Robinson-**S**chensted-**K**nuth algorithm
Start with two empty tableaux. Read letters of the word one after another. With each letter proceed as follows:

1. start with the bottom row of the *insertion tableau P*,
2. insert the letter to the leftmost box in this row which contains a number which is **bigger** than the one which you want to insert,
3. if you had to bump some letter, this bumped letter must be inserted in to the next row according to the rule number 2,
4. if you inserted a letter to an empty box in the *insertion tableau P,* make a mark about the position of this box in the *recording tableau Q* and proceed to the next letter of the word.

74	99		
23	53	70	
16	37	41	82

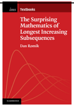
insertion tableau $P(w)$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$$

Further reading



Dan Romik
„*The Surprising Mathematics of Longest Increasing Subsequences*"

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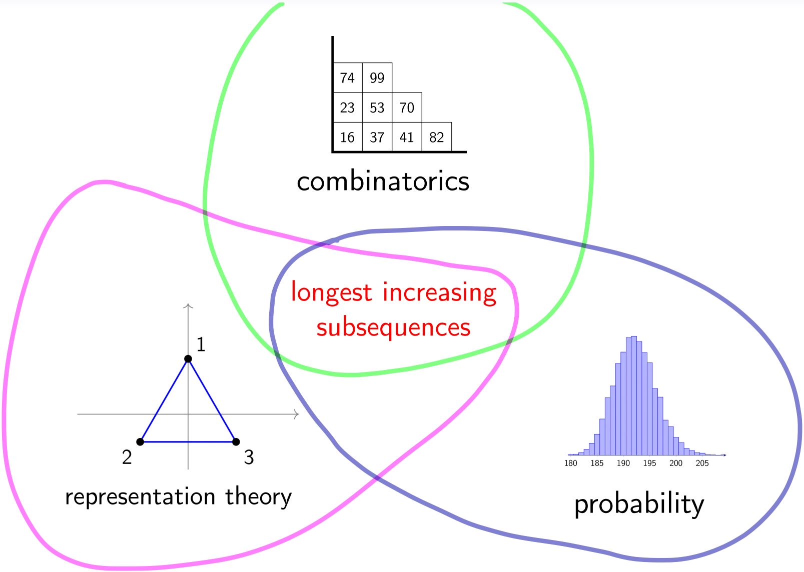
surprising mathematics of longest increasing subsequences

Piotr Śniady

IMPAN Toruń

joint work with Mikołaj Marciniak and Łukasz Maślanka

handout, slides $\longrightarrow$ `psniady.impan.pl/surprising`


74 99
23 53 70
16 37 41 82
combinatorics
longest increasing subsequences
1
2
3
representation theory
180 185 190 195 200 205
probability

Longest Increasing Subsequence

23, 53, 74, 16, 99, 70, 82, 37, 41

what is the length of the longest increasing subsequence?

Longest Increasing Subsequence

23, 53, 74, 16, 99, 70, 82, 37, 41

what is the length of the longest increasing subsequence?

Longest Increasing Subsequence

23, 53, 74, 16, 99, 70, 82, 37, 41

what is the length of the longest increasing subsequence?

Longest Increasing Subsequence

23, 53, 74, 16, 99, 70, 82, 37, 41

what is the length of the longest increasing subsequence?

$$\mathrm{LIS}(23, 53, 74, 16, 99, 70, 82, 37, 41) = 4$$

Longest Increasing Subsequence

23, 53, 74, 16, 99, 70, 82, 37, 41

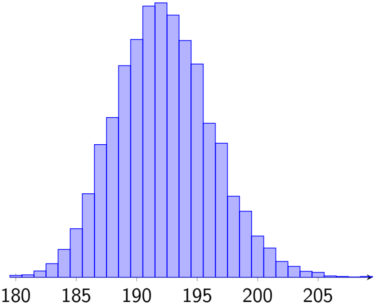
what is the length of the longest increasing subsequence?

$$\mathrm{LIS}(23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41) = 4$$

Stanisław Ulam:

let π_n be a uniformly random permutation of the letters $1, 2, \ldots, n$

what can you say about the random variable $\mathrm{LIS}_n = \mathrm{LIS}(\pi_n)$ in the limit $n \to \infty$?



$n = 10^4$

Robinson-**S**chensted-**K**nuth algorithm
Start with two empty tableaux. Read letters of the word one after another. With each letter proceed as follows:

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74	99		
23	53	70	
16	37	41	82

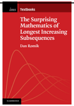
insertion tableau $P(w)$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$$

Further reading



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Robinson–Schensted–Knuth algorithm is a bijection. . .

input:

- sequence $\mathbf{w} = (w_1, \dots, w_n)$

output:

- semistandard tableau P,
- standard tableau Q,

P and Q have the same shape with n boxes

---

example:

$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41)$

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

Robinson–Schensted–Knuth algorith — the induction step

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathrm{w})$

8	9		
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$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

Robinson–Schensted–Knuth algorith — the induction step

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathrm{w})$

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Robinson–Schensted–Knuth algorith — the induction step

74	99		
23	53	70	
16	37	41	82

18

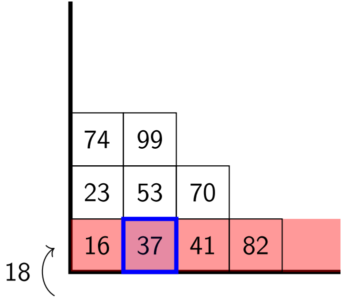
insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

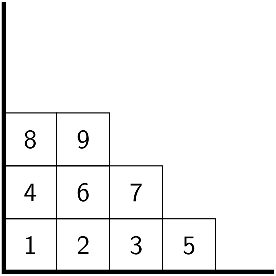
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



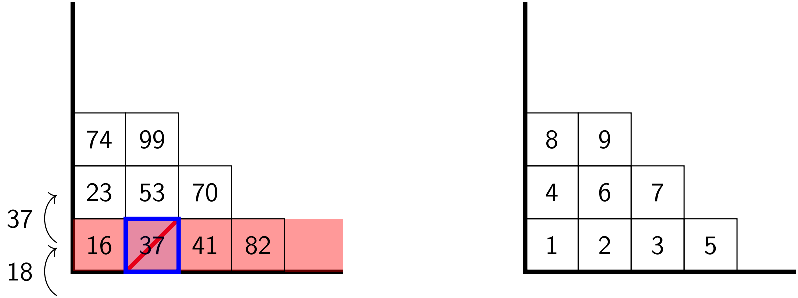
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

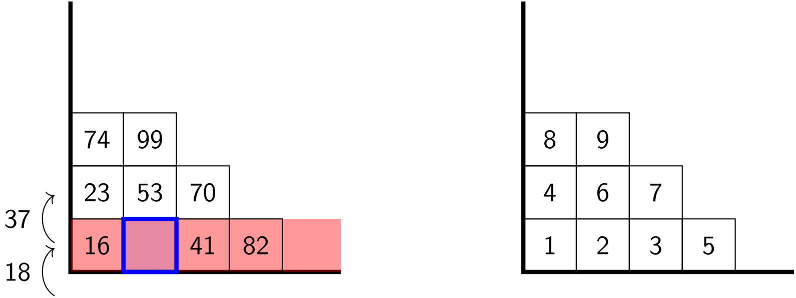
Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$

recording tableau $Q(\mathrm{w})$

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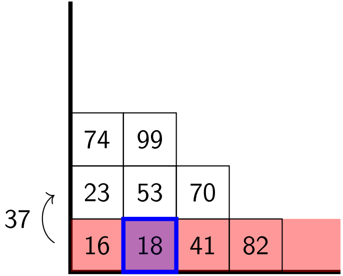
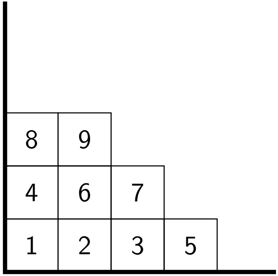
Robinson–Schensted–Knuth algorith — the induction step



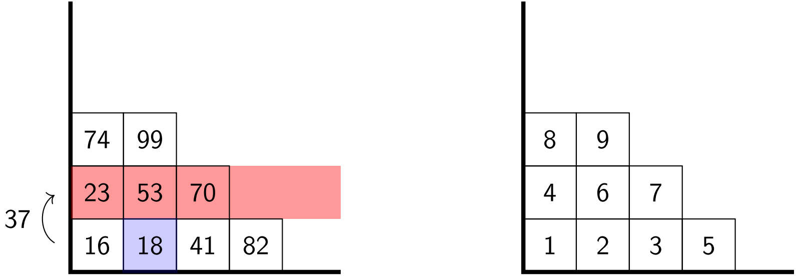
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

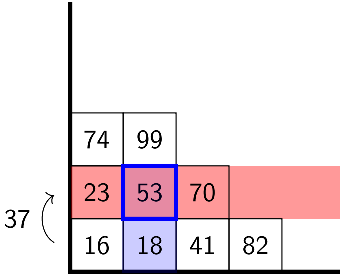
Robinson–Schensted–Knuth algorith — the induction step



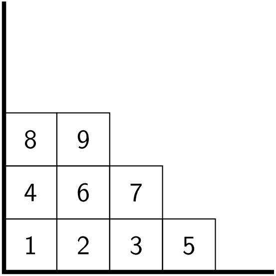
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$

Robinson–Schensted–Knuth algorith — the induction step



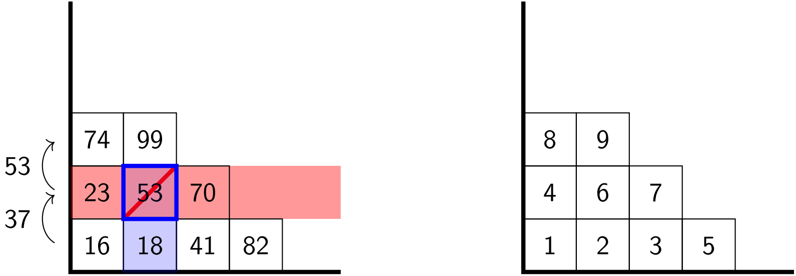
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$

Robinson–Schensted–Knuth algorith — the induction step

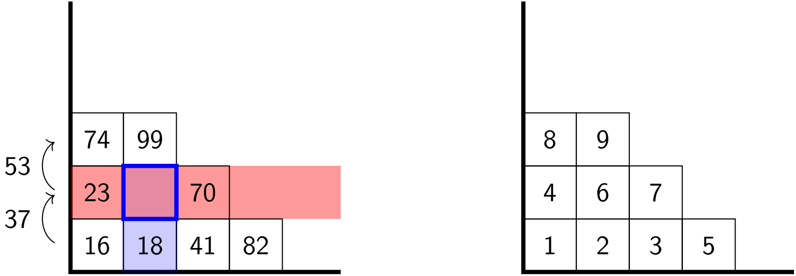


insertion tableau $P(\mathrm{w})$

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step

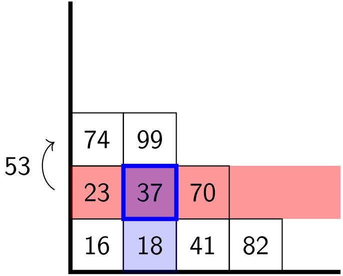


insertion tableau $P(\mathrm{w})$

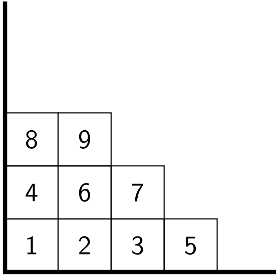
recording tableau $Q(\mathrm{w})$

$$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step

53

74	99		
23	37	70	
16	18	41	82

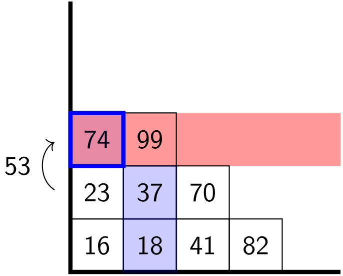
insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

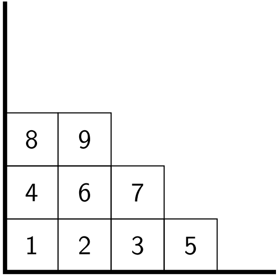
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$



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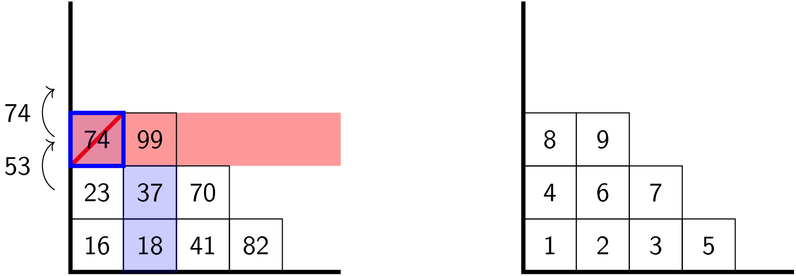
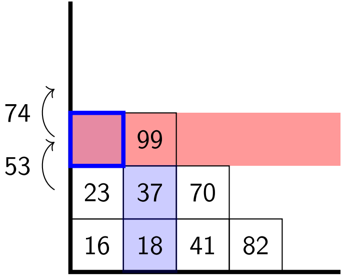
Robinson–Schensted–Knuth algorith — the induction step



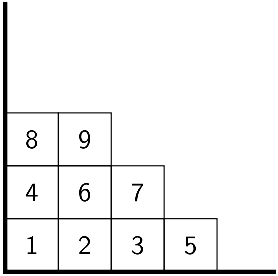
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



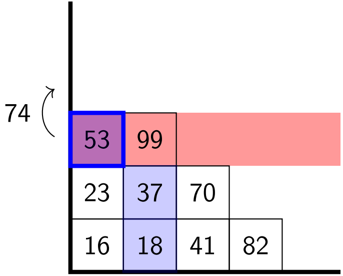
insertion tableau $P(\mathrm{w})$



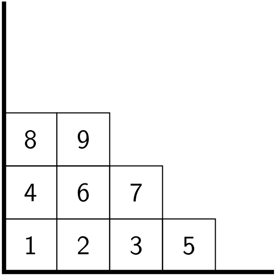
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$

Robinson–Schensted–Knuth algorith — the induction step

74

53	99		
23	37	70	
16	18	41	82

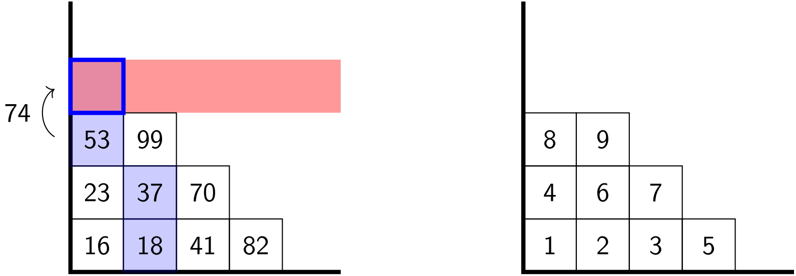
insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

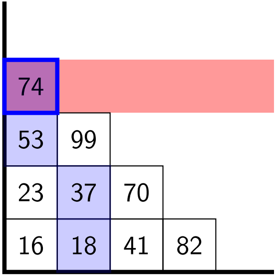
Robinson–Schensted–Knuth algorith — the induction step



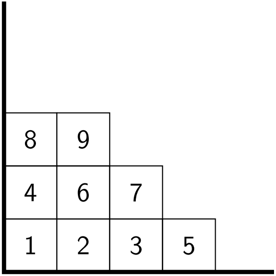
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 18)$

Robinson–Schensted–Knuth algorith — the induction step

74			
53	99		
23	37	70	
16	18	41	82

insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step

74			
53	99		
23	37	70	
16	18	41	82

insertion tableau $P(\mathrm{w})$

new box

10			
8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step

74			
53	99		
23	37	70	
16	18	41	82

insertion tableau $P(\mathrm{w})$

10			
8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

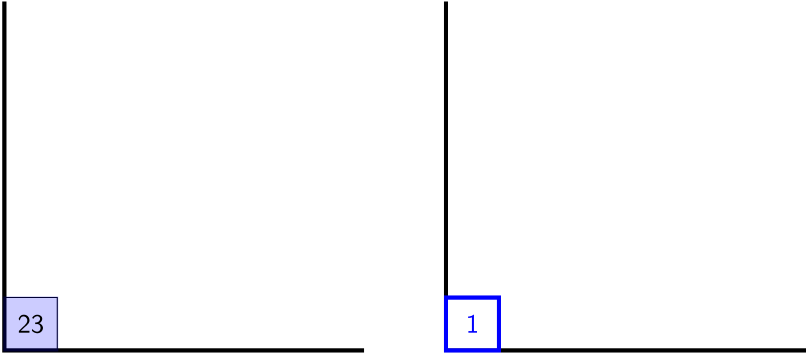
Robinson–Schensted–Knuth algorithm

insertion tableau $P(\mathrm{w})$

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = \emptyset$

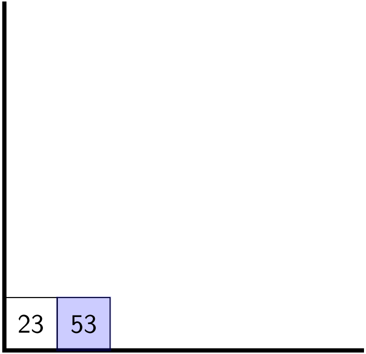
Robinson–Schensted–Knuth algorithm



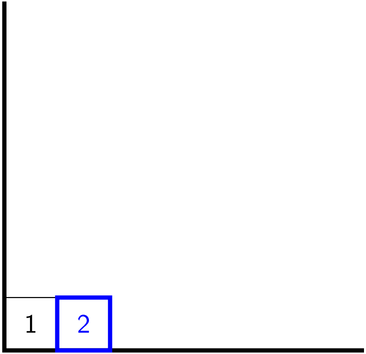
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23)$

Robinson–Schensted–Knuth algorithm



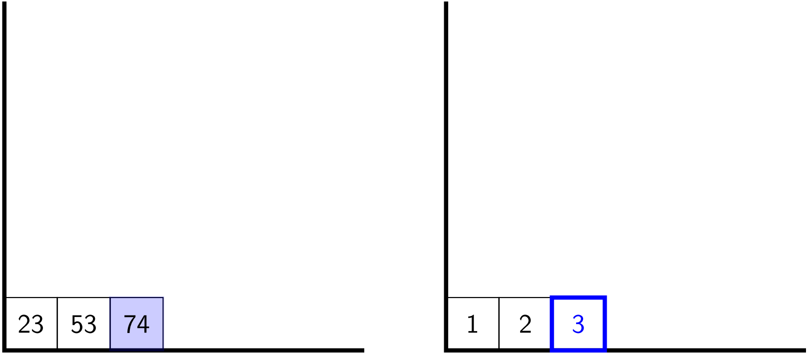
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53)$

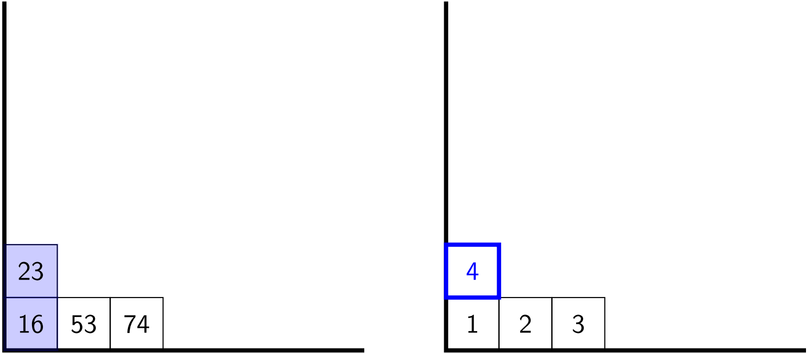
Robinson–Schensted–Knuth algorithm



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74)$

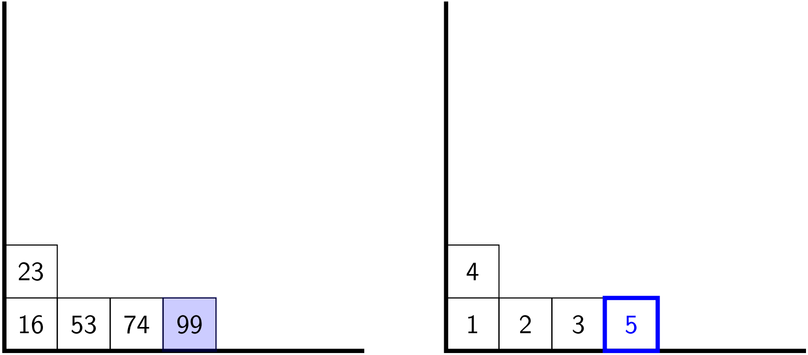
Robinson–Schensted–Knuth algorithm



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16)$

Robinson–Schensted–Knuth algorithm



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99)$

Robinson–Schensted–Knuth algorithm

23	74		
16	53	70	99

insertion tableau $P(\mathrm{w})$

4	6		
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70)$

Robinson–Schensted–Knuth algorithm

23	74	99	
16	53	70	82

insertion tableau $P(\mathrm{w})$

4	6	$\boxed{7}$	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82)$

Robinson–Schensted–Knuth algorithm

74			
23	53	99	
16	37	70	82

insertion tableau $P(\mathrm{w})$

8			
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37)$

Robinson–Schensted–Knuth algorithm

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathrm{w})$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

Robinson–Schensted–Knuth algorithm

74			
53	99		
23	37	70	
16	34	41	82

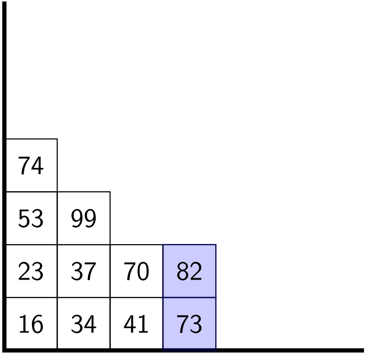
insertion tableau $P(\mathrm{w})$

$\boxed{10}$			
8	9		
4	6	7	
1	2	3	5

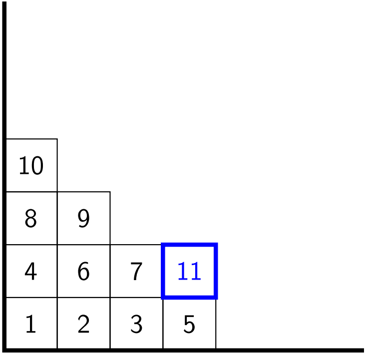
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34)$

Robinson–Schensted–Knuth algorithm



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73)$

Robinson–Schensted–Knuth algorithm

74			
53			
23	99		
16	37	70	82
2	34	41	73

insertion tableau $P(\mathrm{w})$

12			
10			
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2)$

Robinson–Schensted–Knuth algorithm

74			
53	99		
23	37		
16	34	70	82
2	24	41	73

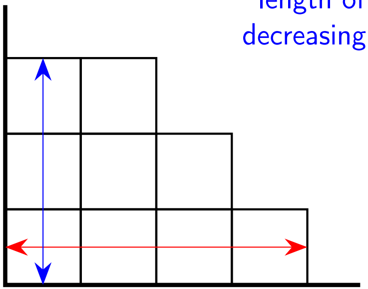
insertion tableau $P(\mathrm{w})$

12			
10	$\boxed{13}$		
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathrm{w})$

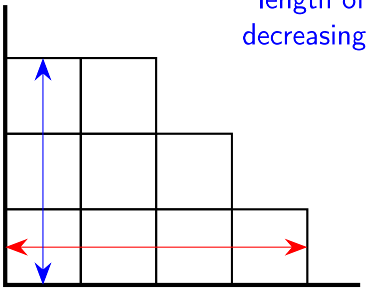
$\mathrm{w} = (23,\ 53,\ 74,\ 16,\ 99,\ 70,\ 82,\ 37,\ 41,\ 34,\ 73,\ 2,\ 24)$

length of the first column
=
length of the longest
decreasing subsequence



length of the first row
=
length of the longest
increasing subsequence

length of the first column
=
length of the longest
decreasing subsequence



length of the first row
=
length of the longest
increasing subsequence

for which funny question
concerning increasing subsequences
the answer is:

"the total length of the first two rows" ?

Robinson–Schensted–Knuth algorithm is a bijection...

input:

- sequence
  $\mathbf{w} = (w_1, \ldots, w_n)$

output:

- semistandard tableau P,
- standard tableau Q,

P and Q have the same shape with n boxes

---

example:
w = (23, 53, 74, 16, 99, 70, 82, 37, 41)

74	99		
23	53	70	
16	37	41	82

insertion tableau P(w)

8	9		
4	6	7	
1	2	3	5

recording tableau Q(w)

Robinson–Schensted–Knuth algorithm is a bijection...

input:

- permutation
  $\mathbf{w} = (w_1, \ldots, w_n)$
  of the letters $1, \ldots, n$

output:

- standard tableau P,
- standard tableau Q,

P and Q have the same shape with n boxes

---

example:

w = (2, 5, 7, 1, 9, 6, 8, 3, 4)

7	9		
2	5	6	
1	3	4	8

insertion tableau P(w)

8	9		
4	6	7	
1	2	3	5

recording tableau Q(w)

(discrete) Fourier transform

instead of a studying a function on the real line $x \mapsto f(x)$
it is better to study its Fourier transform

$$\lambda \mapsto \int e^{i\lambda x} \, f(x) \, dx$$

---

instead of studying a sequence $(a_1, a_2, \dots, a_n)$
it is better to study its discrete Fourier transform

$$k \mapsto \sum_{1 \le m \le n} e^{2\pi i \frac{km}{n}} \, a_m$$

---

how to define Fourier transform on a non-commutative group?

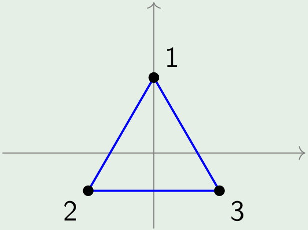
$\longrightarrow$representation theory

representations 1

representation theory: how an abstract group can be concretely realized as a group of matrices?

Example

symmetric group $\mathfrak{S}(3)$
permutations of $\{1, 2, 3\}$



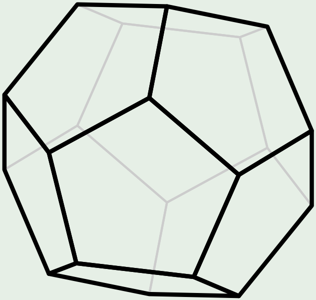
formal definition: representation ρ of a group G is a homomorphism

$$\rho : G \to M_k$$

from the group to invertible matrices

representations 2

Example



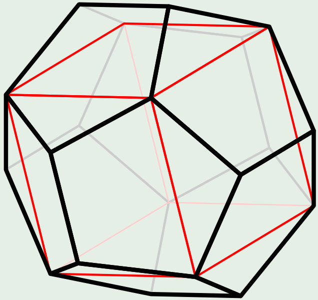
any rotation of the dodecahedron gives
an even permutation of the five cubes,
element of the alternating group $\mathfrak{A}(5)$

this is a bijection

revert the optics:
representation of the alternating group $\mathfrak{A}(5)$

representations 2

Example



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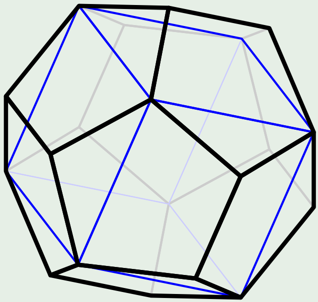
representations 2

Example



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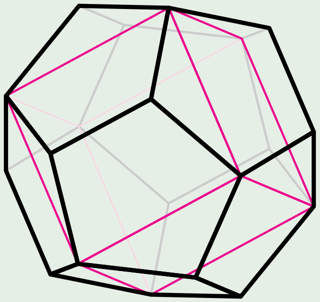
revert the optics: representation of the alternating group $\mathfrak{A}(5)$

representations 2

Example



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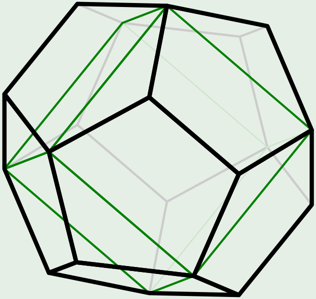
this is a bijection

revert the optics: representation of the alternating group $\mathfrak{A}(5)$

representations 2

Example



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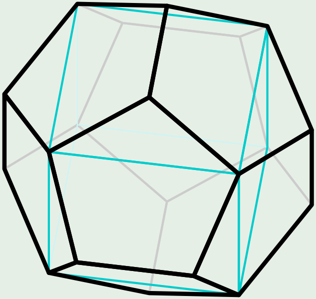
revert the optics: representation of the alternating group $\mathfrak{A}(5)$

representations 2

Example



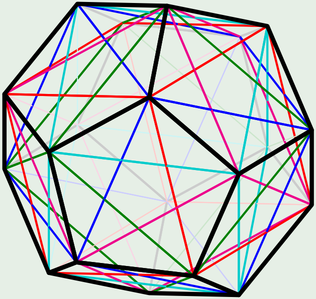
any rotation of the dodecahedron gives an even permutation of the five cubes, element of the alternating group $\mathfrak{A}(5)$

this is a bijection

revert the optics: representation of the alternating group $\mathfrak{A}(5)$

representations 2

Example



any rotation of the dodecahedron gives
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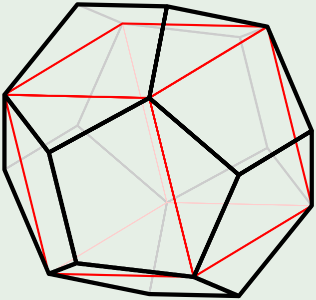
this is a bijection

revert the optics:
representation of the alternating group $\mathfrak{A}(5)$

representations 2

Example

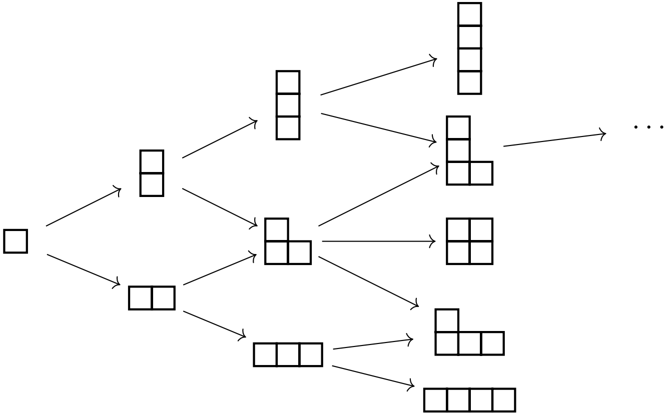


any rotation of the dodecahedron gives an even permutation of the five cubes, element of the alternating group $\mathfrak{A}(5)$

this is a bijection

revert the optics: representation of the alternating group $\mathfrak{A}(5)$

irreducible representations of the symmetric groups $\mathfrak{S}(1) \subset \mathfrak{S}(2) \subset \mathfrak{S}(3) \subset \cdots$



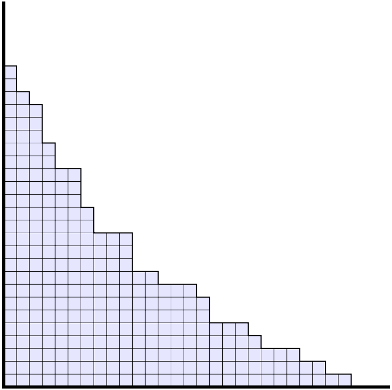
representation theory $\longleftrightarrow$ combinatorics

today: Markov chain

Ulam's problem, on steroids

π_n be a uniformly random permutation of $1, 2, \ldots, n$;

what can you say about the shape of tableaux $P(\pi_n)$ and $Q(\pi_n)$ in the limit $n \to \infty$?



Ulam's problem, on steroids

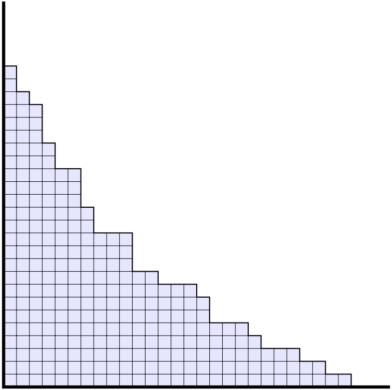
π_n be a uniformly random permutation of $1, 2, \ldots, n$;

what can you say about the shape of tableaux $P(\pi_n)$ and $Q(\pi_n)$ in the limit $n \to \infty$?

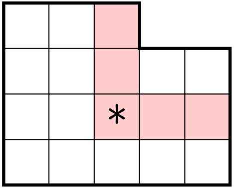
if λ is a diagram with n boxes, its probability is equal to

$$\mathbb{P}(\lambda) = \frac{f^\lambda \times f^\lambda}{n!},$$

where f^λ is the number of standard Young tableaux of shape λ



hook-length formula



$h_* = 5$

if λ is a diagram with n boxes, the number of standard Young tableaux of this shape is equal to

$$f^\lambda = \frac{n!}{\prod_{\Box \in \lambda} h_\Box}$$

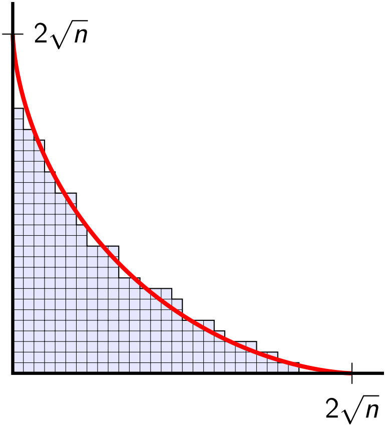
Further steps:

- logarithm changes a product to a sum,
- the sum can be approximated by a (double) integral,
- variational calculus is your friend,

Ulam's problem, on steroids

π_n be a uniformly random permutation of $1, 2, \ldots, n$;

what can you say about the shape of tableaux $P(\pi_n)$ and $Q(\pi_n)$ in the limit $n \to \infty$?

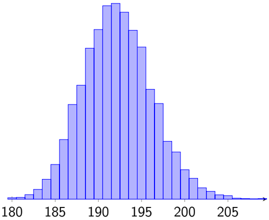


yes, there exists a limit shape!
Logan&Shepp,
Vershik&Kerov 1977

Corollary:

$$\lim_{n \to \infty} \frac{\mathbb{E}\,\mathrm{LIS}_n}{\sqrt{n}} = 2$$

Ulam:
what is the limit distribution of Longest Increasing Subsequence?



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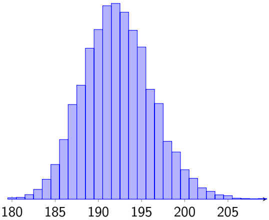
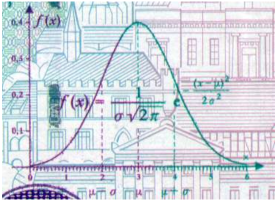
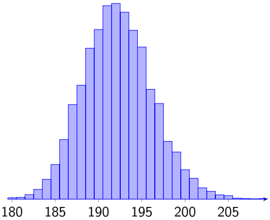
what is the limit distribution of Longest Increasing Subsequence?



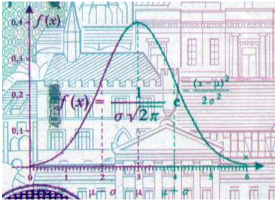
Gauss?



Ulam:
what is the limit distribution of Longest Increasing Subsequence?

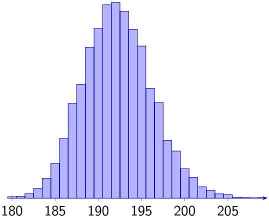


Gauss?

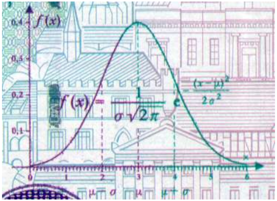


NO!

Ulam:
what is the limit distribution of Longest Increasing Subsequence?

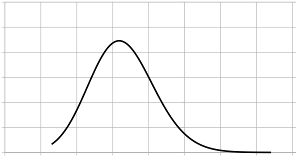


Gauss?

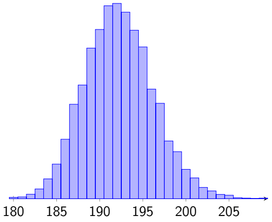


NO!

surprise:
this is
Tracy–Widom distribution



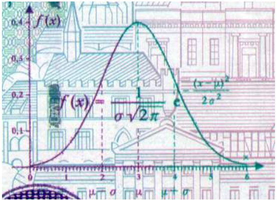
Ulam:
what is the limit distribution of Longest Increasing Subsequence?



$\mathbb{E}\,\mathrm{LIS}_n \approx 2\ n^{1/2}$

$\mathrm{Var}\,\mathrm{LIS}_n \sim n^{1/3} \ll n^{1/2}$

Gauss?



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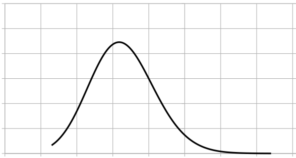
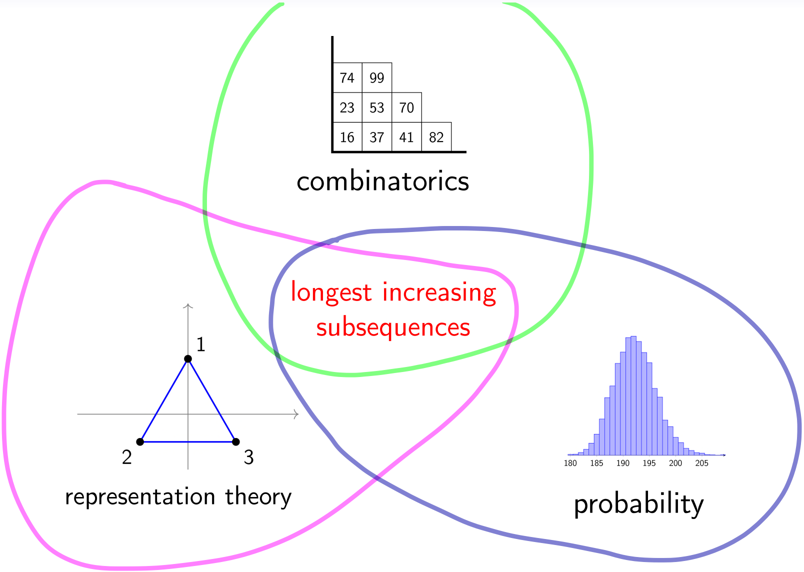


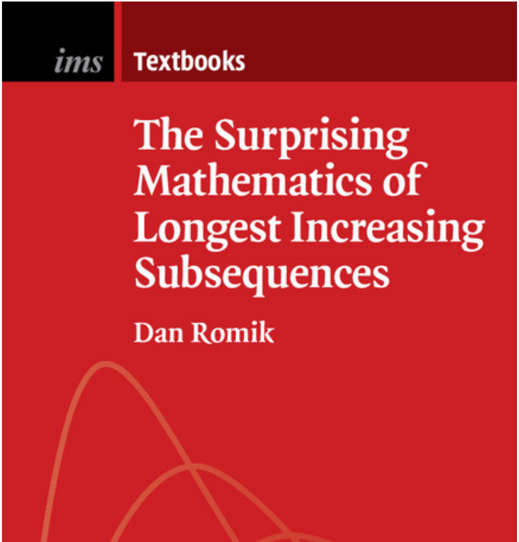
universal, the most important probability distribution

the following random variables
(after shift and rescaling)
have the same distribution
(=Tracy–Widom distribution):

- the length of the longest increasing subsequence LIS(π_n)
  in a random permutation $1, \dots, n$, dla $n \to \infty$
  BAIK, DEIFT, JOHANSSON
  OKOUNKOV
- the largest eigenvalue
  of the most beautiful hermitian random matrix $n \times n$
  for $n \to \infty$,
- e.x. growing interface between turbulences in a liquid cristal
  TAKEUCHI, SANO, SASAMOTO, SPOHN

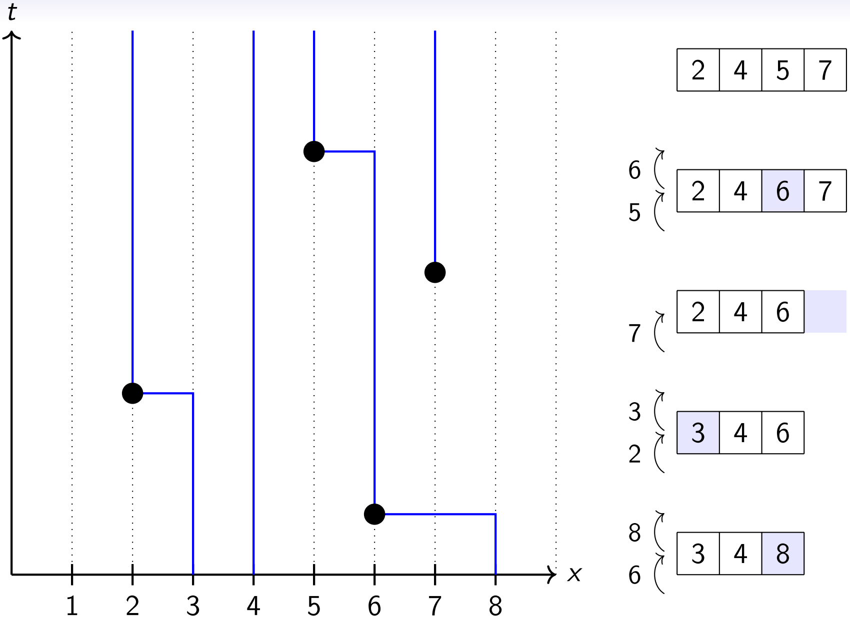

74 99
23 53 70
16 37 41 82
combinatorics
longest increasing subsequences
1
2
3
representation theory
180 185 190 195 200 205
probability

product placement 1



legal PDF file available for free on the author's website

today:
new improved version of a theorem of ALDOUS and DIACONIS about $\longrightarrow$ Hammersley process


t
x
1
2
3
4
5
6
7
8
2 4 5 7
6
5
2 4 6 7
7
2 4 6
3
2
3 4 6
8
6
3 4 8

Hammersley process

sample black points in $[0,1] \times \mathbb{R}_+$
by Poisson point process with unit intensity

let $x_1(t), x_2(t), \dots$ be positions of the particles at time t

Theorem (Aldous, Diaconis 1995; version about P)

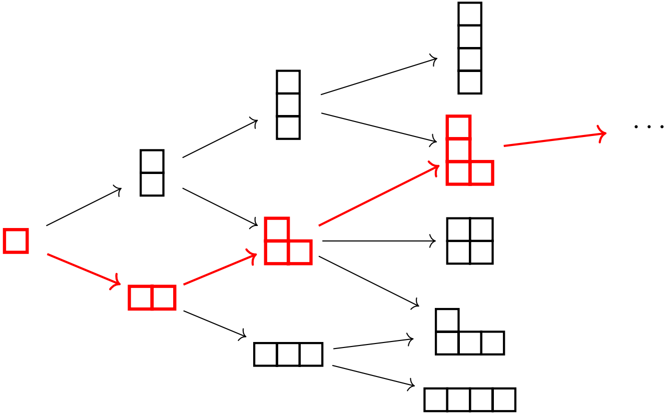
for any $0 < w < 1$ the random set

$$\left\{ \sqrt{t}\, \big(x_i(t) - w\big) \ : i = 1, 2, \dots \right\}$$

converges in distribution to Poisson point process with intensity $\frac{1}{\sqrt{w}}$ in the limit $t \to \infty$

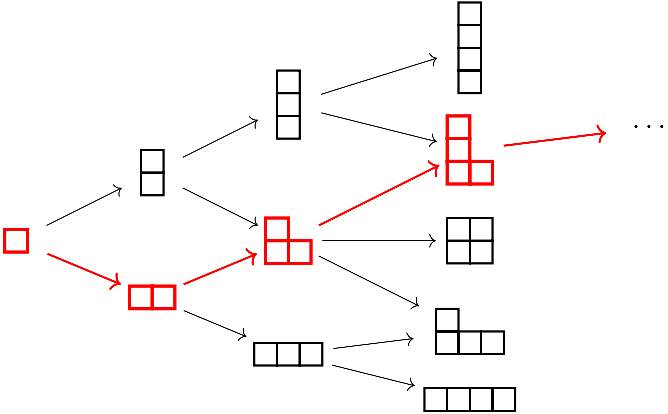
a result about the bottom row of the insertion tableau
after $\approx t$ steps of RSK
applied to independent random variables
with the uniform distribution $U(0,1)$

Plancherel growth process $\lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \cdots$



let $(\pi_1, \ldots, \pi_k)$ be a uniformly random permutation of $1, \ldots, k$;
define $\lambda^{(n)} = \mathrm{RSK}(\pi_1, \ldots, \pi_n)$ to be the common shape of the insertion and recording tableau related to the prefix of π

Plancherel growth process $\lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \cdots$



let $(\xi_1, \xi_2, \dots)$ be i.i.d. $U(0,1)$ random variables from $[0,1]$
define $\lambda^{(n)} = \mathrm{RSK}(\xi_1, \dots, \xi_n)$ to be the common shape of the insertion and recording tableau related to the prefix of ξ

growth of LIS, growth of the bottom row

Theorem (Aldous, Diaconis 1995; version about Q)

the random function

$$\mathbb{R}_+ \ni t \mapsto \lambda_1^{\left(n+\lfloor t\sqrt{n}\rfloor\right)} - \lambda_1^{(n)}$$

converges in distribution to Poisson process

$$\mathbb{R}_+ \ni t \mapsto N_t$$

as $n \to \infty$

Maślanka, Marciniak, Śniady 2020:
extension to more than one row

proof inspired by Vershik and Kerov 1985

Plancherel growth process: probability distribution for fixed time

for any diagram μ with n boxes

$$\mathbb{P}\left[\lambda^{(n)} = \mu\right] = \frac{f^{\mu} \times f^{\mu}}{n!}, \qquad \text{"Plancherel measure of order } n\text{"}$$

where f^{μ} is the number of standard Young tableaux with shape μ

Hint: use RSK bijection; arbitrary P and Q with shape μ

Plancherel growth process: probability distribution of $\left(\lambda^{(n-1)}, \lambda^{(n)}\right)$

for any diagram μ with $n-1$ boxes
and any diagram ν with n boxes
such that $\mu \nearrow \nu$

$$\mathbb{P}\left[\lambda^{(n-1)} = \mu \text{ and } \lambda^{(n)} = \nu\right] = \frac{f^{\mu} \times f^{\nu}}{n!} = \frac{\sqrt{\mathbb{P}\left(\lambda^{(n-1)} = \mu\right)}\sqrt{\mathbb{P}\left(\lambda^{(n)} = \nu\right)}}{\sqrt{n}}$$

where f^{μ} is the number of standard Young tableaux with shape μ

Hint: use RSK bijection;
arbitrary tableau P with shape ν,
$Q \setminus \{n\}$ is an arbitrary tableau with shape μ

distribution of a prefix $\lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(n)}$

$$\mathbb{P}\left[\left(\lambda^{(1)}, \ldots, \lambda^{(n)}\right) = \left(\mu^{(1)}, \ldots, \mu^{(n)}\right)\right] = \frac{f^{\mu^{(n)}} \times 1}{n!}$$

depends only on the endpoint

Hint: use RSK bijection; arbitrary P, specific Q

---

corollary:

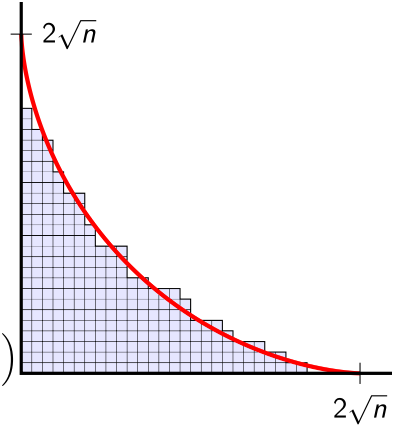
Plancherel growth process $\lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \cdots$ is a Markov chain

Def: $E_1^{(n)}$ is the event that

$$\lambda^{(n)} = \underbrace{\mathrm{grow}_1 \lambda^{(n-1)}}_{\text{one box added in the bottom row}}$$

$$\mathbb{P}\left(E_1^{(1)}\right) \geq \mathbb{P}\left(E_1^{(2)}\right) \geq \cdots$$

$$\mathbb{E}\,\mathrm{LIS}_n = \mathbb{E}\lambda_1^{(n)} = \mathbb{P}\left(E_1^{(1)}\right) + \cdots + \mathbb{P}\left(E_1^{(n)}\right)$$



$$\lim_{n\to\infty} \frac{\mathbb{E}\,\mathrm{LIS}_n}{\sqrt{n}} = 2$$

$$\implies \lim_{n\to\infty} \underbrace{\sqrt{n}\,\mathbb{P}\left(E_1^{(n)}\right)}_{c_n} = 1$$

vector space of functions on the set $\mathbb{Y}_n$ of diagrams with n boxes;

for $A \subseteq \mathbb{Y}_n$ define scalar product $\quad \langle f, g \rangle_A = \sum_{\lambda \in A} f_\lambda g_\lambda$

and the norm $\quad \|f\|_A = \sqrt{\langle f, f \rangle_A}$

$$Y_\mu := \frac{f^\mu}{\sqrt{n!}} = \sqrt{\mathbb{P}(\lambda^{(n)} = \mu)},$$

$$X_\mu := \frac{f^{\operatorname{del}_1 \mu}}{\sqrt{(n-1)!}} = \sqrt{\mathbb{P}(\lambda^{(n-1)} = \operatorname{del}_1 \mu)},$$

$$\langle Y, Y \rangle_A = \mathbb{P}\left(\lambda^{(n)} \in A\right),$$

$$\langle X, X \rangle_A = \mathbb{P}\left(\operatorname{grow}_1 \lambda^{(n-1)} \in A\right),$$

$$\langle X, Y \rangle_A = \sqrt{n}\, \mathbb{P}\left(\lambda^{(n)} \in A \text{ and } E_1^{(n)}\right),$$

$$\langle Y, Y \rangle_A = \mathbb{P}\left(\lambda^{(n)} \in A\right),$$

$$\langle X, X \rangle_A = \mathbb{P}\left(\operatorname{grow}_1 \lambda^{(n-1)} \in A\right),$$

$$\langle X, Y \rangle_A = \sqrt{n}\, \mathbb{P}\left(\lambda^{(n)} \in A \text{ and } E_1^{(n)}\right),$$

$$\lim_{n\to\infty} \underbrace{\sqrt{n}\, \mathbb{P}\left(E_1^{(n)}\right)}_{c_n} = 1$$

$$\lim_{n\to\infty} \left\| c_n^{-1} X - Y \right\|_{\mathbb{Y}_n}^2 = \lim_{n\to\infty} \langle c_n^{-1} X - Y, c_n^{-1} X - Y \rangle_{\mathbb{Y}_n} = 0$$

$$\mathbb{P}\left(\lambda^{(n)} \in A \,\middle|\, E_1^{(n)}\right) - \mathbb{P}\left(\lambda^{(n)} \in A\right) = \left\langle c_n^{-1} X - Y, Y \right\rangle_A$$
$$\leq \left\| c_n^{-1} X - Y \right\|_A \, \|Y\|_A \to 0$$

$$\langle Y, Y \rangle_A = \mathbb{P}\left(\lambda^{(n)} \in A\right),$$
$$\langle X, X \rangle_A = \mathbb{P}\left(\mathrm{grow}_1\, \lambda^{(n-1)} \in A\right),$$
$$\langle X, Y \rangle_A = \sqrt{n}\, \mathbb{P}\left(\lambda^{(n)} \in A \text{ and } E_1^{(n)}\right),$$

$$\lim_{n\to\infty} \underbrace{\sqrt{n}\,\mathbb{P}\left(E_1^{(n)}\right)}_{c_n} = 1$$

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$$\mathbb{P}\left(\lambda^{(n)} \in A \,\middle|\, E_1^{(n)}\right) - \mathbb{P}\left(\lambda^{(n)} \in A\right) = \left\langle c_n^{-1} X - Y, Y \right\rangle_A \leq \left\| c_n^{-1} X - Y \right\|_A \, \|Y\|_A \to 0$$

$$\lim_{n\to\infty} \underbrace{\sqrt{n}\,\mathbb{P}\left(E_1^{(n)}\right)}_{c_n} = 1$$

$$\lim_{n\to\infty} \left\| c_n^{-1} X - Y \right\|^2_{\mathbb{Y}_n} = \lim_{n\to\infty} \langle c_n^{-1} X - Y, c_n^{-1} X - Y \rangle_{\mathbb{Y}_n} = 0$$

$$\mathbb{P}\left(\lambda^{(n)} \in A \,\middle|\, E_1^{(n)}\right) - \mathbb{P}\left(\lambda^{(n)} \in A\right) = \left\langle c_n^{-1} X - Y, Y \right\rangle_A$$
$$\leq \left\| c_n^{-1} X - Y \right\|_A \; \| Y \|_A \to 0$$

---

conclusion: total variation distance between

- probability distribution of $\lambda^{(n)}$, and
- the conditional probability distribution of $\lambda^{(n)}$ under the condition that $E_1^{(n)}$ occured

converges to zero, as $n \to \infty$

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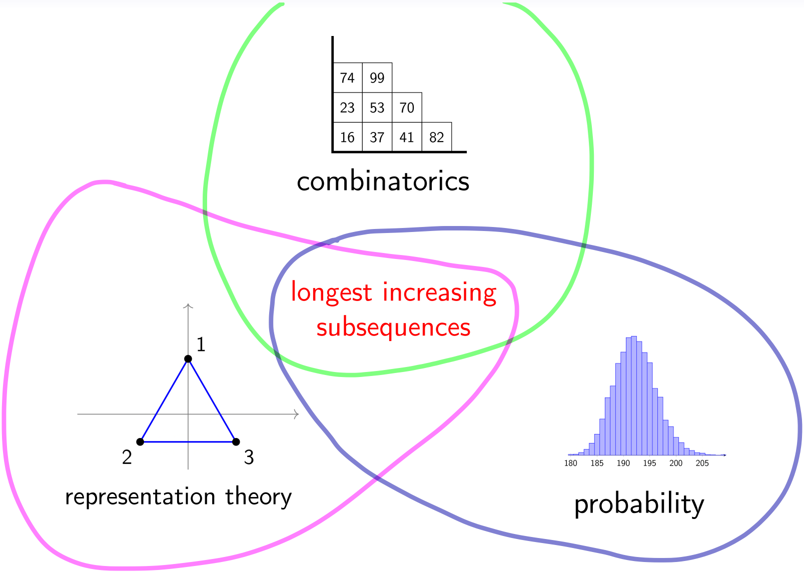
iterate this argument and: total variation distance between

- $\left(E_1^{(n)}, \dots, E_1^{(m)}\right)$, and
- the sequence of *independent* Bernoulli random variables $\left(\tilde{E}_1^{(n)}, \dots, \tilde{E}_1^{(m)}\right)$

is of order $o\left(\frac{m-n}{\sqrt{n}}\right)$ $\implies$ Poisson limit theorem

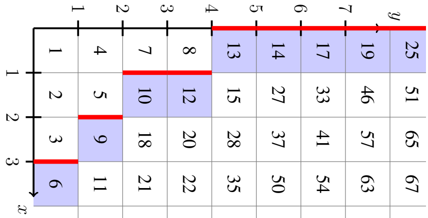

74 99
23 53 70
16 37 41 82
combinatorics
longest increasing subsequences
1
2
3
representation theory
180 185 190 195 200 205
probability

Łukasz Maślanka,
Mikołaj Marciniak,
Piotr Śniady
Poisson limit theorems
for the Robinson–Schensted
correspondence
and the Hammersley
multi-line process
arXiv:2005.13824

the new proof has some hidden extra applications

⟶ *St. Petersburg Seminar on Representation Theory and Dynamical Systems*
June 10, 2020
17.00 MSK (Moscow time),
16.00 CEST (Warsaw time)
Zoom meeting id: 933-433-492
Password: the order of the symmetric group $\mathfrak{S}(6)$

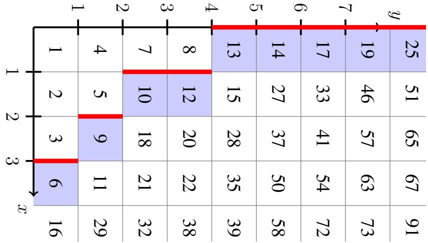
Łukasz Maślanka, Mikołaj Marciniak, Piotr Śniady
Poisson limit of bumping routes in the Robinson–Schensted correspondence
arXiv:2005.14397



the new proof has some hidden extra applications

⟶ *Journée-séminaire de combinatoire CALIN, Laboratoire d'Informatique de Paris Nord*
June 16, 2020
14.00 CEST (Paris time)

Łukasz Maślanka, Mikołaj Marciniak, Piotr Śniady
Poisson limit of bumping routes in the Robinson–Schensted correspondence
arXiv:2005.14397



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