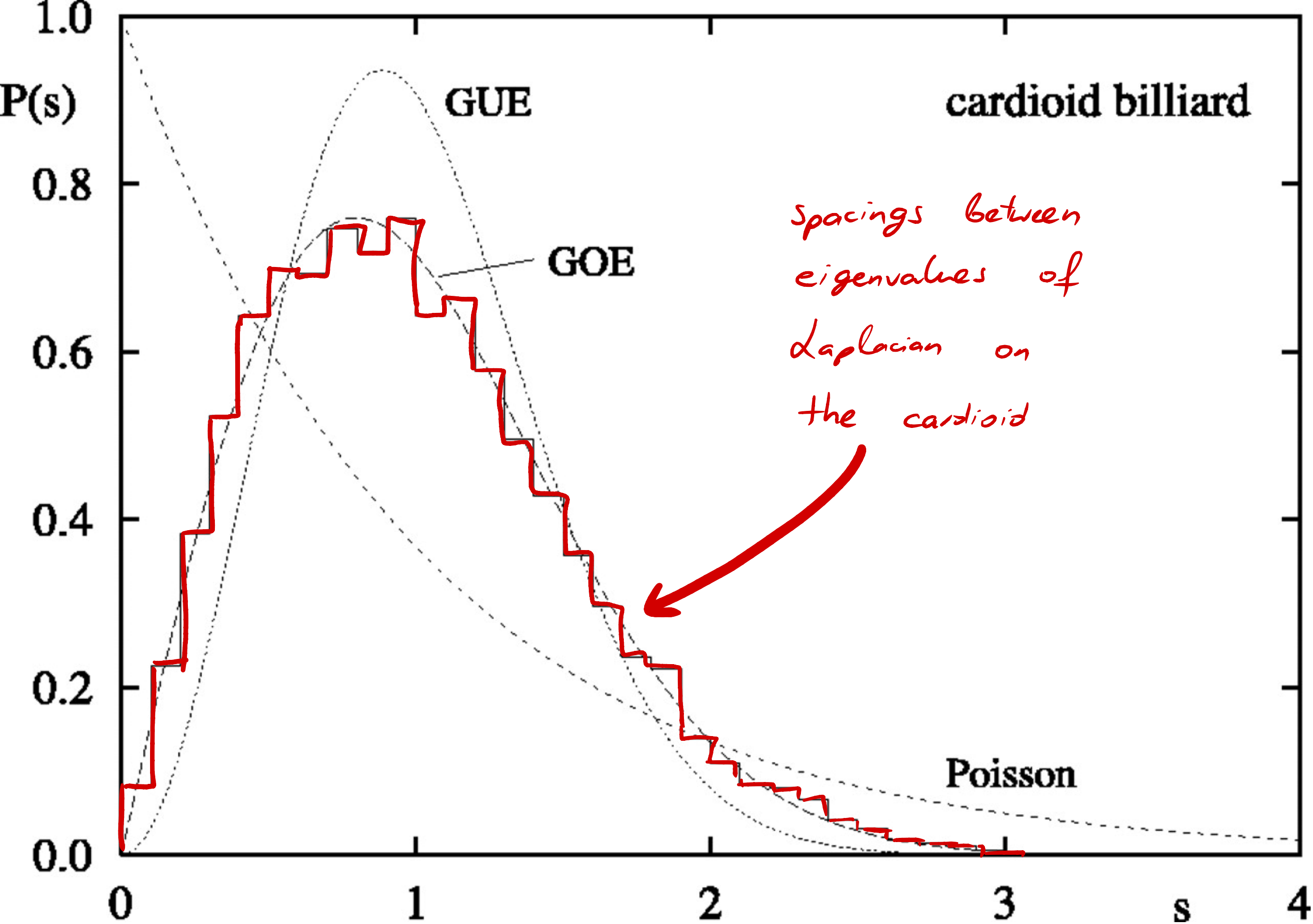
"are eigenvectors regular or chaotic?"

eigenvectors of Laplacian on a cardioid





"quantum biliard"


1.0
0.8
0.6
0.4
0.2
0.0
P(s)
0
1
2
3
4
s
GUE
GOE
Poisson
cardioid billiard
Spacings between eigenvalues of Laplacian on the cardioid

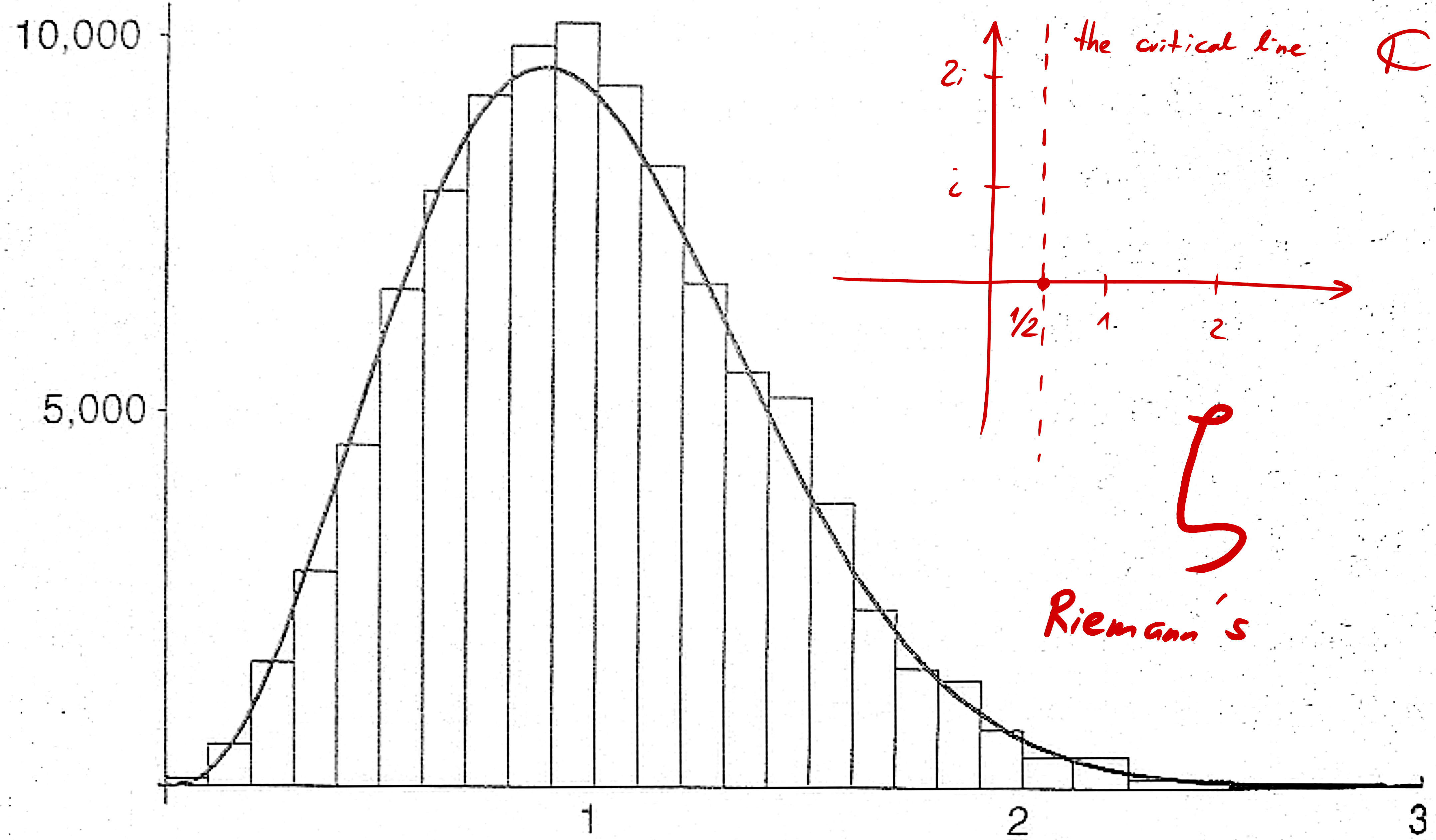


FIGURE 18-5 The Montgomery-Odlyzko Law. (Distribution of spacings for the 90,001st to 100,000th zeta-function zeros.)

Longest Increasing Subsequence

$\sigma = (3, 1, 6, 7, 2, 5, 4)$

$\sigma = (3, 1, 6, 7, 2, 5, 4)$

$\sigma = (3, 1, 6, 7, 2, 5, 4)$

longest increasing subsequence has length 3

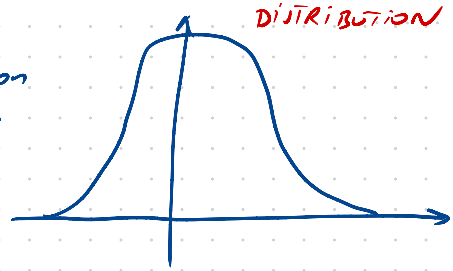
$\mathrm{LIS}(\sigma) = 3$

if σ_n is a uniformly random permutation of $\{1, \dots, n\}$

then

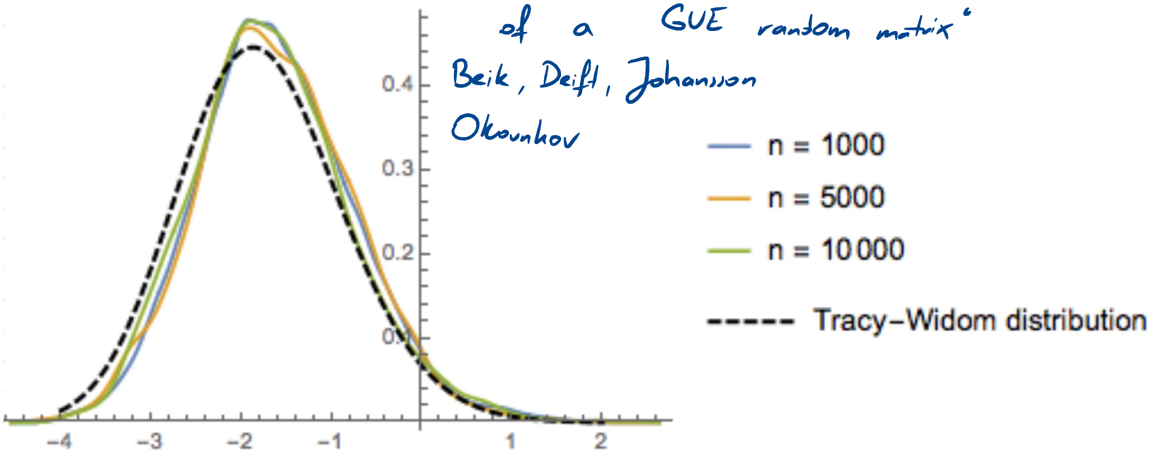
$$\frac{\mathrm{LIS}(\sigma_n) - 2\sqrt{n}}{\sqrt[6]{n}} \xrightarrow{\text{distribution}}$$

NOT A GAUSSIAN DISTRIBUTION



„limit behavior of the largest eigenvalue of a GUE random matrix"

Beik, Deift, Johansson
Okounkov



What this series IS about?

"Let X be a random hermitian matrix with specified eigenvalues $\lambda_1, \dots, \lambda_N \in \mathbb{R}$"...

Explanation:

if $A \in M_N^{s.a.}(\mathbb{C})$ is a hermitian matrix, it has N eigenvalues $\lambda_1, \dots, \lambda_N$ and N eigenvectors $u_1, \dots, u_N$ which are orthogonal;

for simplicity assume $\lambda_i \neq \lambda_j$

we may assume $\|u_i\| = 1$.

$$A = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}}_{U} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_N \end{bmatrix}^{-1}$$

$$U^* U = \text{Gram matrix} = \left[\langle u_i, u_j \rangle \right] = I$$

U is a unitary matrix

the unitary group $U(N) = \{ U \in M_N(\mathbb{C}) : U^* U = I \}$

$$X = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} U^{-1}$$

if you do not know much about the basis of eigenvectors, you should choose it as random as you can.

take Haar measure on $U(N)$

"Let X be a random hermitian matrix with specified eigenvalues $\lambda_1, \dots, \lambda_N \in \mathbb{R}$"...

... and let Y be a random hermitian matrix with specified eigenvalues $\mu_1, \dots, \mu_N \in \mathbb{R}$

what can we say about eigenvalues of

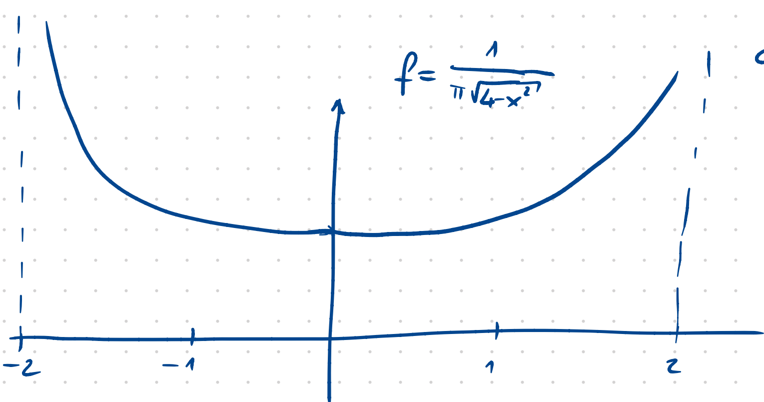
$$X + Y = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} U^* + V \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_N \end{bmatrix} V^* \ ?$$

independent random matrices from $\mathcal{U}(N)$ with distribution = Haar measure.

$\boxed{\text{for } N \to \infty?}$

Example:

$$U \begin{bmatrix} \overbrace{+1 \quad +1 \quad \cdots \quad +1}^{a \text{ times}} & & \\ & \underbrace{-1 \quad \cdots \quad -1}_{a \text{ times}} \end{bmatrix} U^* + V \begin{bmatrix} \overbrace{+1 \quad +1 \quad \cdots \quad +1}^{a \text{ times}} & & \\ & \underbrace{-1 \quad \cdots \quad -1}_{a \text{ times}} \end{bmatrix} V^*$$



arc-sine law

typical histogram of eigenvalues

Answer $\longrightarrow$ Voiculescu's free probability

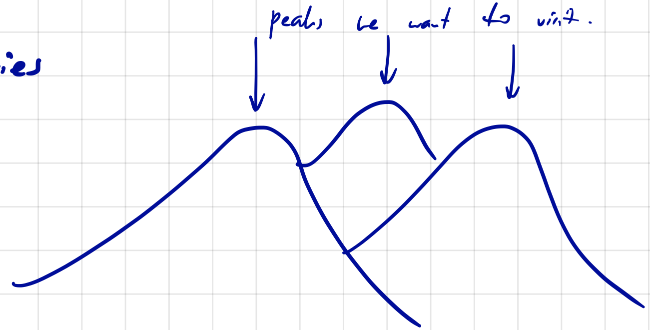
[lots of nice combinatorics]

[a bit of classical probability theory]

[a bit of representation theory of $U(N)$]

Highlights of the series

mathematical landscape



PROVERBIAL
„what happens to eigenvalues of a sum of two matrices".

* what happens to eigenvalues of large random matrices

* freeness - surprising abstract framework for random matrices

* combinatorics of non-crossing partitions and free cumulants

* eigenvalues of nonhermitian matrices and Brown measure.

* operator algebras, free group factors, ...