Generalized Frobenius formula and asymptotics of characters of symmetric groups

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Outline

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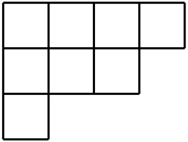
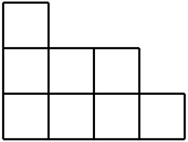
1. Problem: asymptotics of characters of symmetric groups
   - Murnaghan–Nakayama rule
   - Asymptotics of characters
2. Generalized Frobenius formula
3. Upper bounds for characters of symmetric groups

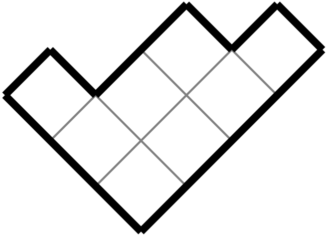
Russian convention for Young diagrams



English convention



French convention

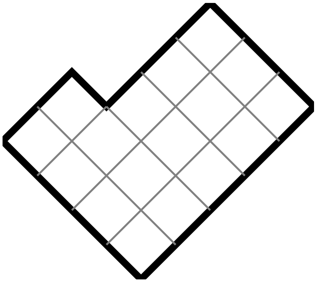


Russian convention

Characters of symmetric groups

Irreducible representations ρ^λ of S_n are indexed by Young diagrams with n boxes. For a given Young diagram λ and permutation $\pi \in S_n$, what is the value of the unnormalized character $\operatorname{Tr} \rho^\lambda(\pi)$?

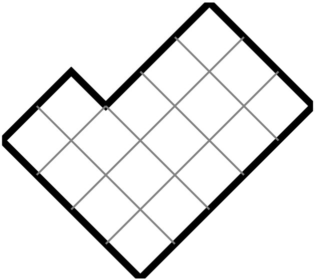
Example:



$$\pi = (1,2,3,4)(5,6,7,8)\times (9,10,11,12)(13,14,15,16,17) = 4^3 5^1$$

Murnaghan–Nakayama rule

Let $l_1, \dots, l_k$ be the lengths of the cycles of π. In order to compute the character $\operatorname{Tr} \rho^\lambda(\pi)$ we need to consider all decompositions of λ into strips of lengths $l_1, \dots, l_k$.
For each strip we get a factor $(-1)^{\text{height}}$...



$\pi = 4^3 5^1$
contribution =
$(-1)^2 \times (-1)^1 \times (-1)^1 \times (-1)^2$

The character is equal to the sum of the contributions over all decompositions.

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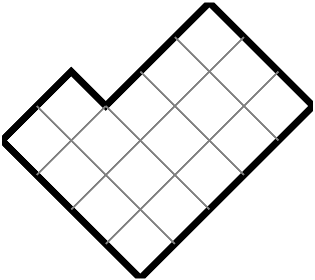

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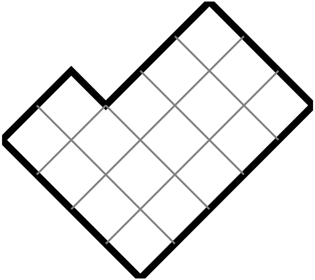

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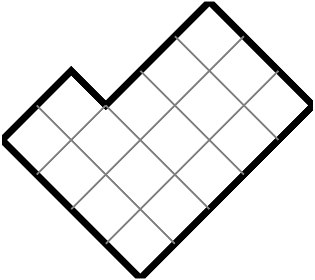

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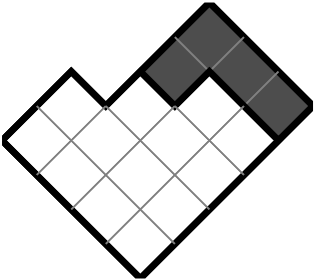

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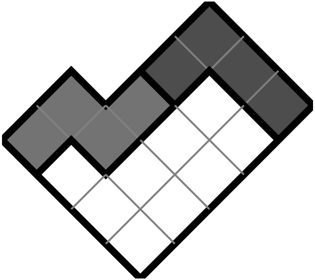

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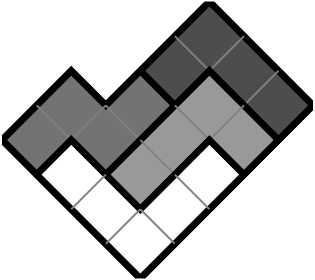

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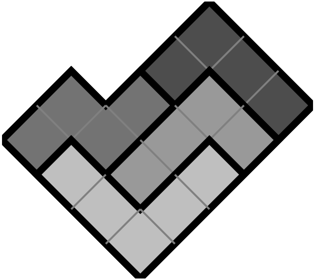

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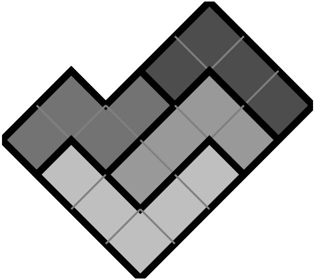

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Asymptotic questions

We would like to study asymptotic questions: how big is the character

$$\chi^\lambda(\pi) = \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\operatorname{Tr} \rho^\lambda(e)}$$

of the symmetric group S_n in the limit as $n \to \infty$. Alternatively, how big are normalized characters

$$\Sigma_{k_1,\dots,k_l} = \frac{\operatorname{Tr} \rho^\lambda(k_1,\dots,k_l,1^{n-k_1-\cdots-k_l})}{\operatorname{Tr} \rho^\lambda(e)} (n)_{k_1+\cdots+k_l},$$

where $(k_1,\dots,k_l,1^{n-k_1-\cdots-k_l})$ is a permutation with a given cycle structure and $(n)_k = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$ denotes the falling power.

Murnaghan–Nakayama rule does not tell us anything useful.

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Balanced Young diagrams

We assume that λ is a balanced diagram, i.e. it has at most $c\sqrt{n}$ rows and columns, where n is the number of boxes.

Kerov, Biane,... proved that for some constant d

$$|\chi^\lambda(\pi)| < \left(\frac{d}{\sqrt{n}}\right)^{|\pi|}$$

if $|\pi|$ is bounded. Is the above inequality true for general $|\pi|$?

Motivations:

- random walks on symmetric group S_n (Diaconis and Shahshahani),
- quantum computations (Moore and Russell).

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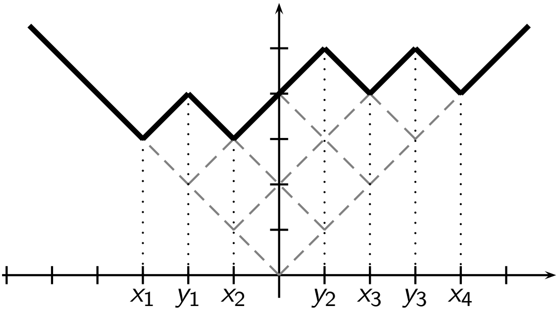
How to encode a Young diagram?



Young diagram can be encoded by the sequences of local minima $(x_1, \dots, x_s)$ and maxima $(y_1, \dots, y_{s-1})$. We define a function

$$H(z) = \frac{(z - x_1) \cdots (z - x_s)}{(z - y_1) \cdots (z - y_{s-1})}.$$

Why is $H(z)$ so nice?

- $H(z)$ is easily determined by the shape of Young diagram λ, good for asymptotic questions;
- $H(z)$ is related to the *transition measure* μ^λ of λ, namely $G(z) = \frac{1}{H(z)}$ is the Cauchy transform of μ^λ;
- the coefficients in the expansion

$$H(z) = z - B_2 z^{-1} - B_3 z^{-2} - \cdots$$

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The usual Frobenius formula

Theorem

$$-k\ \Sigma_k = [z^{-1}]\left[H(z)H(z-1)\cdots H(z-k+1)\right].$$

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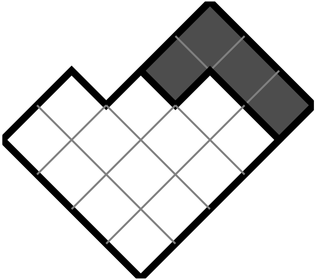


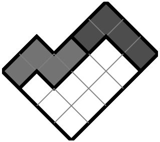
Theorem (Generalized Frobenius formula, simplest case)

$$k_1 k_2 \ \Sigma_{k_1,k_2} =$$

$$[z_1^{-1}][z_2^{-1}] \Bigg[H(z_1) H(z_1 - 1) \cdots H(z_1 - k_1 + 1) \times$$

$$H(z_2) H(z_2 - 1) \cdots H(z_2 - k_2 + 1) \times$$

$$\frac{(z_1 - z_2)(z_1 - z_2 + k_2 - k_1)}{(z_1 - z_2 - k_1)(z_1 - z_2 + k_2)} \Bigg].$$



Theorem (Generalized Frobenius formula)

$$(-1)^l k_1 \cdots k_l \; \Sigma_{k_1,\dots,k_l} =$$

$$[z_1^{-1}] \cdots [z_l^{-1}] \Bigg[\left(\prod_{1 \le r \le l} H(z_r) H(z_r - 1) \cdots H(z_r - k_r + 1) \right) \prod_{1 \le s < t \le l} \frac{(z_s - z_t)(z_s - z_t + k_t - k_s)}{(z_s - z_t - k_s)(z_s - z_t + k_t)} \Bigg].$$

Main advantage: direct expression for characters in terms of Boolean cumulants.

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Shifted Boolean cumulants

Let us fix some constant ζ. The coefficients of the expansion

$$H(z+\zeta) = z + \zeta + \tilde{B}_1 + \tilde{B}_2 z^{-1} + \tilde{B}_3 z^{-2} + \cdots$$

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Positivity of character polynomials

Theorem

Let integers $1 \leq k_1, \dots, k_l \leq \zeta$ *be given.*
Then the normalized character $(-1)^l \Sigma_{k_1,\dots,k_l}$ *is a polynomial in shifted Boolean cumulants* $\tilde{B}_2, \tilde{B}_3, \dots$ *with non-negative coefficients.*

Looks like Kerov conjecture for *free* cumulants.

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Upper bounds for characters

Corollary

If λ and ν are Young diagrams such that $|\tilde{B}_i^\lambda| < \tilde{B}_i^\nu$ then

$$|\Sigma^\lambda_{k_1,\dots,k_l}| < |\Sigma^\nu_{k_1,\dots,k_l}|.$$

Now if we want to prove upper bounds for characters it is enough to prove them for some nice Young diagram ν.

For example, for ν we may take rectangular Young diagrams for which characters were calculated by Stanley.

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The main inequality

Theorem

For every c there exists a constant d such that if a Young diagram with n boxes has at most $c\sqrt{n}$ rows and columns then

$$|\chi^\lambda(\pi)| < \left(\frac{d}{\sqrt{n}}\right)^{|\pi|}.$$

This talk was about an application of power series to representation theory. More such applications are around!

Amarpreet Rattan, Piotr Śniady.
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