representations

shape of Young diagrams

characters

Combinatorics of asymptotic representation theory

Piotr Śniady

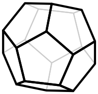
Polish Academy of Sciences
and
University of Wrocław

maps

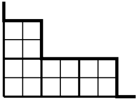
Gaussian fluctuations

open problems

?



representations



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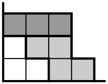

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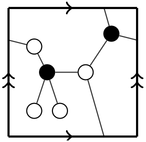
$$\overbrace{\mathrm{Ch}_5}^{\text{character}} = \overbrace{R_6 + 15 R_4 + 5 R_2^2 + 8 R_2}^{\text{shape}}$$

maps



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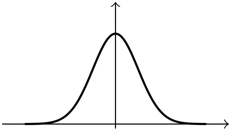

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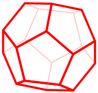

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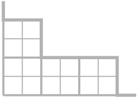

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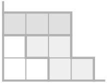

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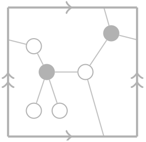

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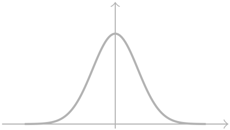

open problems

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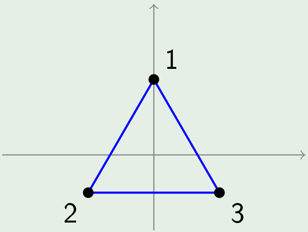
representations 1

representation theory: how an abstract group can be concretely realized as a group of matrices?

Example

symmetric group $\mathfrak{S}(3)$
permutations of $\{1, 2, 3\}$



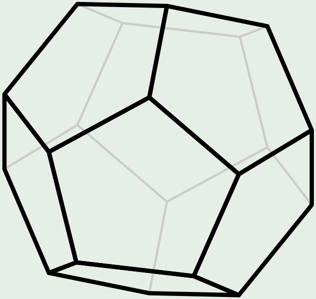
formal definition: representation ρ of a group G is a homomorphism

$$\rho : G \to M_k$$

from the group to invertible matrices

representations 2

Example



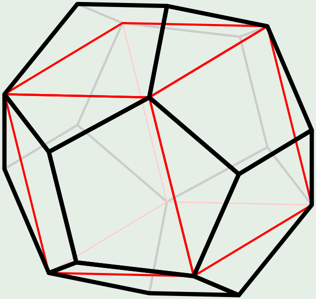
any rotation of the dodecahedron gives
an even permutation of the five cubes,
element of the alternating group $\mathfrak{A}(5)$

this is a bijection

revert the optics:
representation of the alternating group $\mathfrak{A}(5)$

representations 2

Example



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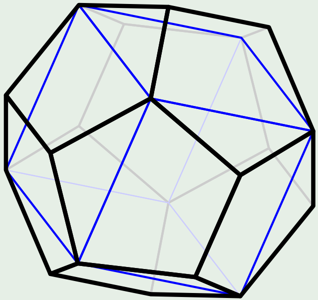

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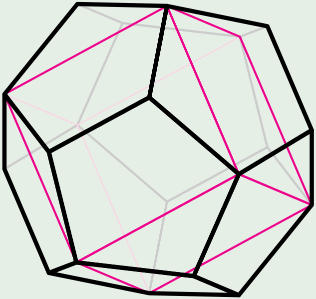

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representations 2

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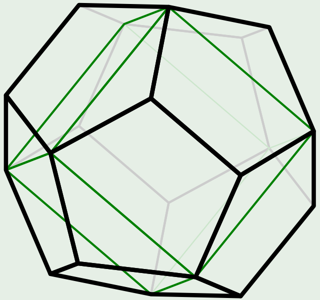

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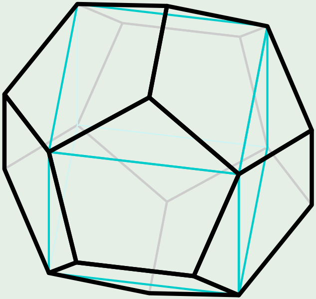

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representations 2

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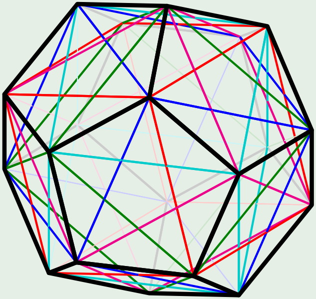

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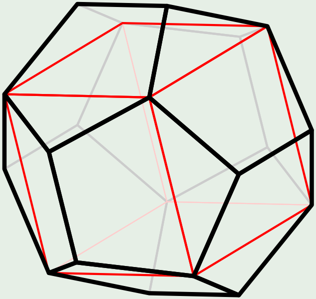

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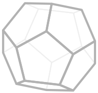

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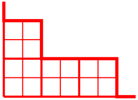
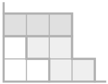


shape of Young diagrams



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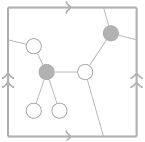

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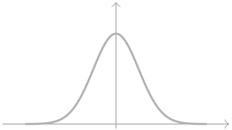

open problems

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irreducible representations

representation ρ is called reducible if can be written as a direct sum of smaller representations:

$$\rho(g) = \begin{bmatrix} \rho_1(g) & \\ & \rho_2(g) \end{bmatrix} \qquad \text{for every } g \in G;$$

we are interested in irreducible representations

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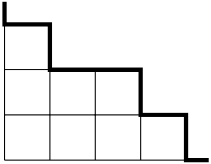
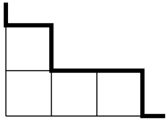
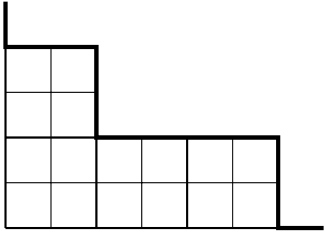
irreducible representation ρ^λ of the symmetric group $\mathfrak{S}(n)$ $\longleftrightarrow$ Young diagram λ with n boxes



shape of Young diagram



Young diagram λ



dilated diagram 2λ

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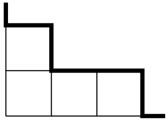
goal for today:

study $\rho^{s\lambda}$ for $s \to \infty$

homogeneous functions



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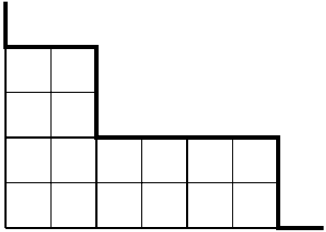

dilated diagram 2λ

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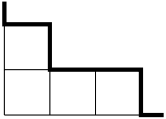
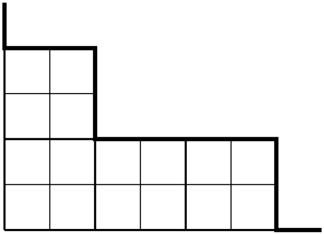


Young diagram λ



dilated diagram 2λ

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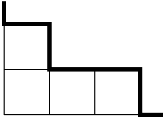
we need *nice* functions on the set of Young diagrams which depend only on shape of λ, not on its size:

$$f(s\lambda) = f(\lambda)$$

homogeneous functions



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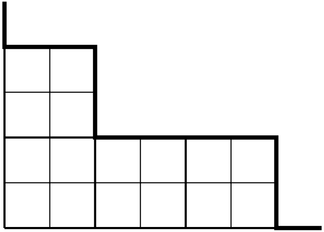

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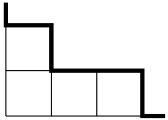

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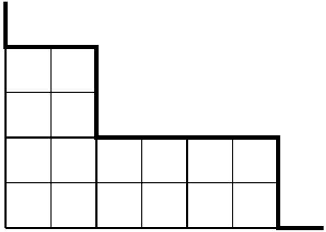

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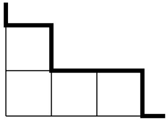

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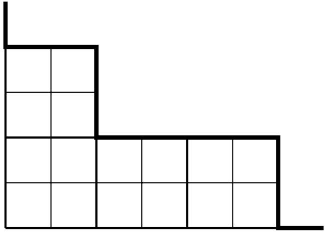

dilated diagram 2λ

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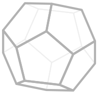
we need *nice* functions on the set of Young diagrams which depend nicely on the size of λ:

$$f(s\lambda) = s^k f(\lambda)$$

homogeneous function of degree k



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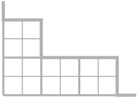
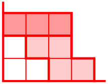


shape of Young diagrams



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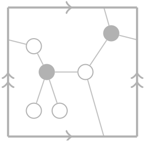

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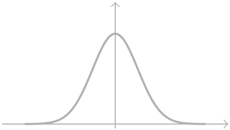


open problems

?

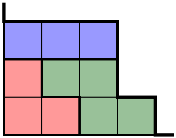
character $\longleftrightarrow$ shape

for irreducible representation

$$\rho^\lambda(\pi) \in M_k \qquad \text{for } \pi \in \mathfrak{S}(n)$$

we define irreducible character

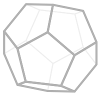
$$\chi^\lambda(\pi) := \operatorname{Tr} \rho^\lambda(\pi) \qquad \text{for } \pi \in \mathfrak{S}(n)$$



classical combinatorics:
Murnaghan-Nakayama rule

$$\pi = (2,7,9)(1,10,8,3)(4,6,5) = 3 \cdot 4 \cdot 3$$

$$\chi^\lambda(\pi) = (-1)^0 \cdot (-1)^1 \cdot (-1)^1 + \cdots$$



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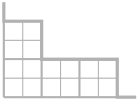

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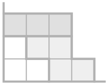

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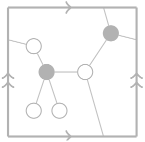

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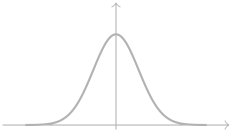

open problems

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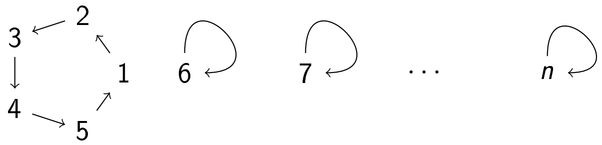
dual combinatorics of the representation theory of $\mathfrak{S}(n)$

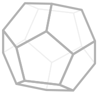
classical combinatorics	dual combinatorics
λ is fixed	conjugacy class is fixed
character $\chi^{\lambda}(\pi)$ — function of π	character $\mathrm{Ch}_k(\lambda)$ — function of λ

$[5] :=$ 

normalized character: → Kerov & Olshanski

$$\mathrm{Ch}_5(\lambda) := \underbrace{n(n-1)\cdots(n-4)}_{5\text{ factors}} \frac{\mathrm{Tr}\,\rho^{\lambda}([5])}{\mathrm{Tr}\,\rho^{\lambda}(e)},$$

n — the number of boxes of λ



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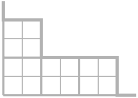

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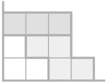

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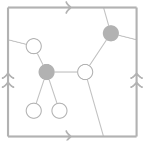

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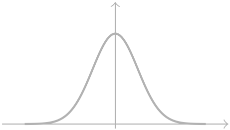

open problems

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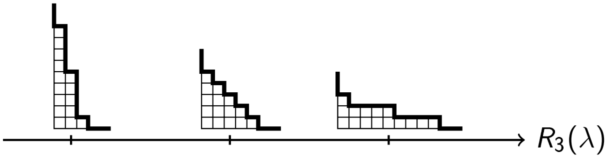
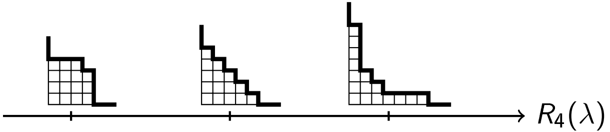
free cumulants $\longleftrightarrow$ shape





$\to$ Biane,
using random matrix theory / Voiculescu's free probability,
Speicher's free cumulants and non-crossing partitions

free cumulants

$s \mapsto \mathrm{Ch}_k(s\lambda)$ is a polynomial of degree $k+1$

free cumulants $R_2(\lambda), R_3(\lambda), \dots$ are top-degree coefficients:

$$R_{k+1}(\lambda) := \lim_{s\to\infty} \frac{1}{s^{k+1}} \mathrm{Ch}_k(s\lambda)$$

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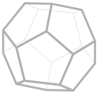
free cumulant R_k is homogeneous with degree k:

$$R_k(s\lambda) = s^k R_k(\lambda)$$

$$R_{k+1} \approx \mathrm{Ch}_k$$



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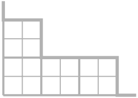
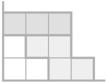


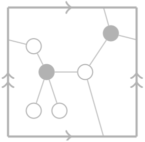
shape of Young diagrams



characters

$$\overbrace{\mathrm{Ch}_5}^{\text{character}} = \overbrace{R_6 + 15R_4 + 5R_2^2 + 8R_2}^{\text{shape}}$$

maps



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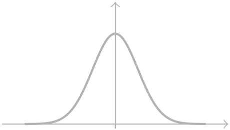


open problems

?

Kerov polynomials

$$\overbrace{\mathrm{Ch}_2}^{\text{character}} = \overbrace{R_3}^{\text{shape}},$$

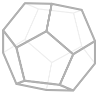
$$\mathrm{Ch}_3 = R_4 + R_2,$$

$$\mathrm{Ch}_4 = R_5 + 5R_3,$$

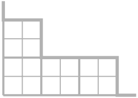
$$\mathrm{Ch}_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2,$$

$$\mathrm{Ch}_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3$$

Kerov positivity conjecture:
the coefficients are non-negative integers;
what is their combinatorial meaning?



representations



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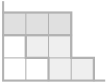

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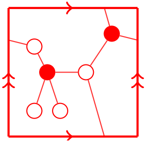

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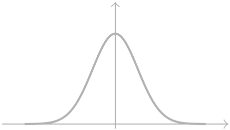


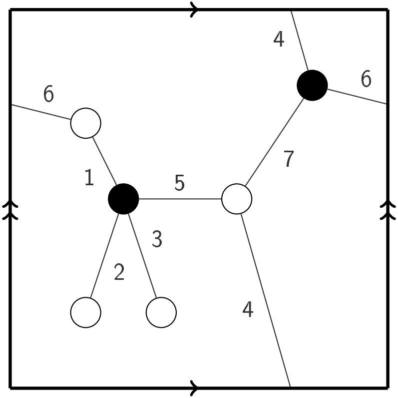
open problems

?

maps

map

- is a graph drawn on an oriented surface,
- bipartite,
- with one face,
- labeled,
- connected



what Kerov polynomials count?

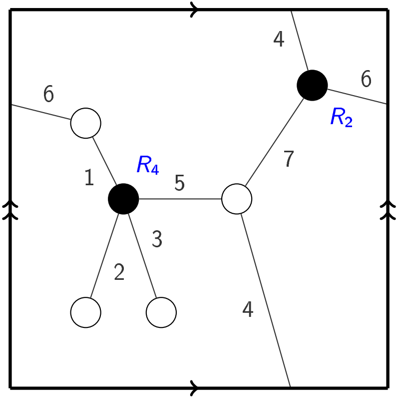
coefficient of $R_{i_1} \cdots R_{i_\ell}$ in Ch_k counts the number of maps with k edges

with black vertices labelled by $R_{i_1}, \dots, R_{i_\ell}$,

each black vertex R_i produces $i-1$ units of liquid,

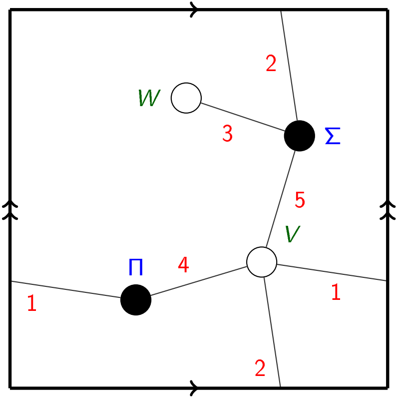
each white vertex demands 1 unit of the liquid,

each edge transports strictly positive amout of liquid from black to white vertex

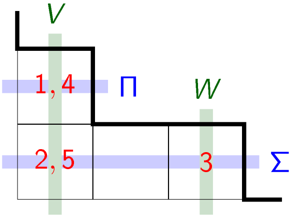


→ Féray, Dołęga & Śniady

embedding of a map to a Young diagram



→Stanley, Féray, Śniady



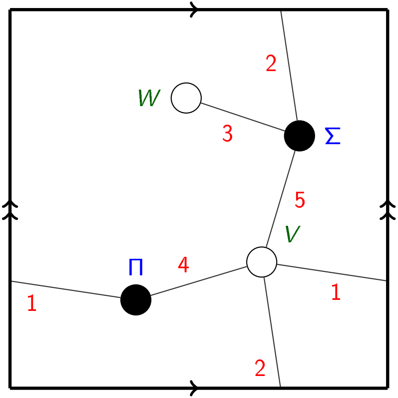
$N_M(\lambda) = \#$ embeddings of M to λ

$N_M(\lambda)$ is a homogeneous function,

$$\deg N_M = k - 1 + \chi(M) = k + 1 - 2\,\text{genus}(M)$$

biggest contribution: planar maps

Stanley's character formula



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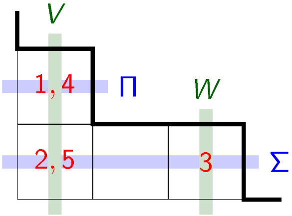

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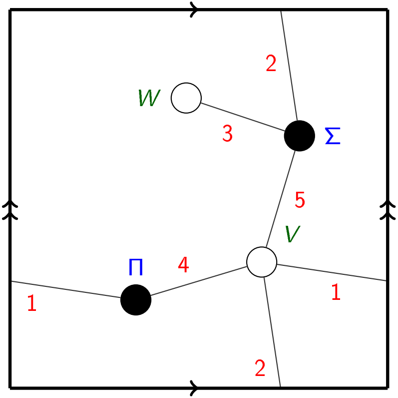

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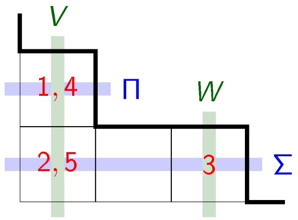
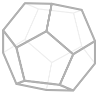


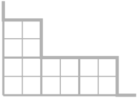
$N_M(\lambda) = \#$ embeddings of M to λ

$$\mathrm{Ch}_k(\lambda) = \sum_M (-1)^{k - \#\text{white vertices}} \, N_M(\lambda),$$

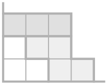
where the sum runs over maps M with k edges



representations



shape of Young diagrams



characters

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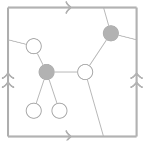
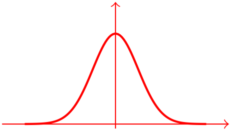
maps



Gaussian fluctuations



open problems

?

characters on two cycles

the normalized character $\mathrm{Ch}_{k,l}(\lambda)$

$$(1,2,\dots,k)(k+1,k+2,\dots,k+l) \in \mathfrak{S}(k+l)$$

Kerov polynomials

$$\mathrm{Ch}_{3,2} = R_3 R_4 - 5R_2 R_3 - 6R_5 - 18R_3$$

not nice!

(abstract) covariance

$$\mathrm{Cov}(\mathrm{Ch}_k, \mathrm{Ch}_l) := \mathrm{Ch}_{k,l} - \mathrm{Ch}_k \mathrm{Ch}_l$$

$$\mathrm{Cov}(\mathrm{Ch}_3, \mathrm{Ch}_2) = -\big(6R_2 R_3 + 6R_5 + 18R_3\big)$$

is nice!

surprising cancellations

$$\mathrm{Ch}_2 = \underbrace{R_3}_{\text{degree } 3},$$

$$\mathrm{Ch}_3 = \underbrace{R_4}_{\text{degree } 4} + R_2,$$

$$\mathrm{Cov}(\mathrm{Ch}_3, \mathrm{Ch}_2) = -\big(6 \underbrace{R_2 R_3}_{\text{degree only } 5} + 6 \underbrace{R_5}_{\text{degree only } 5} + 18 R_3\big)$$

explanation by Kerov polynomials:
$\mathrm{Cov}(\mathrm{Ch}_3, \mathrm{Ch}_2)$ counts connected maps with two cells, such that...

Gaussian fluctuations

(abstract) cumulant

$$k(\mathrm{Ch}_{i_1}, \dots, \mathrm{Ch}_{i_\ell}) = \mathrm{Ch}_{i_1,\dots,i_\ell} - \cdots$$

surprising cancellation:

$$\deg k(\mathrm{Ch}_{i_1}, \dots, \mathrm{Ch}_{i_\ell}) = \deg \mathrm{Ch}_{i_1} + \cdots + \deg \mathrm{Ch}_{i_\ell} - 2(\ell - 1)$$

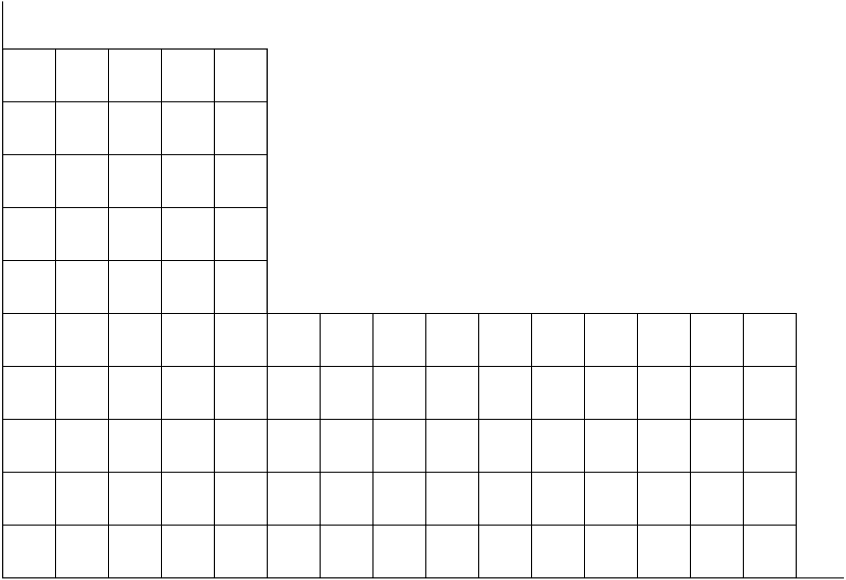
$\mathrm{Ch}_1, \mathrm{Ch}_2, \mathrm{Ch}_3, \dots$ behave asymptotically as (abstract) Gaussian random variables

---

Theorem

for a large class of reducible representations of $\mathfrak{S}(n)$,
if we randomly select an irreducible component ρ^λ, for $n \to \infty$
λ will concentrate around some limit shape →Biane
and the fluctuations are Gaussian →Kerov, Śniady

random Young tableaux 1



random Young tableaux 1

75	81	89	98	100										
58	60	72	94	99										
51	56	62	93	95										
26	38	54	79	92										
18	33	37	59	87										
12	20	35	36	42	46	67	68	70	78	82	84	88	90	97
11	17	19	22	30	43	52	55	64	65	66	74	83	85	96
8	10	13	21	23	29	34	45	47	49	63	71	76	80	91
2	7	9	15	16	24	27	39	41	44	48	57	69	77	86
1	3	4	5	6	14	25	28	31	32	40	50	53	61	73

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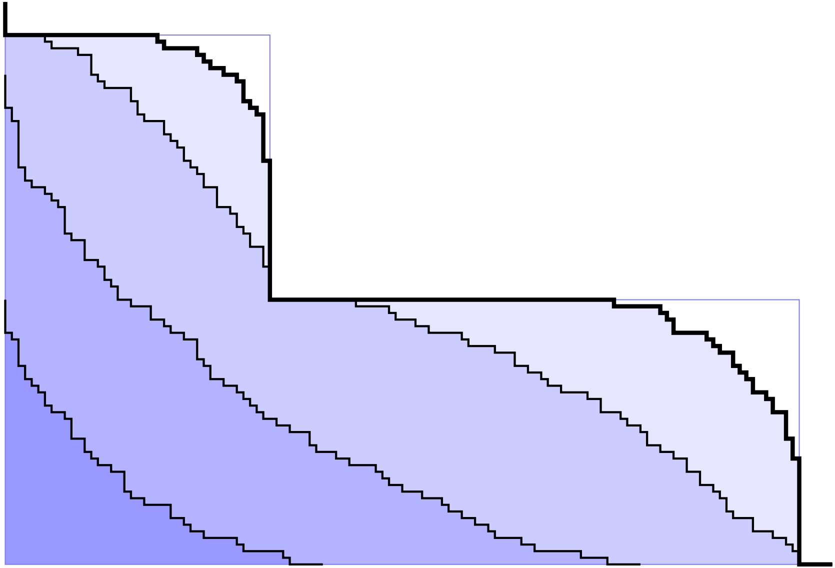
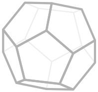
2	7	9	15	16	24	27	39	41	44	48	57	69	77	86
1	3	4	5	6	14	25	28	31	32	40	50	53	61	73

restriction $\rho^{\lambda}\big\downarrow^{\mathfrak{S}(n)}_{\mathfrak{S}(m)}$ to a subgroup

random Young tableaux 2





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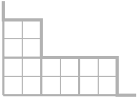
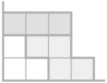


shape of Young diagrams



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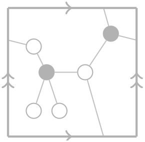

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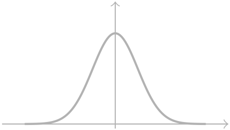


open problems

?

open problems

$$\vdots$$

$$\begin{aligned} \mathrm{Ch}_6 - R_7 &= \frac{35}{4} C_5 + 42 C_3, \\ \mathrm{Ch}_7 - R_8 &= 14 C_6 + \frac{469}{3} C_4 + \frac{203}{3} C_2^2 + 180 C_2. \end{aligned}$$

→GOULDEN & RATTAN

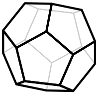
positivity?

$$\vdots$$

$$\begin{aligned} \mathrm{Ch}_3^{(\gamma)} &= R_4 + 3\gamma R_3 + (1 + 2\gamma^2) R_2, \\ \mathrm{Ch}_4^{(\gamma)} &= R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2) R_3 + (7\gamma + 6\gamma^3) R_2, \end{aligned}$$

→LASSALLE

positivity?



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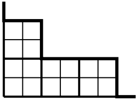

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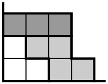

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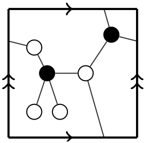

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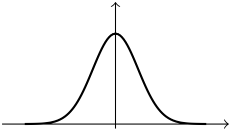

open problems

?

further reading

- Piotr Śniady
  Combinatorics of asymptotic representation theory.
  Proceedings of 6th European Congress of Mathematics
  arXiv:1203.6509

- Valentin Féray, Piotr Śniady
  Asymptotics of characters of symmetric groups related to Stanley character formula.
  Ann. of Math. (2) 173 (2011), no. 2, 887–906

- Maciej Dołęga, Valentin Féray, Piotr Śniady
  Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations.
  Adv. Math. 225 (2010), no. 1, 81–120