Series of lectures: characters, maps, free cumulants

Piotr Śniady

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plan of the series of lectures

Lecture 1: characters, maps, free cumulants...
and Stanley character formula,

Lecture 2: characters, maps, free cumulants...
and random Young diagrams,

Lecture 3: characters, maps, free cumulants...
and Kerov character polynomials,

Lecture 2: characters, maps, free cumulants. . . and random Young diagrams

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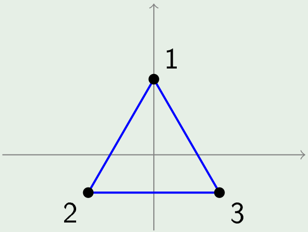
representations

representation theory: how an abstract group
can be concretely realized as a group of matrices?

Example

symmetric group $\mathfrak{S}(3)$
permutations of $\{1, 2, 3\}$



formal definition: representation ρ of a group G is a homomorphism

$$\rho \colon G \to \mathrm{GL}_d$$

from the group to invertible matrices

irreducible representations

representation ρ is called reducible
if can be written as a direct sum of smaller representations:

$$\rho(g) = \begin{bmatrix} \rho_1(g) & \\ & \rho_2(g) \end{bmatrix} \qquad \text{for every } g \in G;$$

we are interested in irreducible representations

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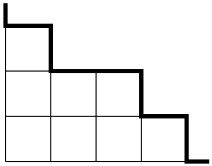
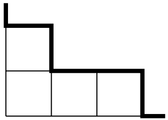
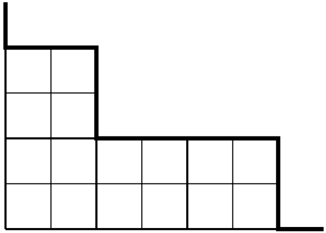
irreducible representation ρ^λ of the symmetric group $\mathfrak{S}(n)$ $\longleftrightarrow$ Young diagram λ with n boxes



shape of Young diagram



Young diagram λ

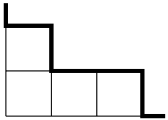


dilated diagram 2λ

goal for today:

study $\rho^{s\lambda}$ for $s \to \infty$

shape of Young diagram



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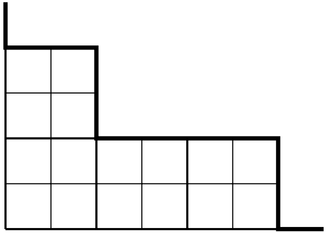

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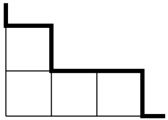

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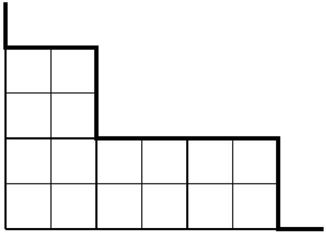

dilated diagram 2λ

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clever choice: for λ with n boxes study

$$\frac{1}{\sqrt{n}}\lambda$$

reducible representations and random Young diagrams

suppose ρ is an interesting reducible representation of $\mathfrak{S}(n)$

$$\rho = \bigoplus_{|\lambda|=n} m^\lambda \rho^\lambda \qquad \text{with } m^\lambda \in \{0,1,2,\dots\};$$

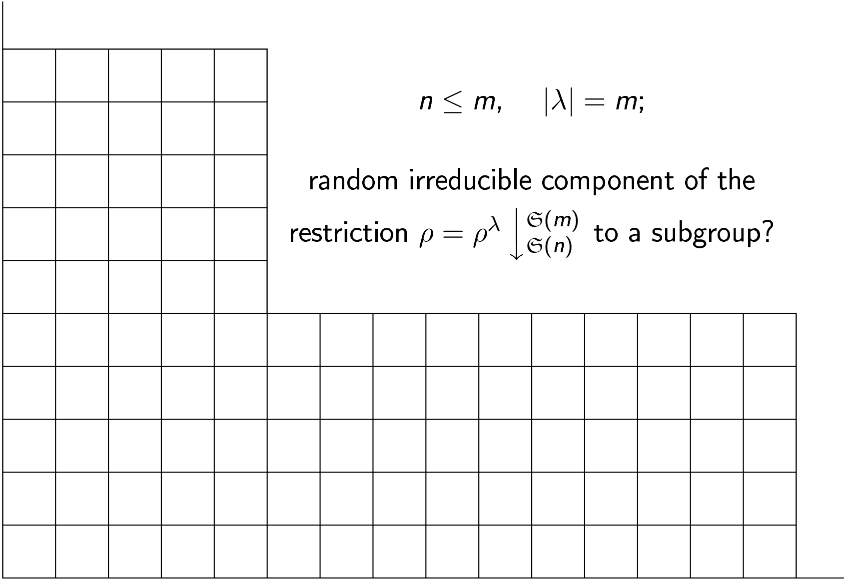
we define a probability measure on Young diagrams with n boxes by:

$$\mathbb{P}_\rho(\lambda) := \frac{m^\lambda \cdot \dim \rho^\lambda}{\dim \rho} \qquad \text{for } \lambda \in \mathbb{Y}_n$$

problem for today

character of ρ $\longleftrightarrow$ probabilistic properties of a random Young diagram

motivating example: restriction to a subgroup



$$n \leq m, \quad |\lambda| = m;$$

random irreducible component of the restriction $\rho = \rho^\lambda \big\downarrow^{\mathfrak{S}(m)}_{\mathfrak{S}(n)}$ to a subgroup?

motivating example: restriction to a subgroup

75	81	89	98	100										
58	60	72	94	99										
51	56	62	93	95										
26	38	54	79	92										
18	33	37	59	87										
12	20	35	36	42	46	67	68	70	78	82	84	88	90	97
11	17	19	22	30	43	52	55	64	65	66	74	83	85	96
8	10	13	21	23	29	34	45	47	49	63	71	76	80	91
2	7	9	15	16	24	27	39	41	44	48	57	69	77	86
1	3	4	5	6	14	25	28	31	32	40	50	53	61	73

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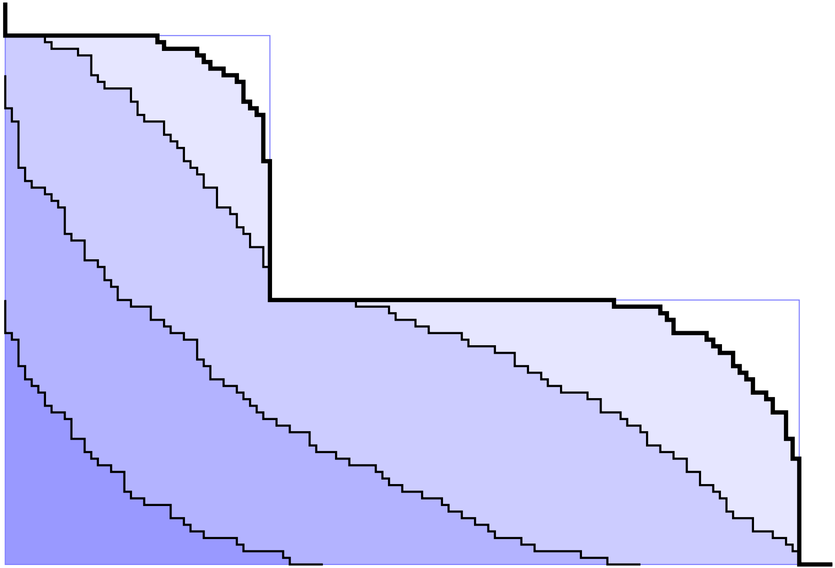

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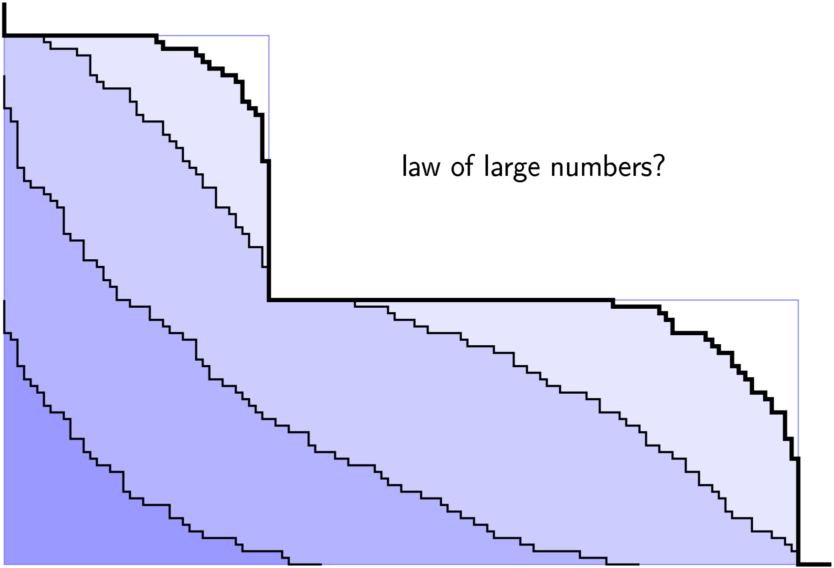


random hermitian matrix with prescribed eigenvalues

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix} = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix} U^{-1},$$

where U is a random matrix from the unitary group $U(d)$

---

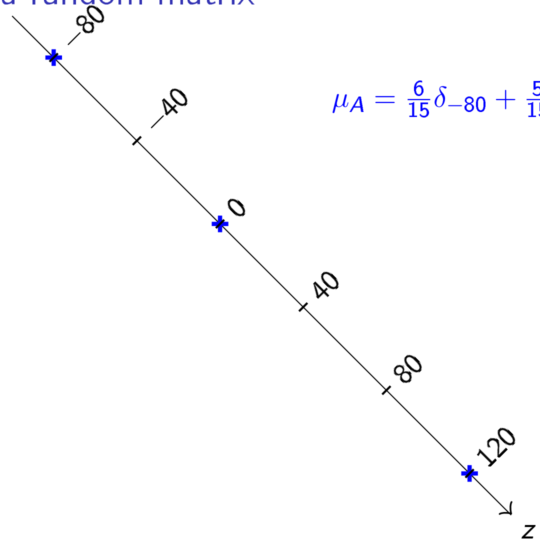
spectral measure: (random) probability measure on $\mathbb{R}$

$$\mu_A = \frac{1}{d} \sum_{1 \leq i \leq d} \delta_{\lambda_i}$$

what happens to eigenvalues if we remove some rows and columns?

$$A\downarrow = \begin{bmatrix} a_{11} & \cdots & a_{1d'} \\ \vdots & \ddots & \vdots \\ a_{d'1} & \cdots & a_{d'd'} \end{bmatrix}$$

truncation of a random matrix



$$\mu_A = \frac{6}{15}\delta_{-80} + \frac{5}{15}\delta_0 + \frac{4}{15}\delta_{120}$$

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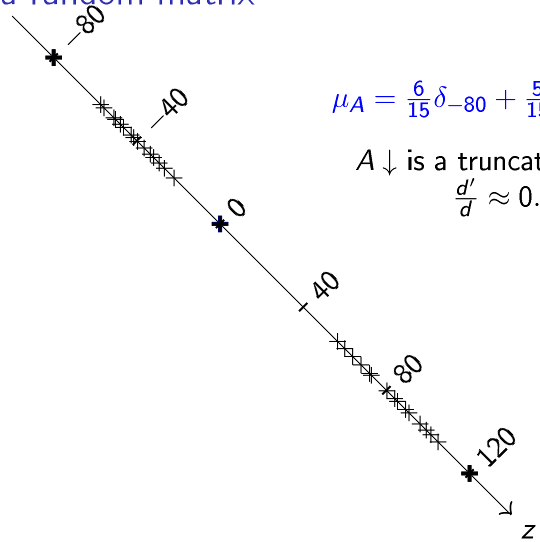

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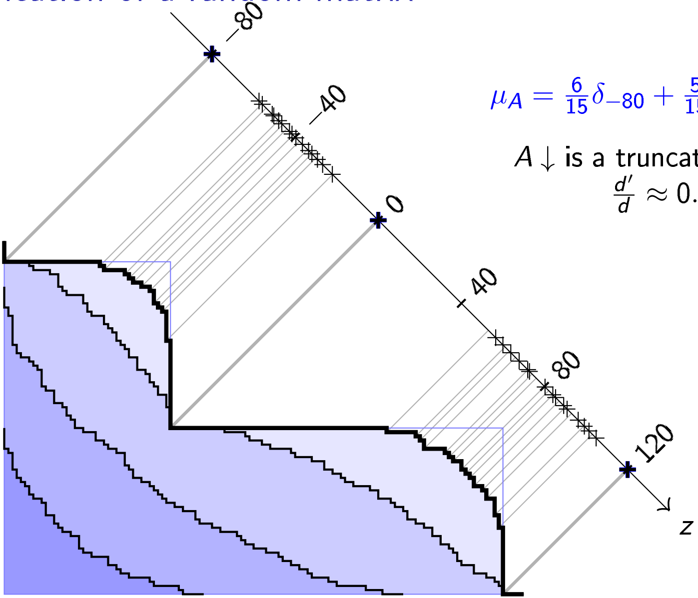

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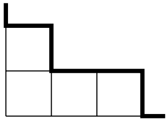
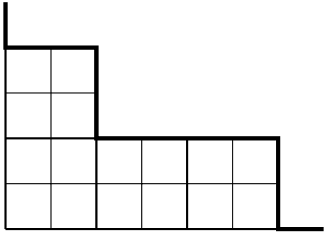
how to parametrize shape of λ?



Young diagram λ



dilated diagram 2λ

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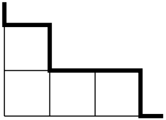

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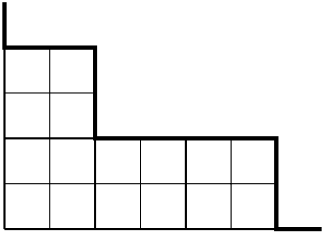

dilated diagram 2λ

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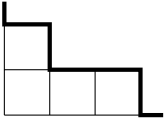
we need *nice* functions on the set of Young diagrams which depend only on shape of λ, not on its size:

$$f(s\lambda) = f(\lambda)$$

how to parametrize shape of λ?



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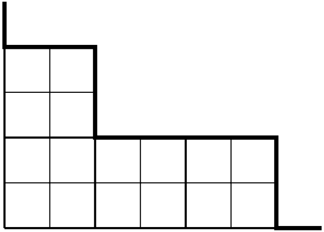

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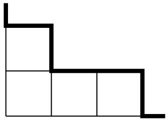

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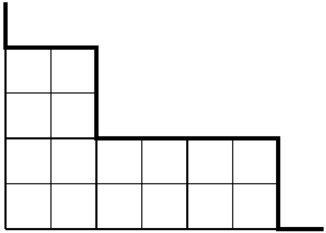

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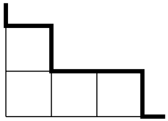

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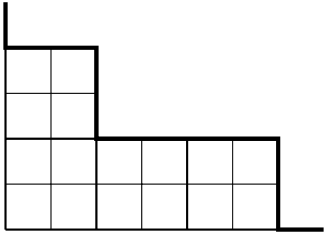

dilated diagram 2λ

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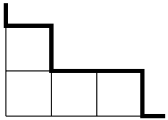
we need *nice* functions on the set of Young diagrams which depend nicely on the size of λ:

$$f(s\lambda) = s^k f(\lambda)$$

homogeneous function of degree k

how to parametrize shape of λ?



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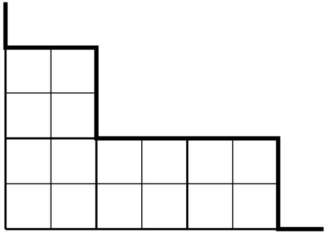
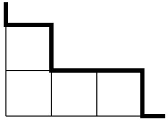


dilated diagram 2λ

ideal parameters of shape should:

- be homogeneous,
- describe shape of λ in a convenient way,
- be effectively computable,
- be related to the character of ρ^λ,

how to parametrize shape of λ?



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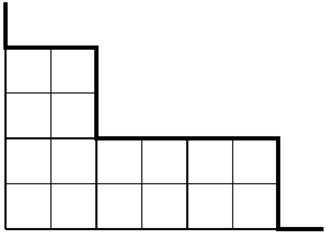

dilated diagram 2λ

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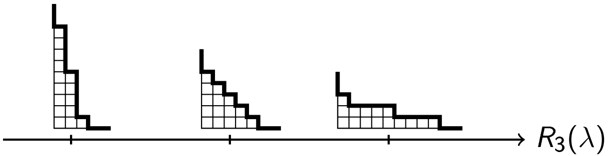
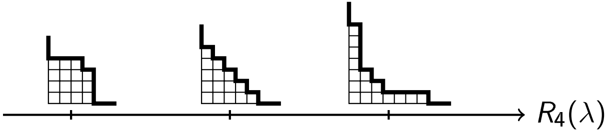
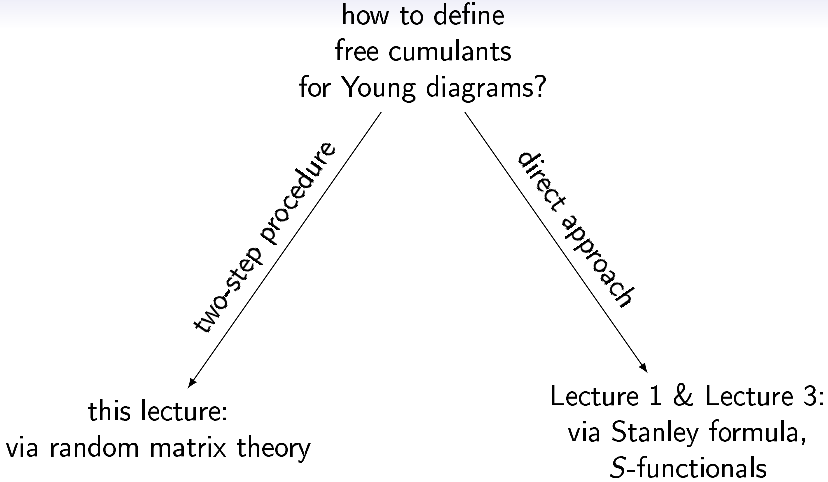
free cumulants:

- are homogeneous,
- describe shape of λ in a convenient way,
- are effectively computable,
- are related to the character of ρ^λ,

free cumulants $\longleftrightarrow$ shape






how to define
free cumulants
for Young diagrams?
two-step procedure
direct approach
this lecture:
via random matrix theory
Lecture 1 & Lecture 3:
via Stanley formula,
S-functionals

step 1: $\rho^\lambda \mapsto$'random matrix'

Biane's matrix

$$\rho^\lambda \mapsto X_{n+1} = \begin{bmatrix} 0 & \rho^\lambda(1,2) & \cdots & \rho^\lambda(1,n) & 1 \\ \rho^\lambda(2,1) & 0 & \cdots & \rho^\lambda(2,n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^\lambda(n,1) & \rho^\lambda(n,2) & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

- directly related to the representation ρ and its character;
- spectrum of the matrix is related to the shape of λ;

$\longrightarrow$ Kerov, Biane

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- directly related to the representation ρ and its character;
- spectrum of the matrix is related to the shape of λ;
- hint: this is the action of Jucys-Murphy element X_{n+1} in the induced representation $\rho^\lambda\big\uparrow^{\mathfrak{S}(n+1)}_{\mathfrak{S}(n)}$
  $\longrightarrow$ bonus material

$\longrightarrow$ Kerov, Biane

step 2: how to parametrize spectrum of a random matrix?

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix} = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix} U^{-1},$$

where U is a random matrix from the unitary group $U(d)$

not so clever

sequence of moments

$$M_\ell = \operatorname{tr} A^\ell = \frac{1}{d} \sum_i \lambda_i^\ell$$
$$= \int z^\ell \, d\mu_A(z)$$

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sequence of free cumulants

$$R_\ell \approx$$

$$d^{\ell-1} \, \mathbb{E} a_{12} a_{23} \cdots a_{\ell-1,\ell} a_{\ell,1}$$

$$= \frac{d^{\ell-1}}{\ell!} \frac{\partial^i}{\partial z^\ell} \log \mathbb{E} e^{z a_{11}} \bigg|_{z=0}$$

$\longrightarrow$Voiculescu, Speicher, Collins

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sequence of free cumulants

$$R_\ell \approx d^{\ell-1}\, \mathbb{E} a_{12} a_{23} \cdots a_{\ell-1,\ell} a_{\ell,1} = \frac{d^{\ell-1}}{\ell!} \frac{\partial^i}{\partial z^\ell} \log \mathbb{E} e^{z a_{11}} \Big|_{z=0}$$

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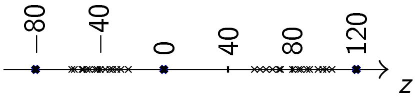

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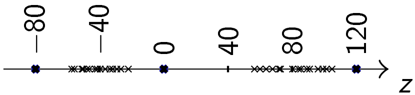

application

$$\mathbb{E} R_\ell(A\!\downarrow) = \left(\frac{d'}{d}\right)^{\ell-1} R_\ell(A)$$

very clever

sequence of free cumulants

$$R_\ell \approx$$

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step 1&2: free cumulants for λ

$$R_\ell(A) \approx d^{\ell-1}\ \mathbb{E} a_{12} a_{23} \cdots a_{\ell-1,\ell} a_{\ell,1}$$

step 1&2: free cumulants for λ

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$$\rho^\lambda \mapsto X_{n+1} = \begin{bmatrix} 0 & \rho^\lambda(1,2) & \cdots & \rho^\lambda(1,n) & 1 \\ \rho^\lambda(2,1) & 0 & \cdots & \rho^\lambda(2,n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^\lambda(n,1) & \rho^\lambda(n,2) & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

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$$R_\ell(\lambda) = R_\ell(\rho^\lambda) \approx (n+1)^{\ell-1} \, \operatorname{tr}\Big[\rho^\lambda(1,2)\rho^\lambda(2,3)\cdots\rho^\lambda(\ell-1,\ell)\rho^\lambda(\ell,1)\Big]$$

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$$R_\ell(A) \approx d^{\ell-1} \ \mathbb{E} a_{12} a_{23} \cdots a_{\ell-1,\ell} a_{\ell,1}$$

$$\rho^\lambda \mapsto X_{n+1} = \begin{bmatrix} 0 & \rho^\lambda(1,2) & \cdots & \rho^\lambda(1,n) & 1 \\ \rho^\lambda(2,1) & 0 & \cdots & \rho^\lambda(2,n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^\lambda(n,1) & \rho^\lambda(n,2) & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} R_\ell(\lambda) = R_\ell(\rho^\lambda) &\approx (n+1)^{\ell-1} \ \mathrm{tr}\Big[\rho^\lambda(1,2)\rho^\lambda(2,3)\cdots\rho^\lambda(\ell-1,\ell)\rho^\lambda(\ell,1)\Big] \\ &\approx n^{\ell-1} \ \mathrm{tr}\rho(1,2,\dots,\ell-1) \end{aligned}$$

step 1&2: free cumulants for λ

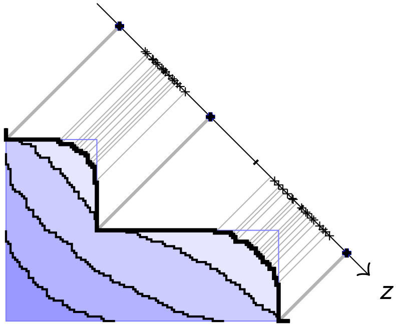
$$R_\ell(\lambda) = R_\ell(\rho^\lambda) \approx$$

$$n^{\ell-1} \operatorname{tr} \rho(1, 2, \dots, \ell - 1)$$

step 1&2: free cumulants for λ

application

$$\mathbb{E} R_\ell \left(\rho^\lambda \Big\downarrow^{\mathfrak{S}(n)}_{\mathfrak{S}(n')} \right) \approx \left(\frac{n'}{n} \right)^{\ell-1} R_\ell(\rho^\lambda)$$



$$R_\ell(\lambda) = R_\ell(\rho^\lambda) \approx n^{\ell-1} \operatorname{tr} \rho(1, 2, \ldots, \ell - 1)$$

algebraic viewpoint on probability

algebra $\mathcal{A}$
of *'random variables'*

$$\mathbb{E}\colon \mathcal{A} \to \mathbb{C}$$

expected value

algebraic viewpoint on probability

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expected value

Example 1

$\mathcal{A} = \mathbb{C}[\mathbb{Y}_n]$ is the algebra of functions on the set of Young diagrams with n boxes, with pointwise product

$$\mathbb{E} F = \sum_{\lambda} \mathbb{P}_\rho(\lambda)\ F(\lambda)$$

is the usual expected value

algebraic viewpoint on probability

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Example 2

$\mathcal{A} = Z\mathbb{C}[\mathfrak{S}(n)]$ is the center
of the group algebra
with convolution product

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is the usual expected value

isomorphism

$$Z\mathbb{C}[\mathfrak{S}(n)] \ni f \mapsto \widehat{f} \in \mathbb{C}[\mathbb{Y}_n]$$

$$\widehat{f}(\lambda) := \operatorname{tr} \rho^{\lambda}(f)$$

normalized conjugacy classes; normalized characters

Example 1, algebra $\mathbb{C}[\mathbb{Y}_n]$

normalized characters

$$\mathrm{Ch}_\pi(\lambda) = n^{\underline{k}} \, \operatorname{tr} \rho^\lambda(\pi)$$

for $n = |\lambda|$ and $k = |\pi|$

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1)$$

normalized conjugacy classes; normalized characters

Example 2, algebra $Z\mathbb{C}[\mathfrak{S}(n)]$

normalized conjugacy classes

$$A_2 = \sum_{\substack{1\leq a,b \leq n,\\ a\neq b}} (a,b) \in \mathbb{C}[\mathfrak{S}(n)]$$

$$A_{3,2} = \sum_{\substack{1\leq a,b,c,d,e \leq n,\\ \text{all different}}} (a,b,c)(d,e)$$

$$\dots$$

Example 1, algebra $\mathbb{C}[\mathbb{Y}_n]$

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...

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for $n = |\lambda|$ and $k = |\pi|$

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1)$$

$$A_\pi \xmapsto{\text{isomorphism}} \mathrm{Ch}_\pi$$

not a toy case

random variable Ch_2
$\mathbb{Y} \ni \lambda \mapsto \mathrm{Ch}_2(\lambda)$

what is its variance?

not a toy case

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$$(A_2)^2 = A_{2,2} + 4A_3 + 2A_{1,1}$$

not a toy case

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$$\operatorname{Var} \mathrm{Ch}_2 = \mathbb{E}\left[(\mathrm{Ch}_2)^2\right] - (\mathbb{E}\,\mathrm{Ch}_2)^2 =$$

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$$\operatorname{tr}\rho\left[(A_2)^2\right] - \left(\operatorname{tr}\rho(A_2)\right)^2 =$$

not a toy case

random variable Ch_2

$\mathbb{Y} \ni \lambda \mapsto \mathrm{Ch}_2(\lambda)$

what is its variance?

$$A_2 = \sum_{\substack{1 \le a,b \le n, \\ a \neq b}} (a,b) \in \mathbb{C}[\mathfrak{S}(n)]$$

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$$\operatorname{Var} \mathrm{Ch}_2 = \mathbb{E}\left[(\mathrm{Ch}_2)^2\right] - (\mathbb{E}\,\mathrm{Ch}_2)^2 =$$

$$\operatorname{tr}\rho\left[(A_2)^2\right] - \left(\operatorname{tr}\rho(A_2)\right)^2 =$$

$$n^{\underline{4}} \operatorname{tr}\rho(2,2) + 4n^{\underline{3}} \operatorname{tr}\rho(3) + 4n^{\underline{2}} \underbrace{\operatorname{tr}\rho(\,\mathrm{id}\,)}_{=1} - \left(n^{\underline{2}} \operatorname{tr}\rho(2)\right)^2 \overset{?}{=}$$

disjoint product

$\mathbb{C}[\mathfrak{S}(n)]$ $\supseteq$ $Z\mathbb{C}[\mathfrak{S}(n)]$

some calculations are simple — no simple calculations

$(1,2)\cdot(3,4) = (1,2)(3,4)$ — $A_2 \cdot A_2 = A_{2,2} + 4A_3 + 2A_{1,1}$

---

new disjoint product

$$A_2 \bullet A_2 = A_{2,2},$$
$$A_2 \bullet A_3 = A_{3,2},$$
$$A_{3,2} \bullet A_3 = A_{3,3,2},$$

$\longrightarrow$ partial permutations of Ivanov & Kerov

algebraic viewpoint on probability

algebra $\mathcal{A}$
of *'random variables'*

$\mathbb{E}\colon \mathcal{A} \to \mathbb{C}$

expected value

Example 3

$\mathcal{A} = Z\mathbb{C}[\mathfrak{S}(n)]$ with disjoint product

$\mathbb{E}f = \operatorname{tr} \rho(f)$

algebraic viewpoint on probability

algebra $\mathcal{A}$
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$\mathbb{E}\colon \mathcal{A} \to \mathbb{C}$

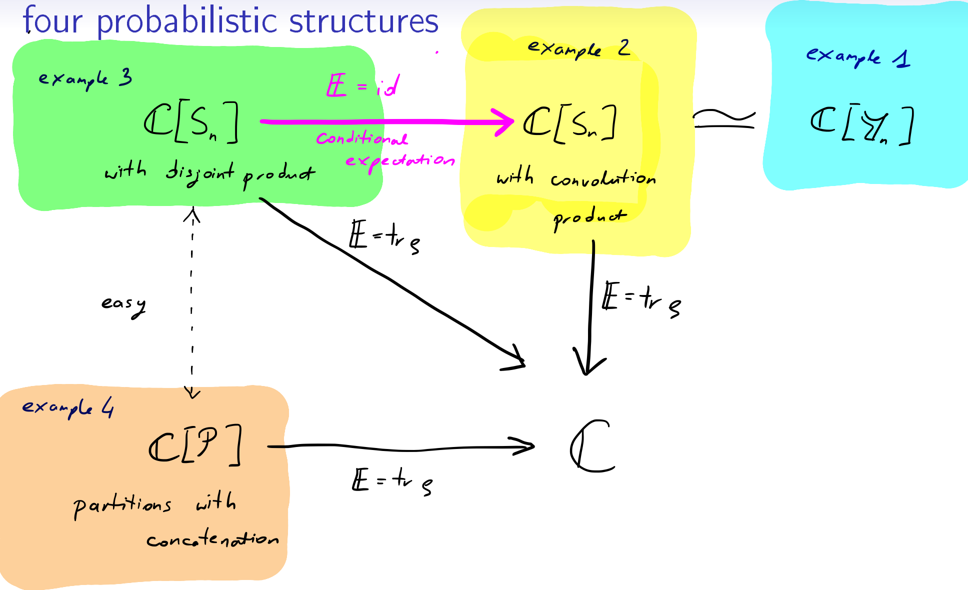
expected value

Example 4

$\mathcal{A} = Z\mathbb{C}[\mathcal{P}]$
partitions with concatenation:
$(3,2,2) \cdot (3) = (3,3,2,2)$

$\mathbb{E}(\pi) = \operatorname{tr} \rho(\pi) = \operatorname{tr} \rho(\pi, 1, 1, \dots, 1)$

four probabilistic structures



four probabilistic structures: covariances

$$\mathrm{Cov}^{id}(A_2, A_2) = A_2 \bullet A_2 - A_2 \cdot A_2 = A_{2,2} - A_2 \cdot A_2$$

example 2

$$\mathrm{Cov}(A_2, A_2) = \operatorname{tr} \rho (A_2 \cdot A_2) - [\operatorname{tr} \rho (A_2)]^2$$

example 3

$$\mathrm{Cov}^{\bullet}(A_2, A_2) = \operatorname{tr} \rho (A_{2,2}) - [\operatorname{tr} \rho (A_2)]^2$$

$\mathbb{C}[S_n]$ with disjoint product $\xrightarrow[\text{conditional expectation}]{\mathbb{E} = id}$ $\mathbb{C}[S_n]$ with convolution product

$\mathbb{C}[S_n]$ with disjoint product $\xrightarrow{\mathbb{E} = \operatorname{tr} \rho}$ $\mathbb{C}$

$\mathbb{C}[S_n]$ with convolution product $\xrightarrow{\mathbb{E} = \operatorname{tr} \rho}$ $\mathbb{C}$

$\mathbb{C}[\mathcal{P}]$ partitions with concatenation $\xrightarrow{\mathbb{E} = \operatorname{tr} \rho}$ $\mathbb{C}$

example 4

$$\mathrm{Cov}((2), (2)) = \operatorname{tr} \rho (2,2) - [\operatorname{tr} \rho (2)]^2$$

Var Ch_2, part 1

$$\operatorname{Var} \mathrm{Ch}_2 = \mathbb{E}(\mathrm{Ch}_2)^2 - (\mathbb{E}\,\mathrm{Ch}_2)^2 =$$

Cov (A_2, A_2) in $(\mathbb{C}[S_n], \operatorname{tr}\varrho)$ with convolution product

$$\operatorname{tr}\rho(A_2^2) - \big(\operatorname{tr}\rho(A_2)\big)^2 =$$

$$\operatorname{tr}\rho(A_{2,2} + 4A_3 + 2A_{1,1}) - \big(\operatorname{tr}\rho(A_2)\big)^2 =$$

$$\underbrace{\operatorname{tr}\rho(A_2 \bullet A_2) - \big(\operatorname{tr}\rho(A_2)\big)^2}_{} + \underbrace{\operatorname{tr}\rho(A_2 A_2 - A_2 \bullet A_2)}_{}$$

$\mathrm{Cov}^{\bullet}(A_2, A_2)$ in $(\mathbb{C}[S_n], \operatorname{tr}\varrho)$ with disjoint product

$\operatorname{tr}\varrho\left(\mathrm{Cov}^{id}(A_2, A_2)\right)$

conditional expectation

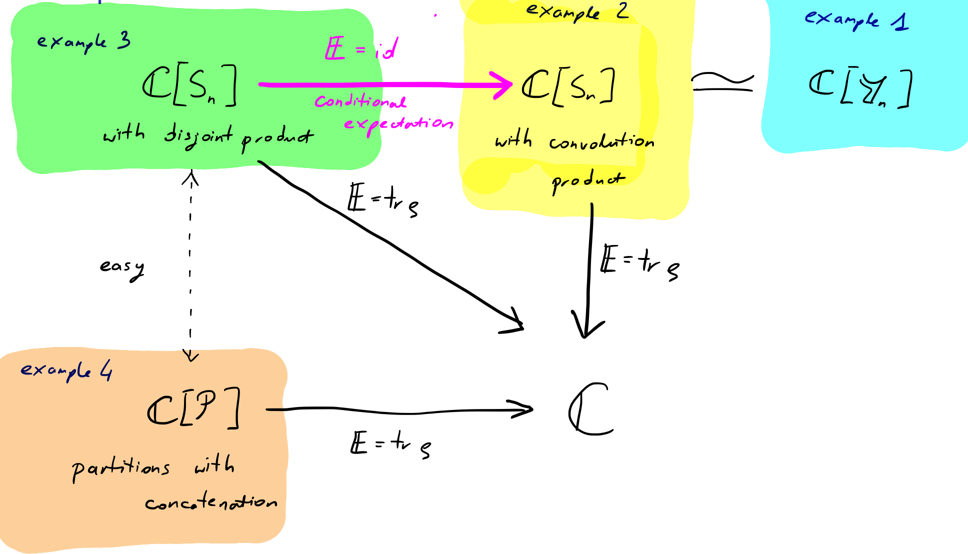
Var Ch_2, part 2

$$\operatorname{tr}\rho(A_2 \bullet A_2) - \left(\operatorname{tr}\rho(A_2)\right)^2 =$$

$$n^{\underline{4}}\operatorname{tr}\rho(2,2) - \left(n^{\underline{2}}\right)^2 \left(\operatorname{tr}\rho(2)\right)^2 =$$

$$n^{\underline{4}}\underbrace{\left[\operatorname{tr}\rho(2,2) - \left(\operatorname{tr}\rho(2)\right)^2\right]}_{\mathrm{Cov}\left((2),(2)\right)} + \underbrace{\left[n^{\underline{4}} - \left(n^{\underline{2}}\right)^2\right]}_{\approx n^3} \left(\operatorname{tr}\rho(2)\right)^2$$

four probabilistic structures



characters $\longleftrightarrow$ properties of random Young diagrams

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let ρ be a reducible representation of $\mathfrak{S}(n)$
with character $\chi(\pi) = \operatorname{tr} \rho(\pi)$
and corresponding random Young diagram λ

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let ρ be a reducible representation of $\mathfrak{S}(n)$
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the main result, very informal version

suppose

$$\chi(\pi_1, \pi_2, \dots, \pi_\ell) \approx \chi(\pi_1) \cdots \chi(\pi_\ell);$$

then

- random Young diagram λ with high probability concentrates around some limit shape;
- fluctuations of λ around this limit shape are Gaussian;

$\longrightarrow$ Biane, Śniady

characters $\longleftrightarrow$ properties of random Young diagrams

let $\rho^{(n)}$ be a reducible representation of $\mathfrak{S}(n)$
with character $\chi^{(n)}(\pi) = \operatorname{tr} \rho^{(n)}(\pi)$
and corresponding random Young diagram $\lambda^{(n)}$

the main result, very informal version

suppose

$$\chi^{(n)}(\pi_1, \pi_2, \dots, \pi_\ell) \approx \chi^{(n)}(\pi_1) \cdots \chi^{(n)}(\pi_\ell);$$

then

- random Young diagram $\lambda^{(n)}$ with high probability concentrates around some limit shape;
- fluctuations of $\lambda^{(n)}$ around this limit shape are Gaussian;

in the limit $n \to \infty$

$\longrightarrow$ Biane, Śniady

characters $\longleftrightarrow$ properties of random Young diagrams

let $\rho^{(n)}$ be a reducible representation of $\mathfrak{S}(n)$
with character $\chi^{(n)}(\pi) = \operatorname{tr} \rho^{(n)}(\pi)$
and corresponding random Young diagram $\lambda^{(n)}$

the main result, very informal version

suppose

$$\chi^{(n)}(\pi_1, \pi_2, \dots, \pi_\ell) \approx \chi^{(n)}(\pi_1) \cdots \chi^{(n)}(\pi_\ell);$$

then

- random Young diagram $\frac{1}{\sqrt{n}}\lambda^{(n)}$ with high probability concentrates around some limit shape;
- fluctuations of $\frac{1}{\sqrt{n}}\lambda^{(n)}$ around this limit shape are Gaussian;

in the limit $n \to \infty$

$\longrightarrow$ Biane, Śniady

covariance and other cumulants

$$k_\ell(A_1, \dots, A_\ell) = \frac{\partial^\ell}{\partial z_1 \cdots \partial z_\ell} \log \mathbb{E} e^{z_1 A_1 + \cdots + z_\ell A_\ell} \Big|_{z_1 = \cdots = z_\ell = 0}$$

---

covariance and other cumulants

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$$\begin{aligned}
\mathbb{E} A_1 &= k_1(A_1), \\
\mathbb{E} A_1 A_2 &= k_2(A_1,A_2) + k_1(A_1)\ k_1(A_2), \\
\mathbb{E} A_1 A_2 A_3 &= k_3(A_1,A_2,A_3) \\
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\end{aligned}$$

---

covariance and other cumulants

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&\quad + k_1(A_1)\ k_1(A_2)\ k_1(A_3)
\end{aligned}$$

$$\begin{aligned}
k_1(A_1) &= \mathbb{E} A_1, \\
k_2(A_1,A_2) &= \mathbb{E} A_1 A_2 - \mathbb{E} A_1\ \mathbb{E} A_2 = \mathrm{Cov}(A_1,A_2)
\end{aligned}$$

cumulants and Gaussianity

$$k_\ell(A_1, \dots, A_\ell) = \frac{\partial^\ell}{\partial z_1 \cdots \partial z_\ell} \log \mathbb{E} e^{z_1 A_1 + \cdots + z_\ell A_\ell} \bigg|_{z_1 = \cdots = z_\ell = 0}$$

---

cumulants and Gaussianity

$$k_\ell(A_1,\dots,A_\ell) = \frac{\partial^\ell}{\partial z_1 \cdots \partial z_\ell} \log \mathbb{E} e^{z_1 A_1 + \cdots + z_\ell A_\ell} \Big|_{z_1=\cdots=z_\ell=0}$$

---

fact

if $A_1, A_2, \dots$ are random variables such that

$$k_\ell(A_{i_1},\dots,A_{i_\ell}) = 0 \qquad \text{for all } \ell \geq 3$$

then the joint distribution of $A_1, A_2, \dots$ is Gaussian

the main result, more formal version

suppose that for each $n \geq 1$
$\rho^{(n)}$ is a representation of $\mathfrak{S}(n)$;
$\chi^{(n)}(\pi) = \operatorname{tr} \rho^{(n)}(\pi)$ is the corresponding character;

$\mathbb{P}^{(n)}$ is the corresponding probability measure on $\mathbb{Y}_n$;
$R_i^{(n)}$ is the corresponding random variable $\mathbb{Y}_n \ni \lambda \mapsto R_i(\lambda)$

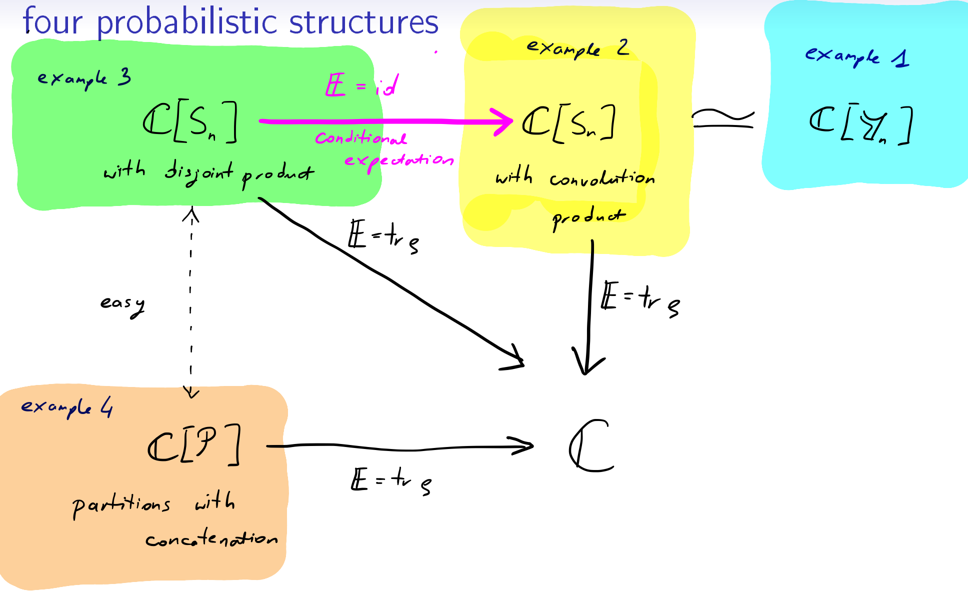
---

the following **four** conditions are equivalent:

- the cumulants which describe the error of the approximation
$$\chi^{(n)}(\pi_1, \pi_2, \dots, \pi_\ell) \approx \chi^{(n)}(\pi_1) \cdots \chi^{(n)}(\pi_\ell)$$
are asymptotically small for $n \to \infty$;
- cumulants which describe the random variables $R_i^{(n)}$ are asymptotically small for $n \to \infty$

$\Longrightarrow$ law of large numbers, central limit theorem

four probabilistic structures



algebraic viewpoint on probability

algebra $\mathcal{A}$
of *'random variables'*

$$\mathbb{E}\colon \mathcal{A} \to \mathbb{C}$$

expected value

algebraic viewpoint on probability

algebra $\mathcal{A}$
of *'random variables'*

$\mathbb{E}\colon \mathcal{A} \to \mathbb{C}$

expected value

example 4, condition 4

$\mathcal{A}$=(linear combinations of) partitions
product of partitions=concatenation

expected value $\mathbb{E}(\pi) = \chi^{(n)}(\pi)$

cumulants:

$$k_2(\pi_1, \pi_2) = \chi^{(n)}(\pi_1, \pi_2) - \chi^{(n)}(\pi_1)\chi^{(n)}(\pi_2)$$

cumulants are asymptotically small if

$$k_\ell(\pi_1, \dots, \pi_\ell) = \left(\frac{1}{\sqrt{n}}\right)^{(\pi_1-1)+\cdots+(\pi_\ell-1)+2(\ell-1)}$$

nonalgebraic viewpoint on probability

example 1, condition 1

$\mathcal{A}$=algebra generated by free cumulants $R_2, R_3, \dots$

product=pointwise product of functions

expected value= the usual expected value

cumulants:

$$k_2(R_i, R_j) = \mathbb{E} R_i R_j - \mathbb{E} R_i \, \mathbb{E} R_j$$

cumulants are asymptotically small if

$$k_\ell(R_{i_1}, \dots, R_{i_\ell}) = \left(\sqrt{n}\right)^{i_1+\cdots+i_\ell-2(\ell-1)}$$

conditional expectation

$$\mathbb{E}\colon (\text{algebra } \mathcal{A}_1 \text{ of random variables}) \to (\text{another algebra } \mathcal{A}_2 \text{ of random variables})$$

$\mathcal{A}_1$ is the linear span of Ch_π
multiplication: disjoint product $\mathrm{Ch}_\pi \bullet \mathrm{Ch}_\sigma = \mathrm{Ch}_{\pi \cup \sigma}$

$\mathcal{A}_2$ is the linear span of Ch_π
multiplication: convolution product

$\mathbb{E}$ is the identity map

difficulty: relate the algebraic and the probabilistic structure

show that cumulants of this conditional expected value are *'small'*:

if λ has at most $O(\sqrt{n})$ rows and columns show that

$$k^{\mathrm{id}}_\ell(\mathrm{Ch}_{\pi_1}, \dots, \mathrm{Ch}_{\pi_\ell})(\lambda) = O\left(\sqrt{n}^{(\pi_1+1)+\cdots+(\pi_\ell+1)-2(\ell-1)}\right)$$

elements of proof 1: cumulants and connected components

$$
\begin{aligned}
\mathbb{E} A_1 &= k_1(A_1), \\
\mathbb{E} A_1 A_2 &= k_2(A_1, A_2) + k_1(A_1)\ k_1(A_2), \\
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&\quad + k_1(A_1)\ k_1(A_2)\ k_1(A_3)
\end{aligned}
$$

$\mathbb{E} A_1 \dots A_n$ is the sum over all set-partitions of $\{1, \dots, n\}$

Feynmann diagram interpretation:
cumulants are related to connected Feynmann diagrams,

elements of proof 2: Stanley character formula

$$\mathrm{Ch}_\pi(\lambda) = (-1)^{\ell(\pi)} \sum_M \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

elements of proof 2: Stanley character formula

$$\mathrm{Ch}_\pi(\lambda) = (-1)^{\ell(\pi)} \sum_M \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

$$k^{id}\big(\mathrm{Ch}_{\pi_1}, \dots, \mathrm{Ch}_{\pi_\ell}\big)(\lambda) = (-1)^{\ell(\pi)} \sum_M \mathfrak{N}_M(\lambda),$$

where the sum runs over connected maps M with face-type π

moral lessons from today, part 0

- for almost any problem
  of the representation theory of symmetric groups $\mathfrak{S}(n)$
  there is a known explicit answer...

moral lessons from today, part 0

- for almost any problem
  of the representation theory of symmetric groups $\mathfrak{S}(n)$
  there is a known explicit answer...
- ...given by some combinatorial algorithm...

moral lessons from today, part 0

- for almost any problem
  of the representation theory of symmetric groups $\mathfrak{S}(n)$
  there is a known explicit answer…
- …given by some combinatorial algorithm…
- … which is too computationally complex
  to be useful for asymptotic questons
  and we need new tools;

moral lessons from today, part 1

- the main result 1: if characters approximately factorize,
  i.e. character on a product of disjoint cycles
  is *approximately* equal
  to the product of characters on each individual cycle
  then the shapes of corresponding Young diagrams
  fulfill law of large numbers and central limit theorem;

moral lessons from today, part 1

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  fulfill law of large numbers and central limit theorem;
- proof, part 1:
  use several algebraic probabilistic structures;
  each describes another aspect of the representation theory;
  relate them to each other;

moral lessons from today, part 1

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- proof, part 2:
  use maps to find bounds on the conditional cumulants;

moral lessons from today, part 1

- the main result 1: if characters approximately factorize, i.e. character on a product of disjoint cycles is *approximately* equal to the product of characters on each individual cycle then the shapes of corresponding Young diagrams fulfill law of large numbers and central limit theorem;
- proof, part 1: use several algebraic probabilistic structures; each describes another aspect of the representation theory; relate them to each other;
- proof, part 2: use maps to find bounds on the conditional cumulants;
- the main result 2: the class of representations with approximate factorization of characters is very large and closed under natural representation-theoretic operations;

moral lessons from today, part 2

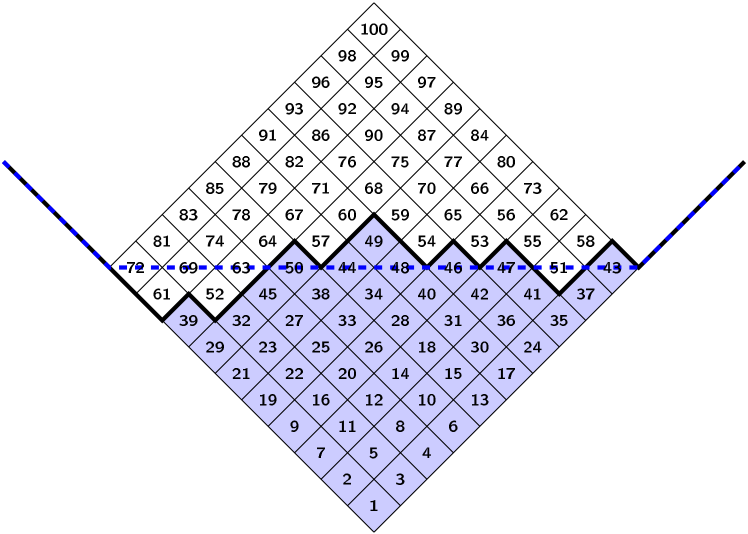
- random Young diagrams $\approx$ random matrix theory because both have very similar combinatorics of maps

  $\longrightarrow$ Okounkov

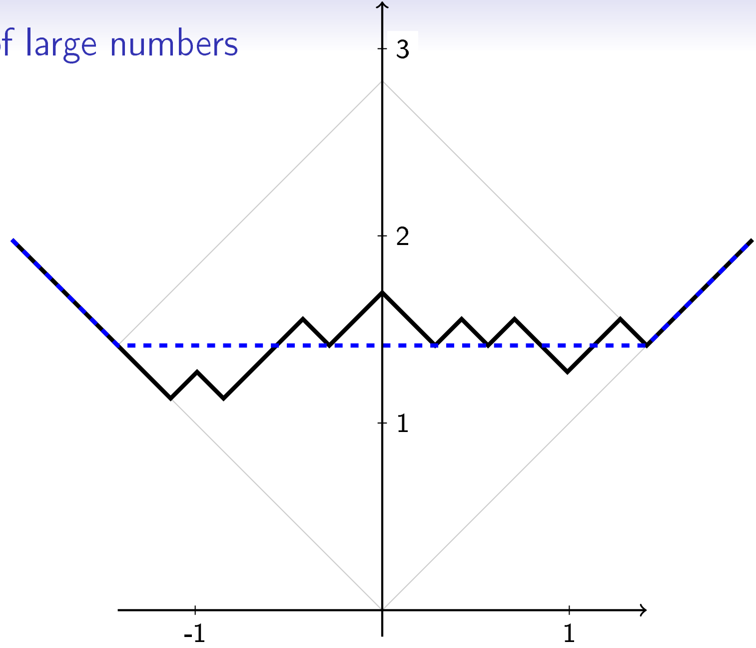
- combinatorics of random Young diagrams $\neq$ combinatorics of random matrics

  $\longrightarrow$ Lecture 3

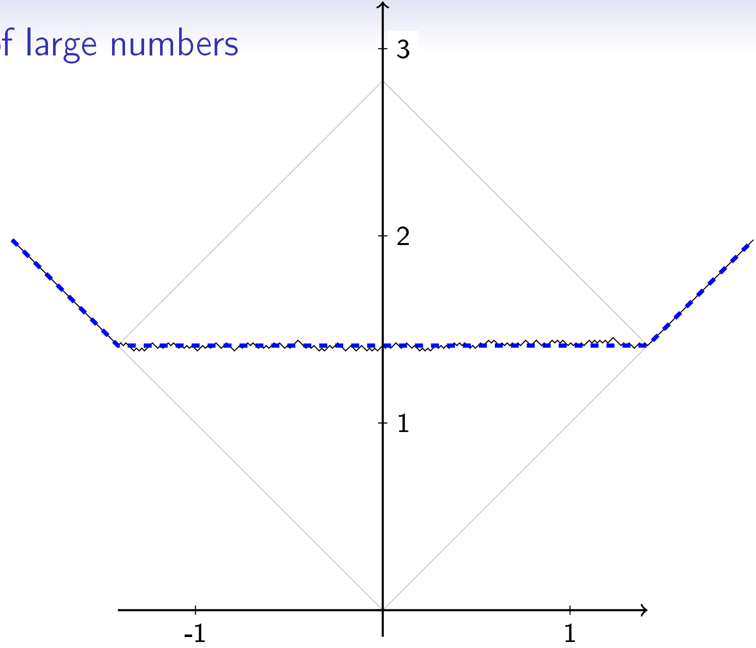
law of large numbers



law of large numbers



law of large numbers



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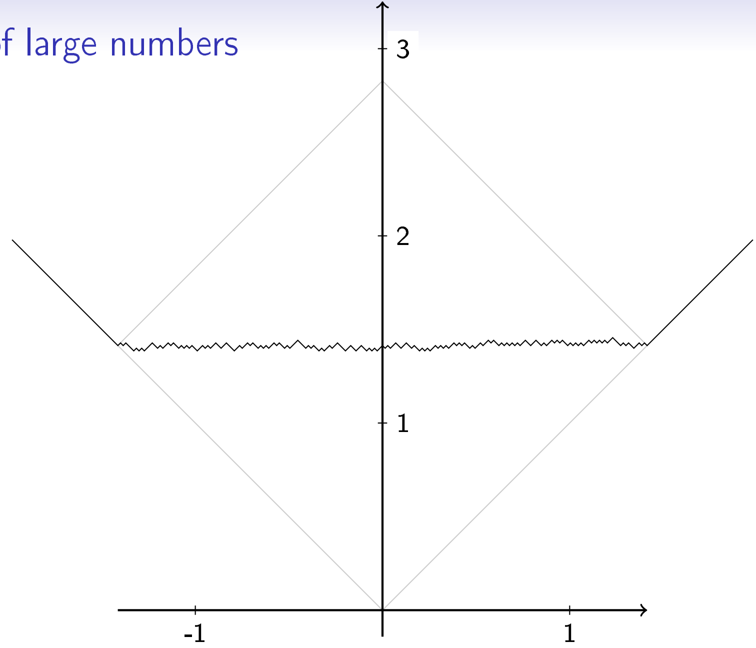
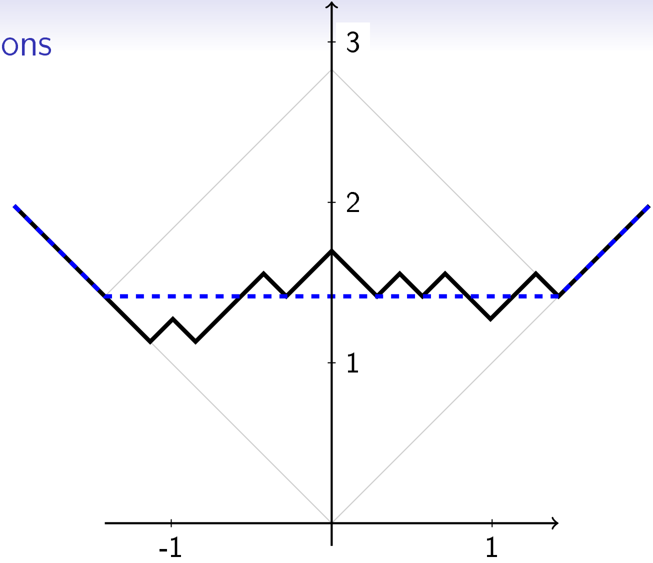


fluctuations



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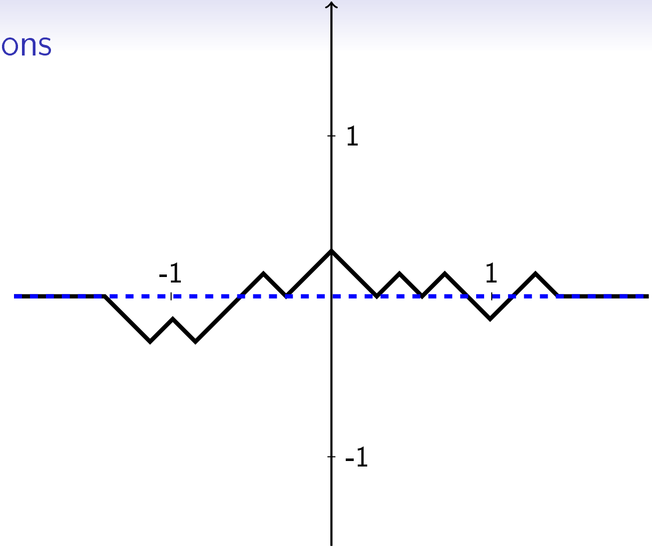

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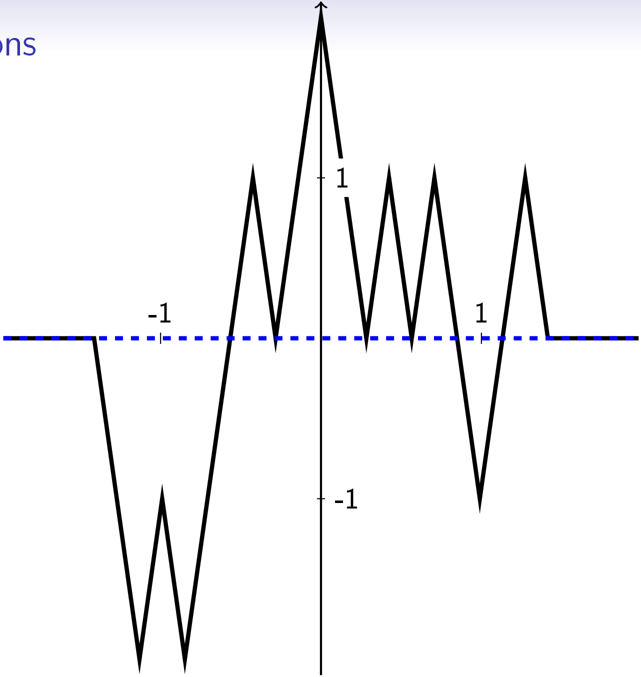

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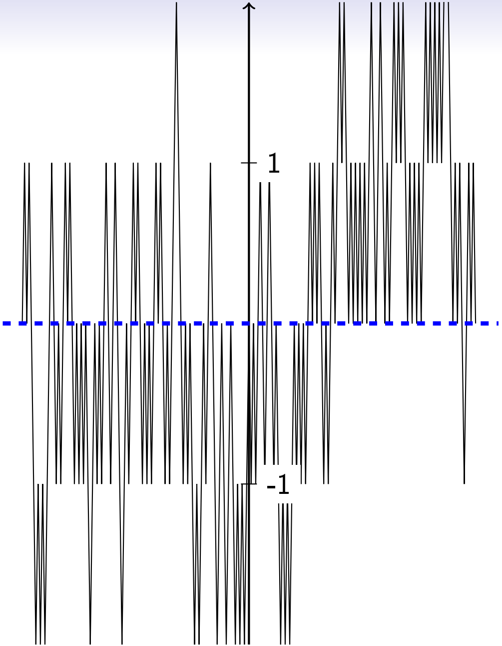


outlook

Lecture 3:
characters, maps,
free cumulants...
and Kerov character polynomials,

$$\overbrace{\mathrm{Ch}_2}^{\text{character}} = \overbrace{R_3}^{\text{shape}},$$

$$\mathrm{Ch}_3 = R_4 + R_2,$$

$$\mathrm{Ch}_4 = R_5 + 5R_3,$$

$$\mathrm{Ch}_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2,$$

$$\mathrm{Ch}_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3$$

further reading

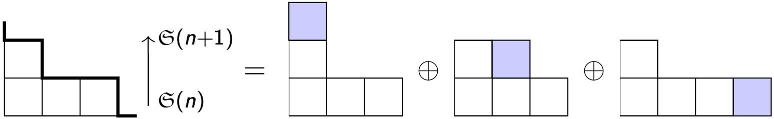
- Jonathan Novak, Piotr Śniady.
  What is. . . free cumulant?
  Notices Amer. Math. Soc. 58 (2011), no. 2, 300-301
- Piotr Śniady.
  Combinatorics of asymptotic representation theory.
  European Congress of Mathematics, 531–545, Eur. Math. Soc., Zürich, 2013
- Piotr Śniady.
  Gaussian fluctuations of characters of symmetric groups and of Young diagrams.
  Probab. Theory Related Fields 136 (2006), no. 2, 263–297

step 1: induced representation $\rho^\lambda\big\uparrow_{\mathfrak{S}(n)}^{\mathfrak{S}(n+1)}$

how to define the action of

$\mathfrak{S}(n+1) = \big\{(1, n+1), (2, n+1), \dots, (n, n+1), \mathrm{id}\big\} \times \mathfrak{S}(n)$

on $\big\{(1, n+1), (2, n+1), \dots, (n, n+1), \mathrm{id}\big\} \times V^\lambda$?



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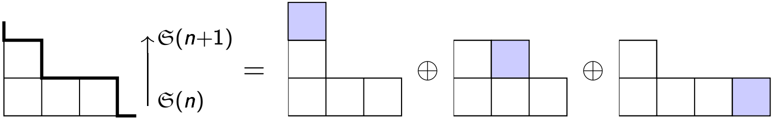

Jucys-Murphy element

$$X_{n+1} = (1,n+1) + (2,n+1) + \cdots + (n,n+1) \in \mathbb{C}[\mathfrak{S}(n+1)]$$

step 1: induced representation $\rho^\lambda \uparrow_{\mathfrak{S}(n)}^{\mathfrak{S}(n+1)}$

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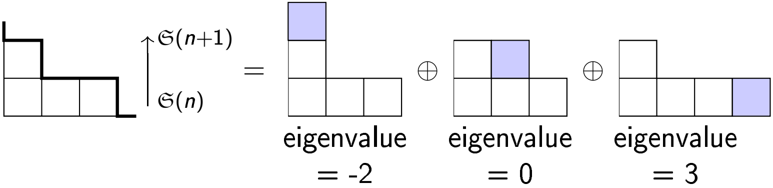

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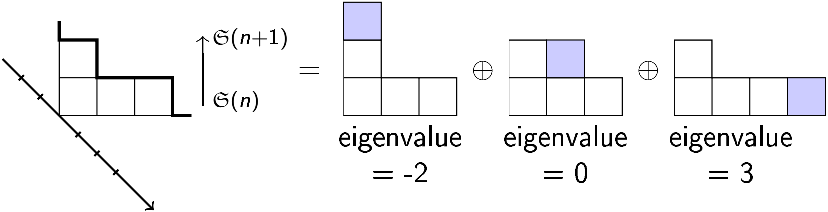

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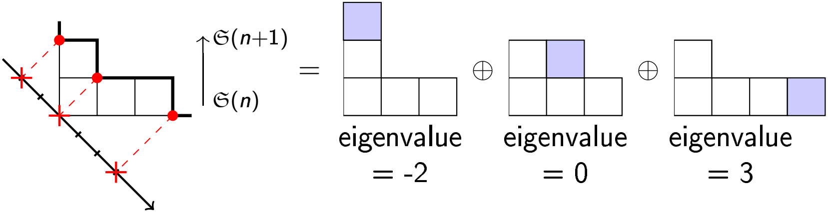

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Free cumulants for free probability people 1

Denote $\star = n+1$. Jucys-Murphy element is defined by

$$J = (1\star) + \cdots + (n\star) \in \mathbb{C}\big(S(n+1)\big).$$

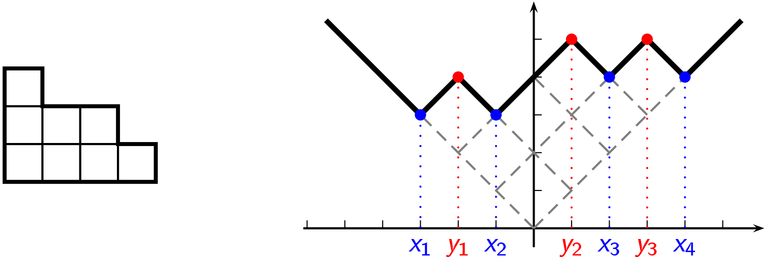
Let ρ^λ be an irreducible representation of $S(n)$.
We equip $\mathbb{C}\big(S(n+1)\big)$ with an expected value:

$$\mathbb{E}X = \chi^\lambda\left(X \downarrow^{S(n+1)}_{S(n)}\right).$$

Free cumulants of Young diagram λ are just free cumulants of Jucys-Murphy element with respect to this expected value:

$$R_k^\lambda = R_k(J).$$

Free cumulants for free probability people 2



Cauchy transform of a Young diagram:

$$G^{\lambda}(z) = \frac{(z - y_1)\cdots(z - y_{s-1})}{(z - x_1)\cdots(z - x_s)}.$$

Free cumulants of λ are the free cumulants related to G^{λ}.