Robinson-**S**chensted-**K**nuth algorithm
Start with two empty tableaux. Read letters of the word one after another. With each letter proceed as follows:

1. start with the bottom row of the *insertion tableau P*,
2. insert the letter to the leftmost box in this row which contains a number which is **bigger** than the one which you want to insert,
3. if you had to bump some letter, this bumped letter must be inserted in to the next row according to the rule number 2,
4. if you inserted a letter to an empty box in the *insertion tableau P,* make a mark about the position of this box in the *recording tableau Q* and proceed to the next letter of the word.

74	99		
23	53	70	
16	37	41	82

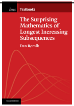
insertion tableau $P(w)$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$

Further reading



Dan Romik
„The Surprising Mathematics of Longest Increasing Subsequences"

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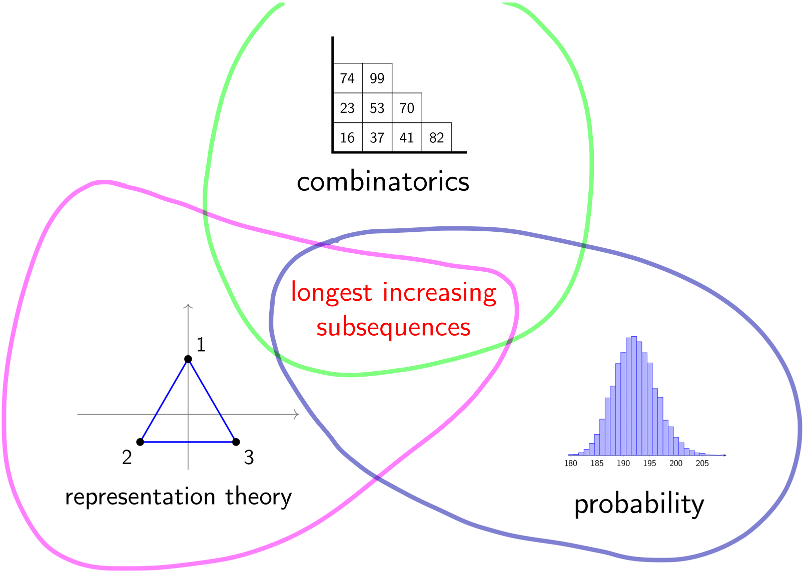
Poisson limit theorems for Robinson–Schensted correspondence

Piotr Śniady

IMPAN Toruń

joint work with Mikołaj Marciniak and Łukasz Maślanka

handout, slides $\longrightarrow$ `psniady.impan.pl/Poisson`


74 99
23 53 70
16 37 41 82
combinatorics
longest increasing subsequences
1
2
3
representation theory
180 185 190 195 200 205
probability

Longest Increasing Subsequence

23, 53, 74, 16, 99, 70, 82, 37, 41

what is the length of the longest increasing subsequence?

Longest Increasing Subsequence

23, 53, 74, 16, 99, 70, 82, 37, 41

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Longest Increasing Subsequence

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what is the length of the longest increasing subsequence?

$$\text{LIS}(23, 53, 74, 16, 99, 70, 82, 37, 41) = 4$$

Longest Increasing Subsequence

23, 53, 74, 16, 99, 70, 82, 37, 41

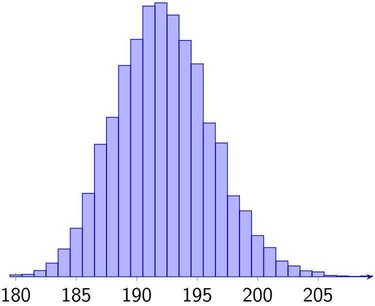
what is the length of the longest increasing subsequence?

$$\mathrm{LIS}(23, 53, 74, 16, 99, 70, 82, 37, 41) = 4$$

Stanisław Ulam:

let π_n be a uniformly random permutation of the letters $1, 2, \ldots, n$

what can you say about the random variable $\mathrm{LIS}_n = \mathrm{LIS}(\pi_n)$ in the limit $n \to \infty$?



$n = 10^4$

Robinson-**S**chensted-**K**nuth algorithm

Start with two empty tableaux. Read letters of the word one after another. With each letter proceed as follows:

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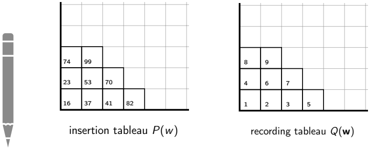
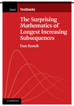


insertion tableau $P(w)$ recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Further reading



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Robinson–Schensted–Knuth algorithm is a bijection. . .

input:

- sequence $\mathbf{w} = (w_1, \dots, w_n)$

output:

- semistandard tableau P,
- standard tableau Q,

P and Q have the same shape with n boxes

---

example:

w $= (23, 53, 74, 16, 99, 70, 82, 37, 41)$

74	99		
23	53	70	
16	37	41	82

insertion tableau P(w)

8	9		
4	6	7	
1	2	3	5

recording tableau Q(w)

Robinson–Schensted–Knuth algorith — the induction step

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
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recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

Robinson–Schensted–Knuth algorith — the induction step

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Robinson–Schensted–Knuth algorith — the induction step

74	99		
23	53	70	
16	37	41	82

18

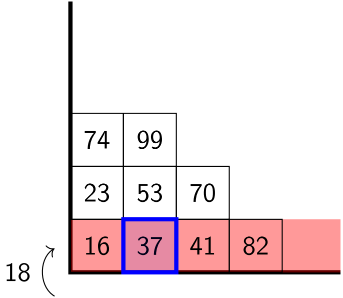
insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

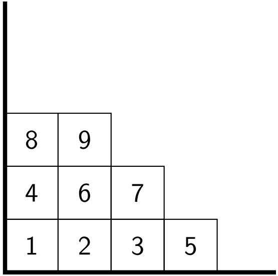
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$



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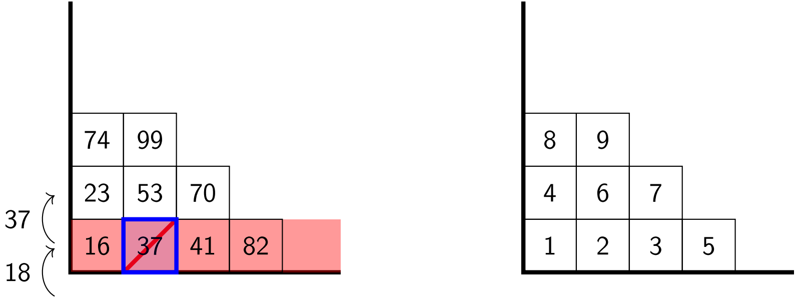
Robinson–Schensted–Knuth algorith — the induction step



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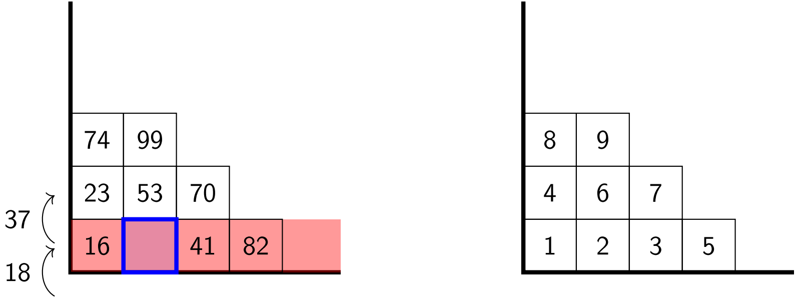

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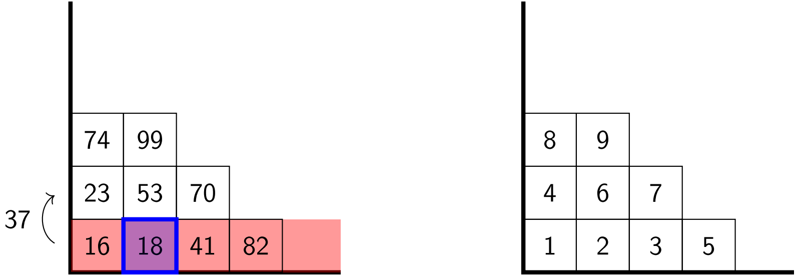

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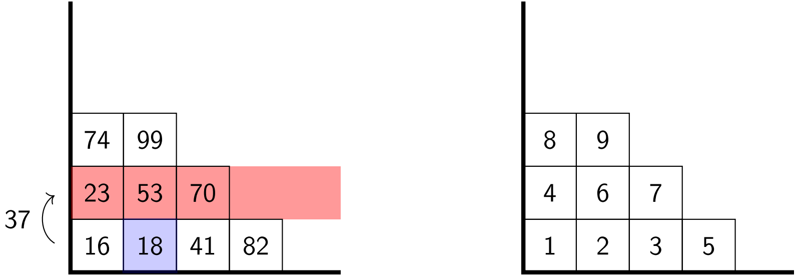
Robinson–Schensted–Knuth algorith — the induction step



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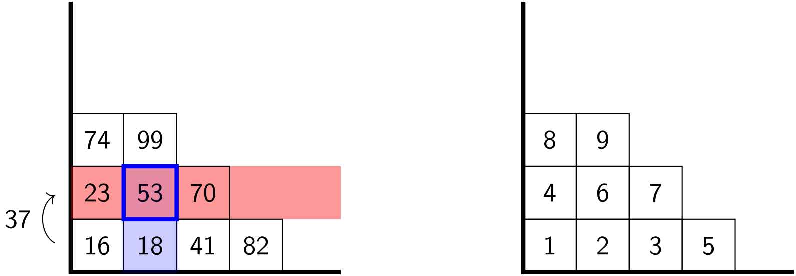

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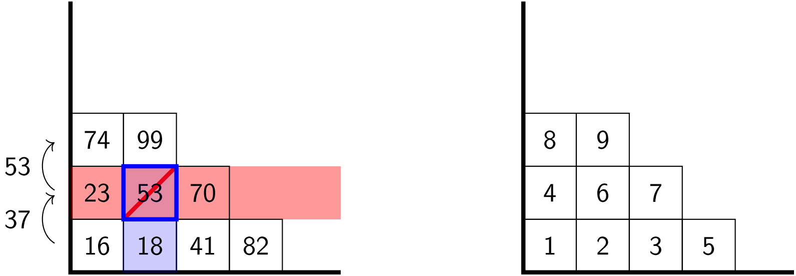

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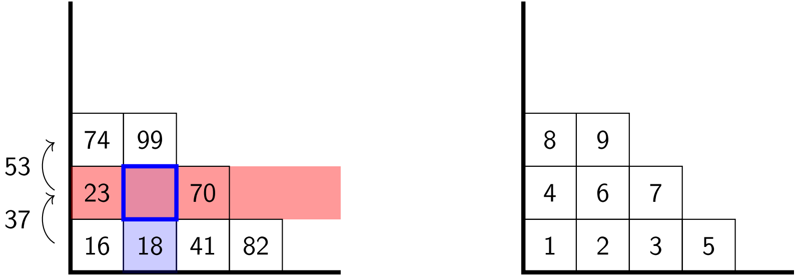
Robinson–Schensted–Knuth algorith — the induction step



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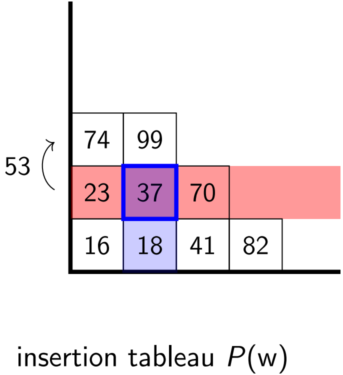
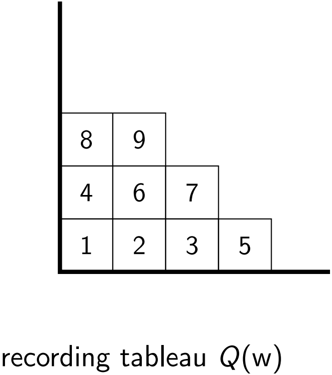
Robinson–Schensted–Knuth algorith — the induction step





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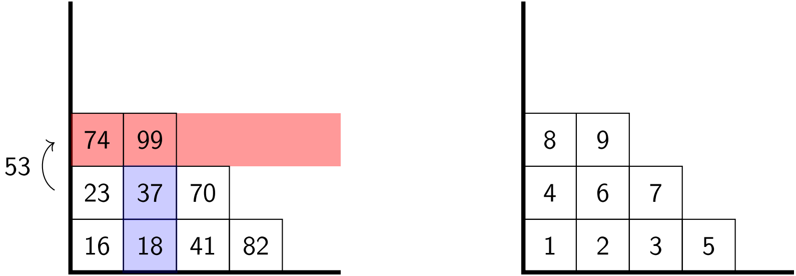
Robinson–Schensted–Knuth algorith — the induction step



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recording tableau $Q(\mathrm{w})$

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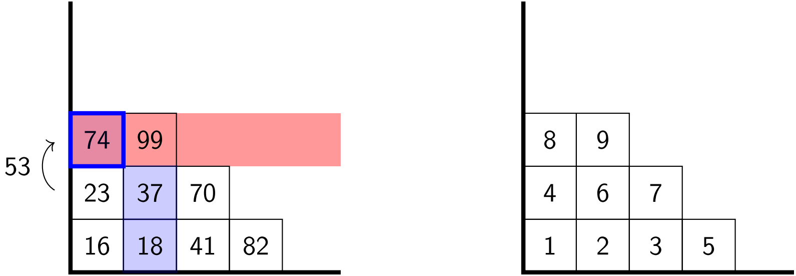
Robinson–Schensted–Knuth algorith — the induction step



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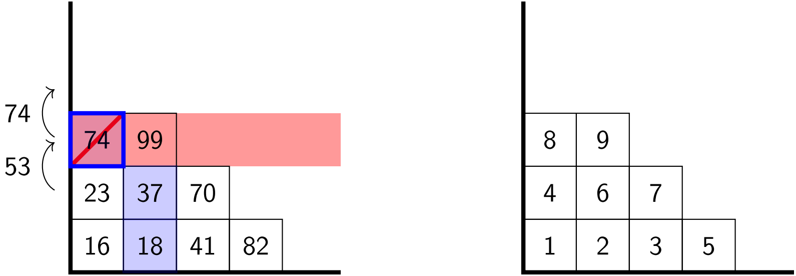
Robinson–Schensted–Knuth algorith — the induction step



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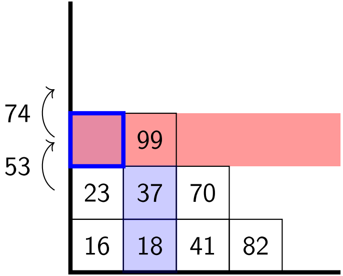

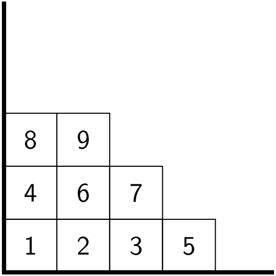

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recording tableau $Q(\mathrm{w})$

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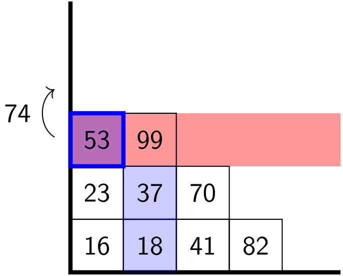
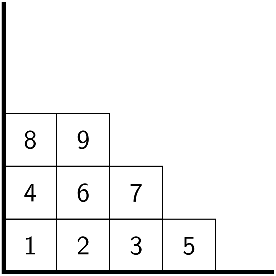
Robinson–Schensted–Knuth algorith — the induction step



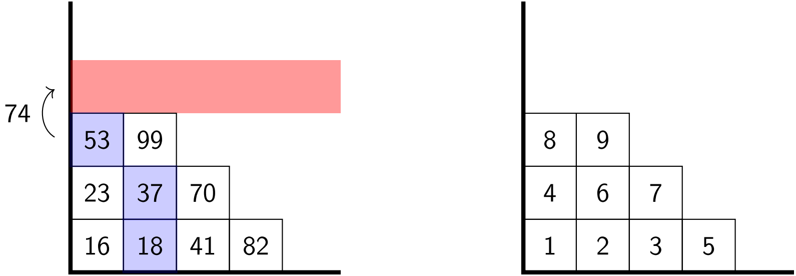
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

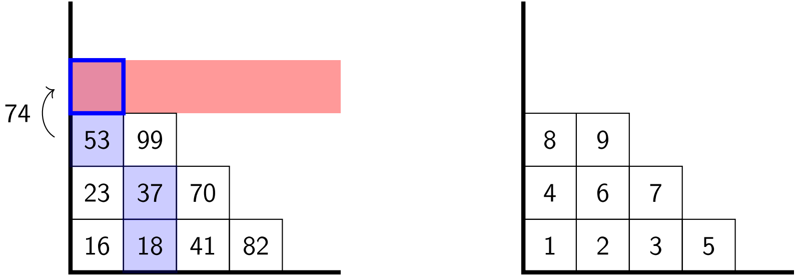
Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

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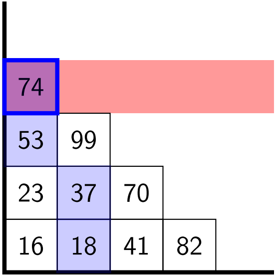
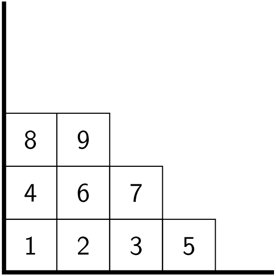


insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step

74			
53	99		
23	37	70	
16	18	41	82

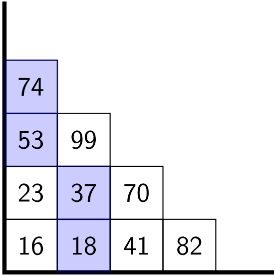
insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

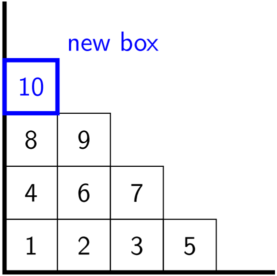
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



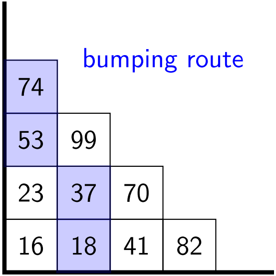
insertion tableau $P(\mathrm{w})$

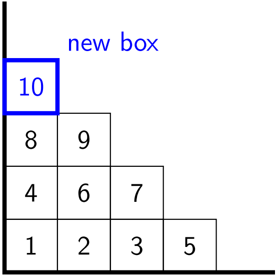


recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step





insertion tableau $P(\mathrm{w})$

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step

74			
53	99		
23	37	70	
16	18	41	82

insertion tableau $P(\mathrm{w})$

10			
8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$$

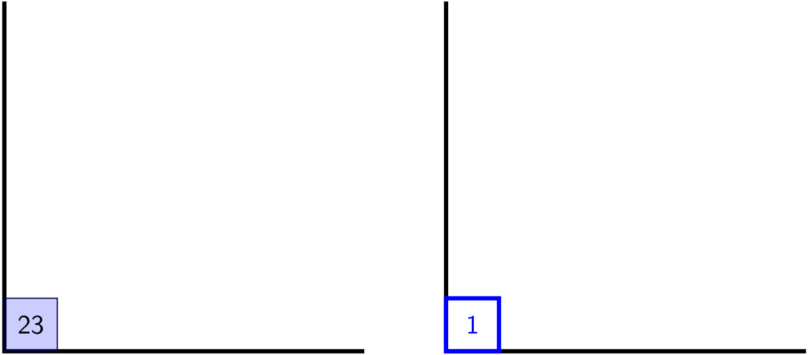
Robinson–Schensted–Knuth algorithm

insertion tableau $P(\mathrm{w})$

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = \emptyset$

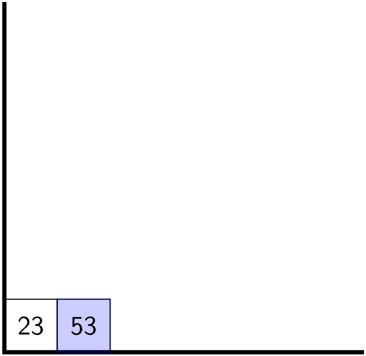
Robinson–Schensted–Knuth algorithm



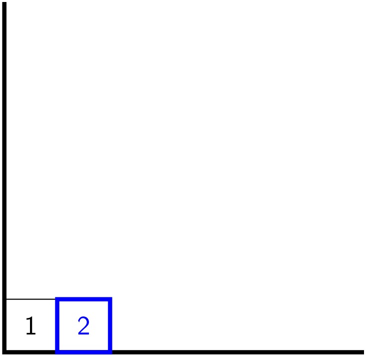
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23)$

Robinson–Schensted–Knuth algorithm



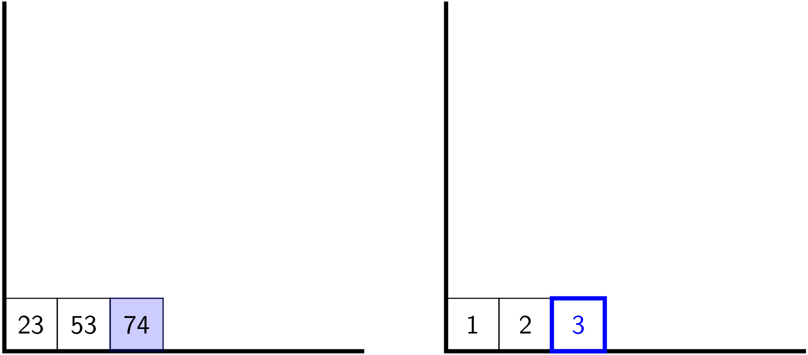
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53)$

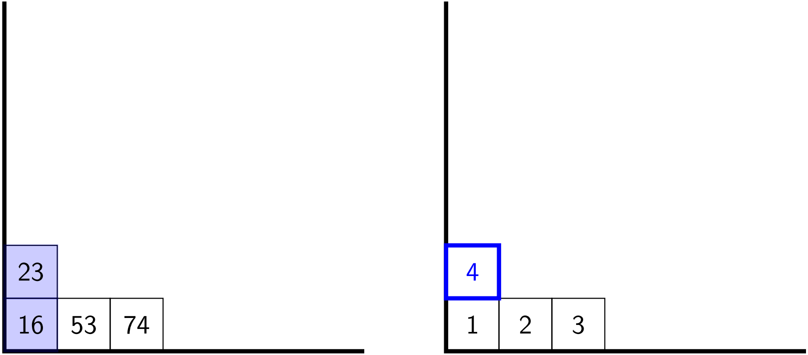
Robinson–Schensted–Knuth algorithm



insertion tableau $P(w)$ recording tableau $Q(w)$

$w = (23, 53, 74)$

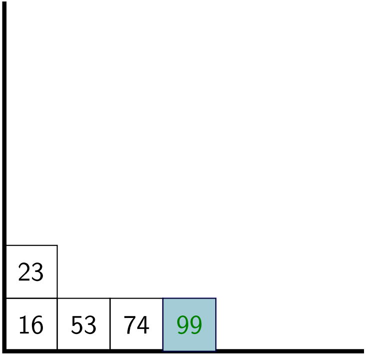
Robinson–Schensted–Knuth algorithm



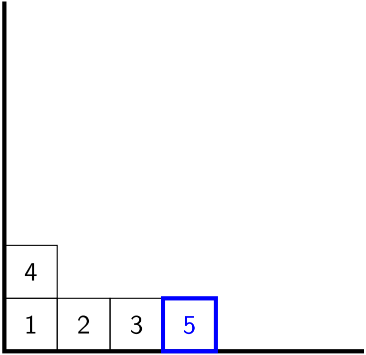
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16)$

Robinson–Schensted–Knuth algorithm



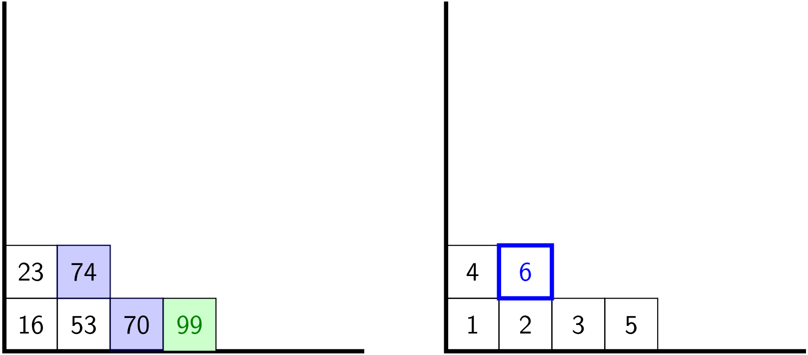
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99)$

Robinson–Schensted–Knuth algorithm



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70)$

Robinson–Schensted–Knuth algorithm

23	74	99	
16	53	70	82

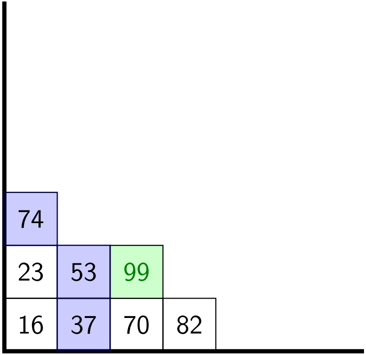
insertion tableau $P(\mathrm{w})$

4	6	7	
1	2	3	5

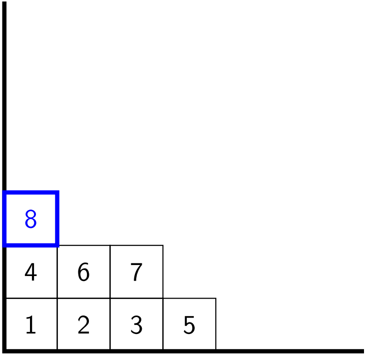
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82)$

Robinson–Schensted–Knuth algorithm



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

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Robinson–Schensted–Knuth algorithm

74	99		
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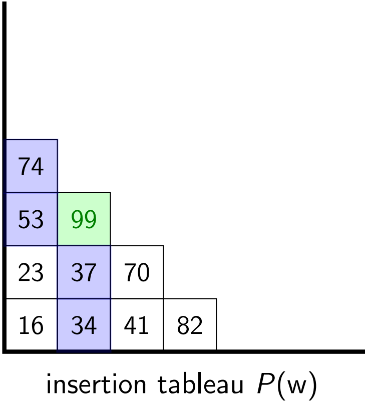
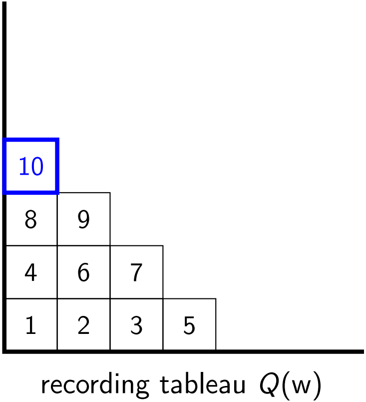
$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

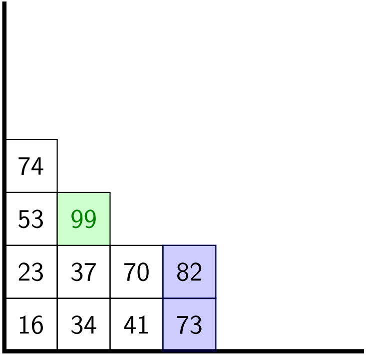
Robinson–Schensted–Knuth algorithm



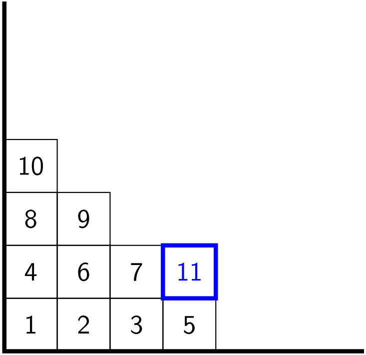


$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34)$

Robinson–Schensted–Knuth algorithm



insertion tableau $P(w)$



recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73)$

Robinson–Schensted–Knuth algorithm

74			
53			
23	99		
16	37	70	82
2	34	41	73

insertion tableau $P(\mathrm{w})$

12			
10			
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2)$

Robinson–Schensted–Knuth algorithm

74			
53	99		
23	37		
16	34	70	82
2	24	41	73

insertion tableau $P(\mathrm{w})$

12			
10	13		
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2, 24)$

Robinson–Schensted–Knuth algorithm is a bijection. . .

input:

- sequence
  $\mathbf{w} = (w_1, \dots, w_n)$

output:

- semistandard tableau P,
- standard tableau Q,

P and Q have the same shape with n boxes

---

example:
w $= (23, 53, 74, 16, 99, 70, 82, 37, 41)$

74	99		
23	53	70	
16	37	41	82

insertion tableau P(w)

8	9		
4	6	7	
1	2	3	5

recording tableau Q(w)

Robinson–Schensted–Knuth algorithm is a bijection. . .

input:

- permutation
  $\mathbf{w} = (w_1, \ldots, w_n)$
  of the letters $1, \ldots, n$

output:

- standard tableau P,
- standard tableau Q,

P and Q have the same shape with n boxes

---

example:
w = (2, 5, 7, 1, 9, 6, 8, 3, 4)

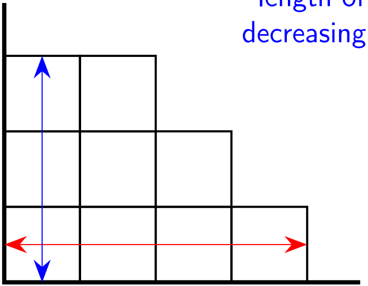
7	9		
2	5	6	
1	3	4	8

insertion tableau P(w)

8	9		
4	6	7	
1	2	3	5

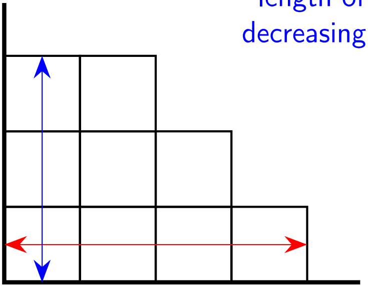
recording tableau Q(w)

length of the first column
=
length of the longest
decreasing subsequence



length of the first row
=
length of the longest
increasing subsequence

length of the first column
=
length of the longest
decreasing subsequence

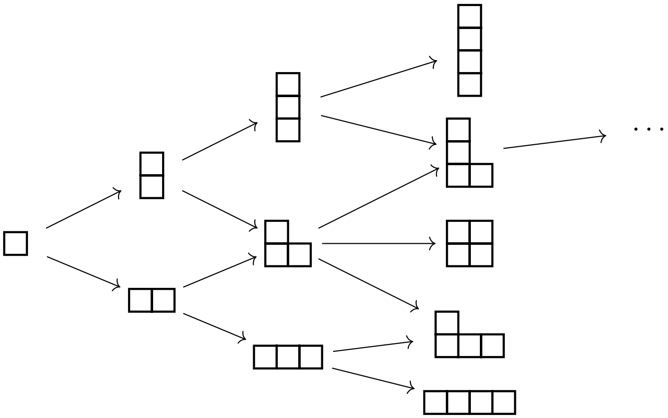


length of the first row
=
length of the longest
increasing subsequence

for which funny question
concerning increasing subsequences
the answer is:

"the total length of the first two rows" ?

irreducible representations of the symmetric groups $\mathfrak{S}(1) \subset \mathfrak{S}(2) \subset \mathfrak{S}(3) \subset \cdots$



representation theory $\longleftrightarrow$ combinatorics

today: Markov chain

problem

what can you say about RSK
applied to random input

asymptotically, as the size of the input tends to infinity?

today: concrete questions about asymptotics of:

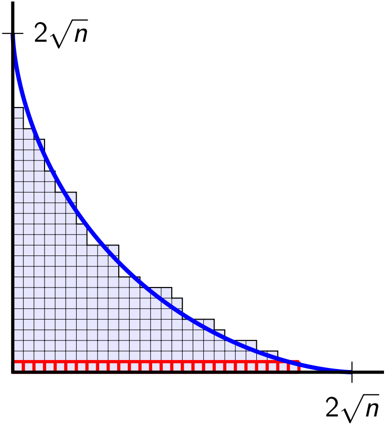
- Longest Increasing Subsequences,
- bottom rows of the insertion/recording tableau,
- bumping routes,
- $\ldots$,

Ulam's problem, on steroids

π_n be a uniformly random permutation of $1, 2, \ldots, n$;

what can you say about the shape $\lambda^{(n)}$ of tableaux $P(\pi_n)$ and $Q(\pi_n)$ in the limit $n \to \infty$?

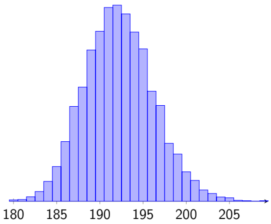
yes, there exists a limit shape! Logan&Shepp, Vershik&Kerov 1977



Corollary: the length of the bottom row $\lambda_0^{(n)}$ fulfils

$$\lim_{n\to\infty} \frac{\mathbb{E}\lambda_0^{(n)}}{\sqrt{n}} = 2$$

Ulam:
what is the limit distribution of the length of the bottom row $\lambda_0^{(n)}$



Ulam:
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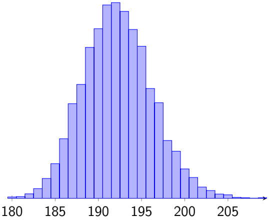
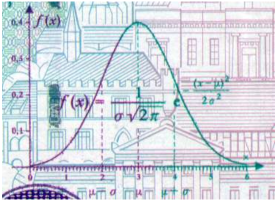


Gauss?



Ulam:
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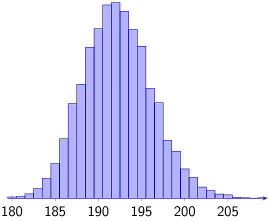
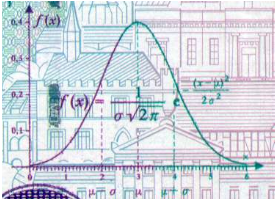
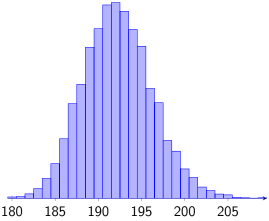


Gauss?

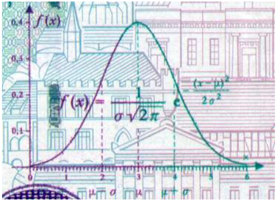


NO!

Ulam:
what is the limit distribution of the length of the bottom row $\lambda_0^{(n)}$



Gauss?

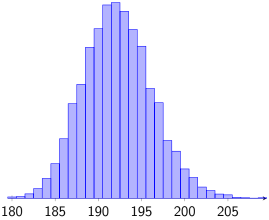


NO!

surprise:
this is
Tracy–Widom distribution

BAIK, DEIFT, JOHANSSON 1999

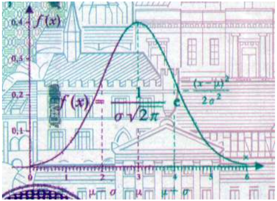
Ulam:
what is the limit distribution of the length of the bottom row $\lambda_0^{(n)}$



$\mathbb{E}\lambda_0^{(n)} \approx 2\ n^{1/2}$

$\mathrm{Var}\ \lambda_0^{(n)} \sim n^{1/3} \ll n^{1/2}$

Gauss?



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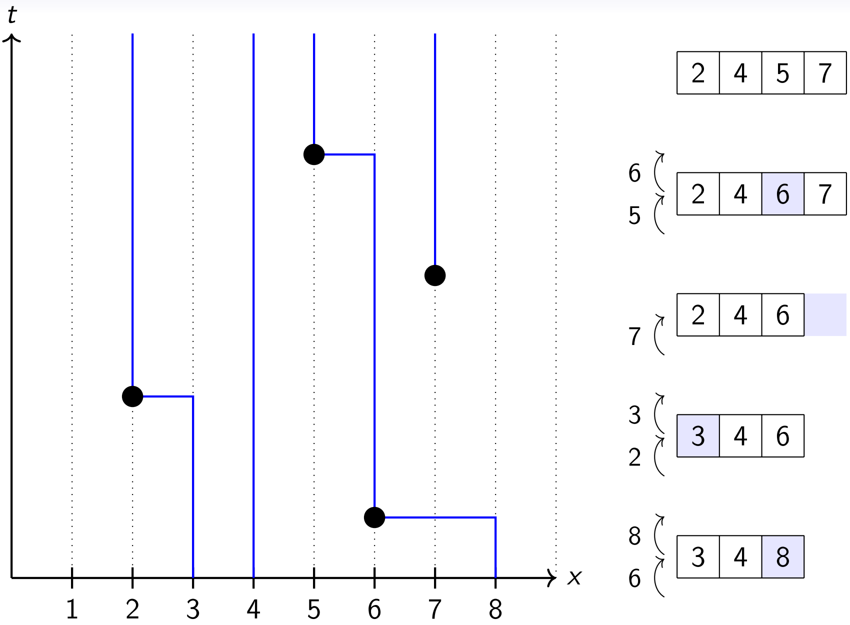

t
x
1
2
3
4
5
6
7
8
2 4 5 7
6
5
2 4 6 7
7
2 4 6
3
2
3 4 6
8
6
3 4 8

Hammersley interacting particle process

sample black points in $[0,1] \times \mathbb{R}_+$
by Poisson point process with unit intensity

let $x_1(t), x_2(t), \dots$ be positions of the particles at time t

Theorem (Aldous, Diaconis 1995; version about P)

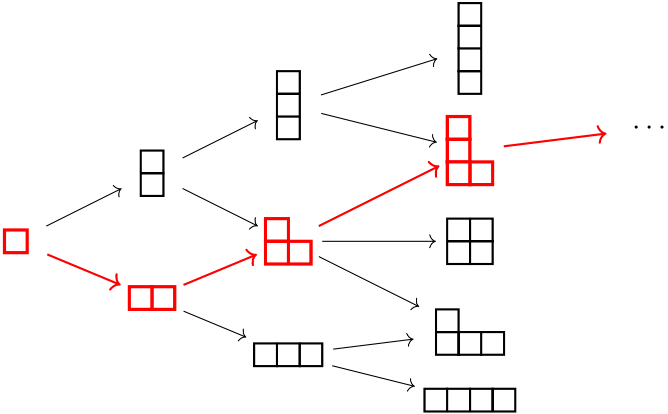
for any $0 < w < 1$ the random set

$$\left\{ \sqrt{t}\,\left(x_i(t) - w\right) \; : i = 1, 2, \dots \right\}$$

converges in distribution to Poisson point process with intensity $\frac{1}{\sqrt{w}}$ in the limit $t \to \infty$

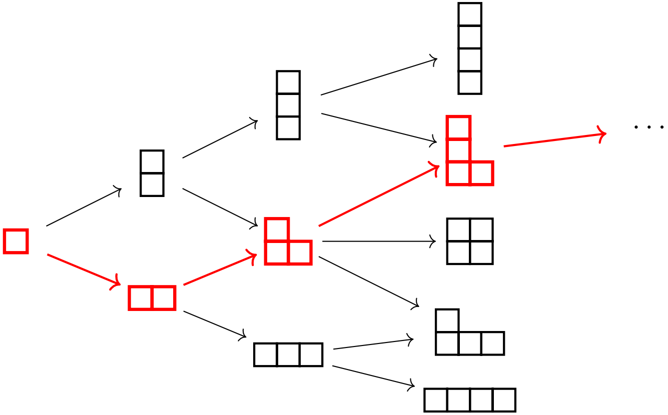
a result about the bottom row of the insertion tableau P
after $\approx t$ steps of RSK
applied to independent random variables
with the uniform distribution on $[0,1]$

Plancherel growth process $\lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \cdots$



let $(\pi_1, \ldots, \pi_k)$ be a uniformly random permutation of $1, \ldots, k$;
define $\lambda^{(n)} = \mathrm{RSK}(\pi_1, \ldots, \pi_n)$ to be the common shape
of the insertion and recording tableau related to the prefix of π

Plancherel growth process $\lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \dots$



let $(\xi_1, \xi_2, \dots)$ be i.i.d. $U(0,1)$ random variables from $[0,1]$
define $\lambda^{(n)} = \mathrm{RSK}(\xi_1, \dots, \xi_n)$ to be the common shape
of the insertion and recording tableau related to the prefix of ξ

growth of the bottom row

Theorem (Aldous, Diaconis 1995; version about Q)

the random function

$$\mathbb{R}_+ \ni t \mapsto \lambda_0^{\left(n+\lfloor t\sqrt{n}\rfloor\right)} - \lambda_0^{(n)}$$

converges in distribution to Poisson process

$$\mathbb{R}_+ \ni t \mapsto N_t$$

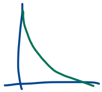
as $n \to \infty$

Maślanka, Marciniak, Śniady 2020: extension to more than one row

proof inspired by Vershik and Kerov 1985

Ulam's problem and Poisson local limit

$$c = \lim_{n\to\infty} \frac{LIS_n}{\sqrt{n}}$$

	$c \geq 2$ „HARD"	$c \leq 2$ „EASY"	Poisson local limit
Plancherel growth process „algebraic combinatorics" „discrete"	 Logan & Shepp Vershik & Kerov 1977 „hook-length formula + + variational calculus"	Vershik & Kerov 1985 „Cauchy-Schwarz inequality"	MMŚ 2020 „bottom rows of P and Q $\overset{d}{\approx}$ Poisson"
⇓ Poissonization Hammersley process „probability" „continuous space-time"	Aldous & Diaconis 1995 [HARD]	Aldous & Diaconis 1995 „compare Hammersley on $\mathbb{R}_+$ to $\mathbb{R}$"	Aldous & Diaconis 1995 Hammersley process on $\mathbb{R}_+$ converges locally to a stationary distribution

Plancherel growth process: probability distribution for fixed time

for any diagram μ with n boxes

$$\mathbb{P}\left[\lambda^{(n)} = \mu\right] = \frac{f^{\mu} \times f^{\mu}}{n!}, \qquad \text{“Plancherel measure of order } n\text{”}$$

where f^{μ} is the number of standard Young tableaux with shape μ

Hint: use RSK bijection; arbitrary P and Q with shape μ

Plancherel growth process: probability distribution of $\left(\lambda^{(n-1)}, \lambda^{(n)}\right)$

for any diagram μ with $n-1$ boxes
and any diagram ν with n boxes
such that $\mu \nearrow \nu$

$$\mathbb{P}\left[\lambda^{(n-1)} = \mu \text{ and } \lambda^{(n)} = \nu\right] = \frac{f^{\nu} \times f^{\mu}}{n!} = \frac{\sqrt{\mathbb{P}\left(\lambda^{(n-1)} = \mu\right)}\sqrt{\mathbb{P}\left(\lambda^{(n)} = \nu\right)}}{\sqrt{n}}$$

where f^{μ} is the number of standard Young tableaux with shape μ

Hint: use RSK bijection;
arbitrary tableau P with shape ν,
$Q \setminus \{n\}$ is an arbitrary tableau with shape μ

distribution of a prefix $\lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(n)}$

$$\mathbb{P}\left[\left(\lambda^{(1)}, \ldots, \lambda^{(n)}\right) = \left(\mu^{(1)}, \ldots, \mu^{(n)}\right)\right] = \frac{f^{\mu^{(n)}} \times 1}{n!}$$

depends only on the endpoint

Hint: use RSK bijection; arbitrary P, specific Q

---

corollary:

Plancherel growth process $\lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \cdots$ is a Markov chain

Def: $E_0^{(n)}$ is the event that

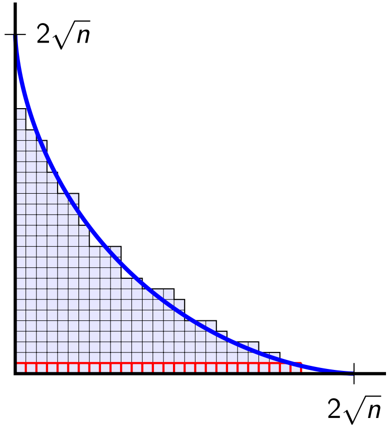
$$\lambda^{(n)} = \underbrace{\text{grow}_0\,\lambda^{(n-1)}}_{\text{one box added in the bottom row}}$$

$$\mathbb{P}\left(E_0^{(1)}\right) \geq \mathbb{P}\left(E_0^{(2)}\right) \geq \cdots$$

$$\mathbb{E}\lambda_0^{(n)} = \mathbb{P}\left(E_0^{(1)}\right) + \cdots + \mathbb{P}\left(E_0^{(n)}\right)$$

$$\lim_{n\to\infty} \frac{\lambda_0^{(n)}}{\sqrt{n}} = 2$$

$$\implies \lim_{n\to\infty} \underbrace{\sqrt{n}\,\mathbb{P}\left(E_0^{(n)}\right)}_{c_n} = 1$$



total variation distance

if $\mathbb{P}$ and $\mathbb{Q}$ are probability distributions on the same finite set X, their total variation distance is defined as

$$\frac{1}{2}\left\|\mathbb{P}-\mathbb{Q}\right\|_{\ell^1} = \max_{S\subset X}\left|\mathbb{P}(X)-\mathbb{Q}(X)\right|$$

vector space of functions on the set $\mathbb{Y}_n$ of diagrams with n boxes;

for $A \subseteq \mathbb{Y}_n$ define scalar product $\langle f, g \rangle_A = \sum_{\lambda \in A} f_\lambda g_\lambda$

and the norm $\|f\|_A = \sqrt{\langle f, f \rangle_A}$

$$Y_\mu := \frac{f^\mu}{\sqrt{n!}} = \sqrt{\mathbb{P}(\lambda^{(n)} = \mu)},$$

$$X_\mu := \frac{f^{\mathrm{del}_1 \mu}}{\sqrt{(n-1)!}} = \sqrt{\mathbb{P}(\lambda^{(n-1)} = \mathrm{del}_0 \mu)},$$

$$\langle Y, Y \rangle_A = \mathbb{P}\left(\lambda^{(n)} \in A\right),$$

$$\langle X, X \rangle_A = \mathbb{P}\left(\mathrm{grow}_0 \lambda^{(n-1)} \in A\right),$$

$$\langle X, Y \rangle_A = \sqrt{n}\, \mathbb{P}\left(\lambda^{(n)} \in A \text{ and } E_0^{(n)}\right),$$

$$\langle Y, Y\rangle_A = \mathbb{P}\left(\lambda^{(n)} \in A\right),$$
$$\langle X, X\rangle_A = \mathbb{P}\left(\mathrm{grow}_0\, \lambda^{(n-1)} \in A\right),$$
$$\langle X, Y\rangle_A = \sqrt{n}\, \mathbb{P}\left(\lambda^{(n)} \in A \text{ and } E_0^{(n)}\right),$$

$$\lim_{n\to\infty} \underbrace{\sqrt{n}\,\mathbb{P}\left(E_0^{(n)}\right)}_{c_n} = 1$$

$$\lim_{n\to\infty} \left\| c_n^{-1} X - Y \right\|_{\mathbb{Y}_n}^2 = \lim_{n\to\infty} \langle c_n^{-1} X - Y, c_n^{-1} X - Y \rangle_{\mathbb{Y}_n} = 0$$

$$\mathbb{P}\left(\lambda^{(n)} \in A \,\middle|\, E_0^{(n)}\right) - \mathbb{P}\left(\lambda^{(n)} \in A\right) = \left\langle c_n^{-1} X - Y, Y \right\rangle_A$$

$$\leq \left\| c_n^{-1} X - Y \right\|_A \; \| Y \|_A \to 0$$

$$\langle Y, Y \rangle_A = \mathbb{P}\left(\lambda^{(n)} \in A\right),$$

$$\langle X, X \rangle_A = \mathbb{P}\left(\operatorname{grow}_0 \lambda^{(n-1)} \in A\right),$$

$$\langle X, Y \rangle_A = \sqrt{n}\,\mathbb{P}\left(\lambda^{(n)} \in A \text{ and } E_0^{(n)}\right),$$

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$$\lim_{n\to\infty} \left\| c_n^{-1} X - Y \right\|^2_{\mathbb{Y}_n} = \lim_{n\to\infty} \langle c_n^{-1} X - Y, c_n^{-1} X - Y \rangle_{\mathbb{Y}_n} = 0$$

$$\mathbb{P}\left(\lambda^{(n)} \in A \,\middle|\, E_0^{(n)}\right) - \mathbb{P}\left(\lambda^{(n)} \in A\right) = \langle c_n^{-1} X - Y, Y \rangle_A \leq \left\| c_n^{-1} X - Y \right\|_A \, \|Y\|_A \to 0$$

---

conclusion: total variation distance between

- probability distribution of $\lambda^{(n)}$, and
- the conditional probability distribution of $\lambda^{(n)}$ under the condition that $E_0^{(n)}$ occured

converges to zero, as $n \to \infty$

---

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moral lesson: information that the event $E_0(n)$ occurred (or did not occur) gives us no additional information about the probability distribution of $\lambda^{(n)}$

---

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moral lesson: information that the event $E_0(n)$ occurred (or did not occur) gives us no additional information about the probability distribution of $\lambda^{(n)}$

iterate this argument and: total variation distance between

- $\left(E_0^{(n)}, \dots, E_0^{(m)}, \lambda^{(m)}\right)$, and
- the sequence of *independent* random variables $\left(\tilde{E}_0^{(n)}, \dots, \tilde{E}_0^{(m)}, \tilde{\lambda}^{(m)}\right)$

is of order $o\left(\frac{m-n}{\sqrt{n}}\right) \implies$ Poisson limit theorem

moral lesson: information that the event $E_0(n)$ occurred (or did not occur) gives us no additional information about the probability distribution of $\lambda^{(n)}$

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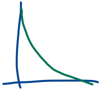
Plancherel growth process „algebraic combinatorics" „discrete" ⇓ Poissonization	 Logan & Shepp Vershik & Kerov 1977 „hook-length formula + + variational calculus"	Vershik & Kerov 1985 „Cauchy-Schwarz inequality"	MMŚ 2020 „bottom rows of P and Q $\stackrel{d}{\approx}$ Poisson"
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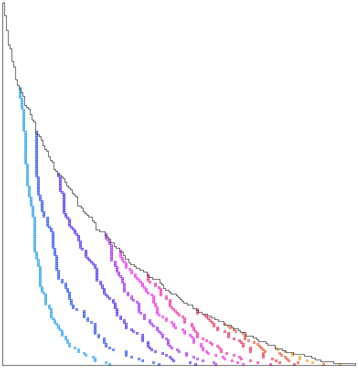
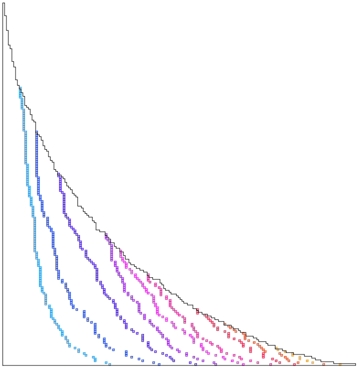
bumping routes



problem ⟶MOORE 2006

what can we say about the shapes of the bumping routes?

bumping routes

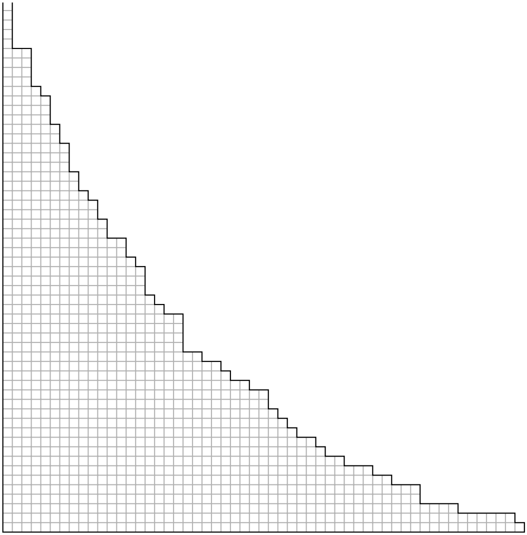


problem $\longrightarrow$ Moore 2006

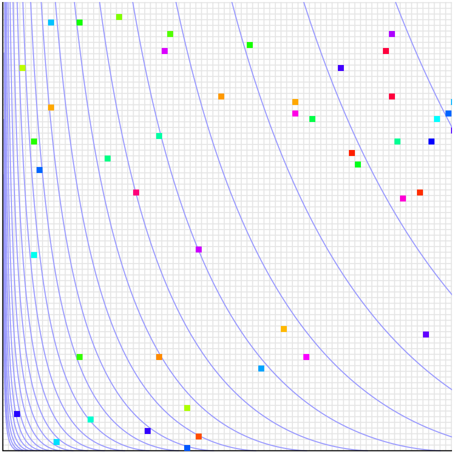
what can we say about the shapes of the bumping routes?

if $w = (w_1, \dots, w_n)$ are i.i.d. uniform $U(0,1)$,
the rescaled bumping route
obtained by adding a new entry w_{n+1}
converges in probability (as $n \to \infty$) to a deterministic curve

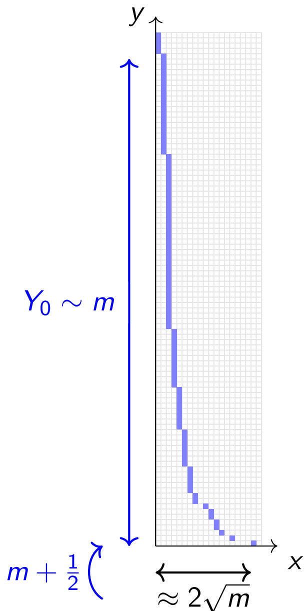
diffusion of a box in the insertion tableau $P(w)$

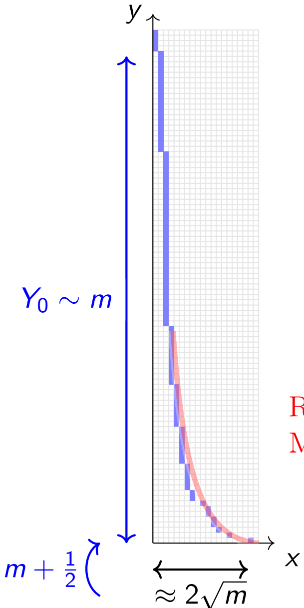


hydrodynamics of the insertion tableau $P(w)$



$m = 1.6 \cdot 10^4$

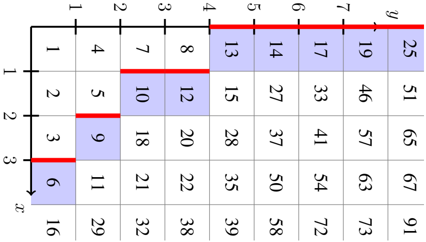

y
$Y_0 \sim m$
$m + \frac{1}{2}$
x
$\approx 2\sqrt{m}$

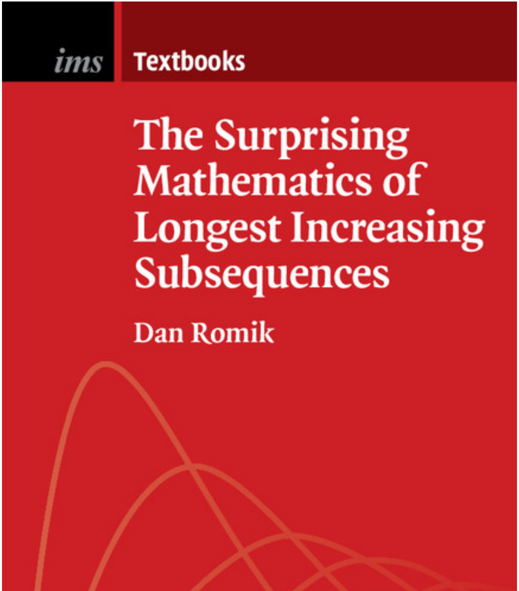


Romik and Śniady 2016
Marciniak 2020

Journée-séminaire de combinatoire CALIN, Laboratoire d'Informatique de Paris Nord
June 16, 2020
14.00 CEST (Paris time)
⟶
`psniady.impan.pl/bumping`

Łukasz Maślanka, Mikołaj Marciniak, Piotr Śniady
Poisson limit of bumping routes in the Robinson–Schensted correspondence
arXiv:2005.14397





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