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Lectures on random matrices and free probability theory

Lecture 2.
GUE random matrices, part 1.

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Empirical eigenvalues distribution.

suppose X is a $N \times N$ random matrix with (random) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{C}$

$$\mu_X = \frac{1}{N} \sum_i \delta_{\lambda_i}$$

Counting measure on the set of the eigenvalues.

<u>RANDOM</u> probability measure on $\mathbb{C}$
gives full information about the eigenvalues of X.
(the joint distribution of)

TYPICAL GOAL: show some "law of large numbers" for μ_X as $N \to \infty$

Moments method:

"k-th moment"

$$M_k := \underbrace{\int_{\mathbb{C}} z^k \, d\mu_X}_{\text{random variable.}} = \underbrace{\overbrace{\frac{1}{N} \mathrm{Tr}}^{\mathrm{tr}\text{ the normalized trace}} X^k}_{\text{random variable!}} = \frac{1}{N} \sum_i \lambda_i^k$$

if X is not hermitian, this kind of information is NOT helpful.

if $X = X^*$ is hermitian, $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ and there is some chance of success

$$\mathbb{E} \int_{\mathbb{C}} z^k \, d\mu_X = \int_{\mathbb{C}} z^k \, \underbrace{\mathbb{E}\mu_X}_{\substack{\text{non-random} \\ \text{probability} \\ \text{measure on } \mathbb{C}}} = \mathbb{E} \frac{1}{N} \mathrm{Tr} X^k$$

„mean eigenvalues distribution"

good for understanding the average eigenvalues distribution.

$$\mathrm{Var} \int_{\mathbb{C}} z^k \, d\mu_X = \mathrm{Var} \frac{1}{N} \mathrm{Tr} X^k$$

if $X = X^*$ is hermitian, this is a real-valued random variable.

Small variance = law of large numbers.

Goal for today:

$X^{(1)}, X^{(2)}, \ldots$ sequence of hermitian random matrices.

$X^{(N)}$ is $N \times N$ random matrix.

~~We will show that~~

ν - probability measure on $\mathbb{R}$.

Goal: $\forall k \in \{0,1,2,\ldots\}$

$$\mathbb{E} \int_{\mathbb{R}} z^k \, d\mu_{X^{(N)}} \xrightarrow[N \to \infty]{} \int_{\mathbb{R}} z^k \, d\nu$$

$$\shortparallel$$

$$\int_{\mathbb{R}} z^k \, d\mathbb{E}\mu_{X^{(N)}}$$

very weak result which will be improved.

$$\boxed{\text{“} \; \mathbb{E}\mu_{X^{(N)}} \xrightarrow{\text{in moments}} \nu \; \text{”}}$$

moments and convergence of measures

$\longrightarrow$ [MS] Section 2.1.

probability measures on $\mathbb{R}$

moments

convergence in moments

weak convergence of probability measures

measures determined by moments

Carleman's criterion

Hint: if $\sum \frac{1}{\sqrt[2n]{m_{2n}}} = +\infty$ then moment problem is determinate

Convergence in moments vs weak convergence

$\longrightarrow$ next lecture.

convergence in moments is usually disdained by probabilists. For example: the limit might be not unique.

However, in the non-commutative setup we cannot really formulate "weak convergence of probability measures" and we have to stick to moments.

→ next lecture.

complex Gaussian distribution.

random variable Z has complex Gaussian distribution if joint distribution of $X = \operatorname{Re} Z$ and $Y = \operatorname{Im} Z$ is Gaussian.

our favorite example: $Z = X + iY$
with $X, Y \sim N(0, \sigma^2)$ and independent.

$$\mathbb{E} Z = 0$$

$$\mathbb{E} Z^2 = \mathbb{E}\, X^2 - Y^2 + 2i\, XY = 0$$

$$\mathbb{E} \bar{Z}^2 = 0$$

$$\mathbb{E}\, Z\bar{Z} = \mathbb{E}\, |Z|^2 = \mathbb{E}\, X^2 + Y^2 = 2\sigma^2.$$

if you work with complex-valued random variables, covariances involve random variables AND their complex conjugates.

afraid of complex random variables? always can translate everything to Re... and Im...

Magic of centered Gaussian distributions.

[works both for real and complex Gaussian]

Claim. Suppose joint distribution of the family (X_α) of random variables is centered Gaussian. Then

$$\mathbb{E}\, X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_n} = \sum_{\substack{\pi = \{\{\pi_{1,1},\, \pi_{1,2}\}, \\ \vdots \quad \}}} \prod_i \underbrace{\mathbb{E}\, X_{\pi_{i,1}} X_{\pi_{i,2}}}_{\text{covariance}}$$

π is a pairing of $[n] := \{1, 2, \dots, n\}$

Exercise : prove it!

→ [MS] Section 1.4.

Example: if $X \sim N(0,1)$ is standard Gaussian then...

$$\mathbb{E}\, X^n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)(n-3)(n-5)\cdots 7\cdot 5\cdot 3\cdot 1 = \\ = (n-1)!! & \text{if } n \text{ is even.} \end{cases}$$

Example:

$$\mathbb{E}\, X_1 X_2 X_3 X_4 = \dots$$

Ginibre ensemble.

our favorite normalization:

$2c = 4\sigma^2 = \frac{1}{N}$

$\sigma^2 = \frac{1}{4N}$

X is a $N \times N$ matrix with entries (f_{ij})

$(\operatorname{Re} f_{ij}, \operatorname{Im} f_{ij})_{ij}$ – family of independent Gaussian random variables $\sim N(0, \sigma^2)$

~~**Exercise.** Write a computer program in some fancy language which → generates a Ginibre random matrix, → calculates its (complex!) eigenvalues, and → plots them on the plane. Play with the program for large N~~

~~Hint: use SageMath~~

(the distribution of) Ginibre ensemble can be uniquely determined by saying that ① the joint distribution of $(\operatorname{Re} f_{ij}, \operatorname{Im} f_{ij})$ is centered Gaussian, ② specifying the covariance:

$$\mathbb{E}\, f_{ij}\, f_{kl} = 0$$

$$\mathbb{E}\, \overline{f_{ij}}\, \overline{f_{kl}} = 0 \qquad c = 2\sigma^2$$

$$\mathbb{E}\, f_{ij}\, \overline{f_{kl}} = [i=k]\,[j=l]\; c$$

$\longrightarrow$ [MS] Section 4.3, exercise 2.

Exercise. Suppose that X is a random matrix from the Ginibre ensemble and U, V are unitary matrices.

Prove that UXV is also a random matrix from Ginibre ensemble.

<u>Hint.</u>

Calculate the covariance of the Gaussian random variables of the form

$$\langle u, Xv \rangle$$

GUE

suppose X is a random matrix from Ginibre ensemble.

we say that $Y := X + X^*$ is a

GUE (Gaussian Unitary Ensemble) random matrix.

The distribution of a GUE random matrix can be uniquely determined by saying that the entries

(g_{ij}) of Y form a complex centered Gaussian random vector with

① $g_{ij} = \overline{g_{ji}}$

② $\mathbb{E}\, g_{ij}\, g_{kl} = \mathbb{E}\left(f_{ij} + \overline{f_{ji}}\right)\left(f_{kl} + \overline{f_{lk}}\right) =$

$= [i = l]\,[j = k]\ 2c$

$\boxed{2c = \frac{1}{N}}$ preferred normalization used by Mingo and Speicher.

$\boxed{2c = 1}$ another normalization used by MCS

Exercise.
Suppose Y is a GUE and
U is a unitary matrix.

Show that $U Y U^{-1}$ is a GUE.

"every orthonormal base of eigenvectors has equal probability".

Exercise. Describe the joint probability distribution of the random variables
$(\mathrm{Re}\; g_{ij}, \mathrm{Im}\; g_{ij})_{i<j}, (g_{ii})_i$

Meditate about the difference between the diagonal and the off-diagonal entries.

Exercise. Traditional computer experiment concerning the eigenvalues.

Hint: use HISTOGRAMS

Genus expansion for GUE

→ [MS] Section 1.7
Section 1.8.

$$\mathbb{E} \int_{\mathbb{R}} z^{2k} \, d\mu_Y =$$

$$= \mathbb{E} \, \underbrace{\mathrm{tr}} \, Y^{2k} = \mathbb{E} \, \frac{1}{N} \mathrm{Tr} \, Y^{2k} = \quad \ldots$$

normalized trace

Hint:

$$\mathbb{E} \, \mathrm{tr} \, Y^{2k+1} = 0$$

for obvious reasons.

Genus expansion for GUE

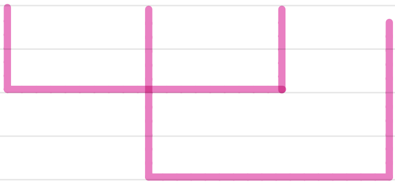
→ [MS] Section 1.7
Section 1.8.

normalized trace

γ

$$\frac{1}{N}\,\mathbb{E}\sum_{i_1,\dots,i_{2k}} g_{i_1 i_2}\, g_{i_2 i_3}\, g_{i_3 i_4}\, g_{i_4 i_5} \cdots\cdots g_{i_{2k} i_1} =$$

Pairing π

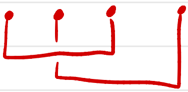


normalized trace

$$= \sum_{\pi} N^{\#\text{closed loops} \;-\; k \;-\; 1}$$

produced by covariance.

Concrete example : $2k = 4$



and

for normalization

$\mathbb{E}\,|g_{ij}|^2 = 1/N = 2C$

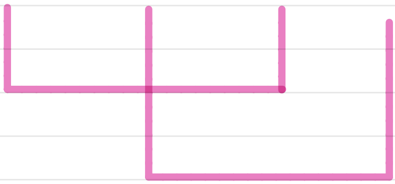
Genus expansion for GUE

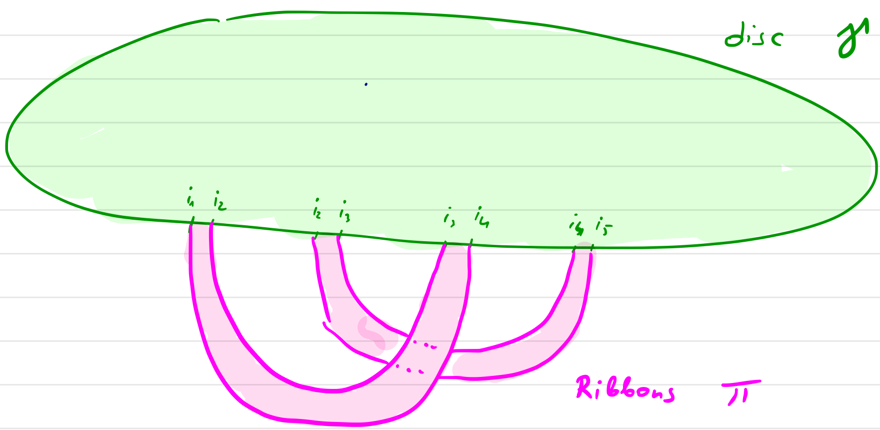
Mingo and Speicher [Section 1.8] have alternative technology.

γ

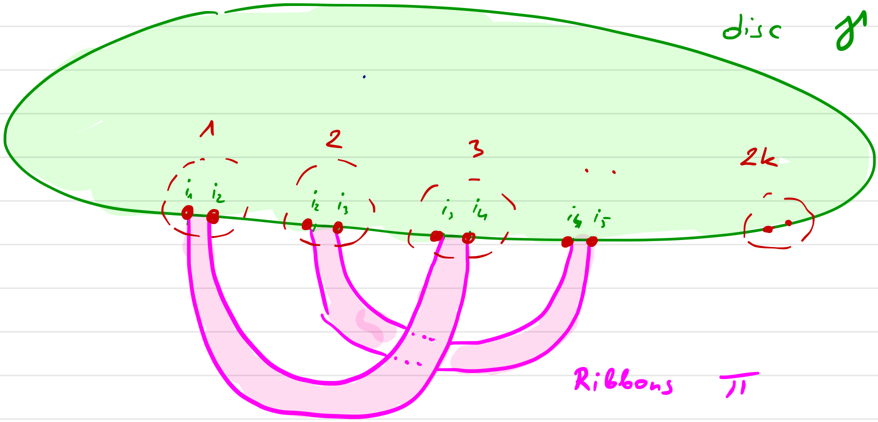
$$\mathbb{E}\sum_{i_1,\dots,i_{2k}} g_{i_1 i_2}\, g_{i_2 i_3}\, g_{i_3 i_4}\, g_{i_4 i_5} \cdots\cdots g_{i_{2k} i_1} =$$

π





oriented surface with boundary.



oriented
surface with boundary.

boundary = a number of circles S^1.
don't like surfaces with a boundary?
glue a disk to each circle S^1.

$\longrightarrow$ connected, oriented
surface without boundary.

Euler characteristic

$$\chi = 2 - 2g = V - E + F =$$

genus

$$= 4k - 4k - 2k + 1 + k + \#\text{loops} =$$

$$= 1 - k + \#\text{loops}$$

$V = 4k$
$E = 4k + 2k$
$F = 1 + k + \#\text{loops}$

$$\#\text{loops} - k - 1 = -2g$$

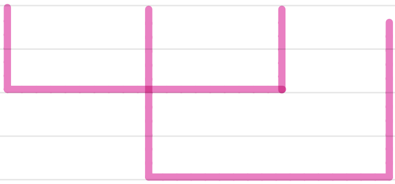
✓.

normalized trace

γ_1

$$\frac{1}{N}\ \mathbb{E}\sum_{i_1,\dots,i_{2k}} g_{i_1 i_2}\, g_{i_2 i_3}\, g_{i_3 i_4}\, g_{i_4 i_5} \cdots\cdots g_{i_{2k} i_1} =$$

π



normalized trace

$$= \sum_{\pi} N^{\#\text{closed loops} - k - 1} =$$

produced by covariance.

$$= \sum_{\pi} \frac{1}{N^{2-\chi}} = \sum_{\pi} \frac{1}{N^{2\,\text{genus}(\pi)}}$$

leading contribution — genus = 0
sphere

requires some arguments → [MS, Section 1.3]

$$\lim_{N\to\infty} \int_{\mathbb{R}} x^{2k} \, d\mathbb{E}\mu_Y(x) =$$

$$= \underbrace{\lim_{n\to\infty} \mathbb{E} \operatorname{tr} Y^{2k}}_{\text{the limit exists and is finite}} = \#\text{ Non-crossing partitions on } [2k]$$

$$= C_k \quad \text{Catalan number}$$

$$= \frac{1}{k+1}\binom{2k}{k} =$$

exercise.

$$= \int_{-2}^{2} \frac{1}{2\pi}\sqrt{4-x^2}\, x^{2k} \, dx =$$

$$= \int_{\mathbb{R}} x^{2k} \, d\mu_{sc}(x)$$

this shows that the MEAN eigenvalues distribution $\mathbb{E}\mu_X$ converges to semicircular law μ_{sc} in moments which is nice, but not exactly what we want.

$$\mathbb{E}\,\mathrm{tr}\, Y^{2k} = C_k + O\left(\frac{1}{N^2}\right)$$

<u>**Exercise.**</u> GOE is a ~~hermitian~~ symmetric, real valued matrix defined similarly as GUE except that ~~COMPLEX GAUSSIAN~~ is replaced by REAL GAUSSIAN.

What changes in the above calculations when GUE is replaced by GOE?