Series of lectures: characters, maps, free cumulants

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plan of the series of lectures

Lecture 1: characters, maps, free cumulants...
and Stanley character formula,

Lecture 2: characters, maps, free cumulants...
and random Young diagrams,

Lecture 3: characters, maps, free cumulants...
and Kerov character polynomials,

Lecture 3a: characters, maps, free cumulants... and Kerov polynomials

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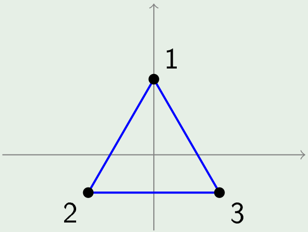
representations 1

representation theory: how an abstract group can be concretely realized as a group of matrices?

Example

symmetric group $\mathfrak{S}(3)$
permutations of $\{1, 2, 3\}$



formal definition: representation ρ of a group G is a homomorphism

$$\rho \colon G \to \mathsf{GL}_d$$

from the group to invertible matrices

irreducible representations

representation ρ is called reducible
if can be written as a direct sum of smaller representations:

$$\rho(g) = \begin{bmatrix} \rho_1(g) & \\ & \rho_2(g) \end{bmatrix} \qquad \text{for every } g \in G;$$

we are interested in irreducible representations

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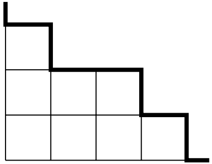
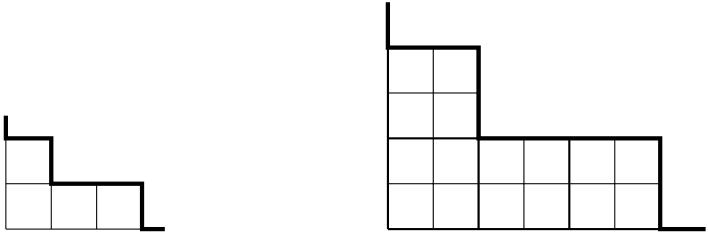
irreducible representation ρ^λ of the symmetric group $\mathfrak{S}(n)$ $\longleftrightarrow$ Young diagram λ with n boxes



shape of Young diagram



Young diagram λ dilated diagram 2λ

goal for today:

$$\text{study } \rho^{s\lambda} \text{ for } s \to \infty$$

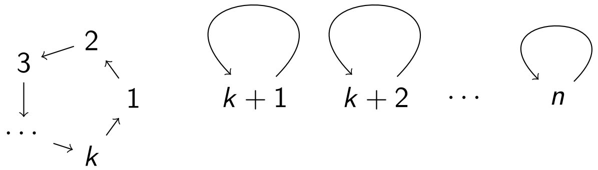
characters and free cumulants

normalized character: for a Young diagram λ with n boxes

$$\mathrm{Ch}_k(\lambda) = \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda(k,1,1,\dots,1)}{\text{dimension of } \rho^\lambda} =$$

$$n^{\underline{k}} \operatorname{tr} \rho^\lambda(k,1,1,\dots,1)$$

$$(k,1,\dots,1) =$$



characters and free cumulants

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$$n^{\underline{k}} \operatorname{tr} \rho^\lambda(k,1,1,\dots,1)$$

$s \mapsto \mathrm{Ch}_k(s\lambda)$ is a polynomial of degree $k+1$

free cumulants $R_2(\lambda), R_3(\lambda), \dots$ are top-degree coefficients:

$$R_{k+1}(\lambda) := \lim_{s\to\infty} \frac{1}{s^{k+1}} \mathrm{Ch}_k(s\lambda)$$

Kerov polynomials

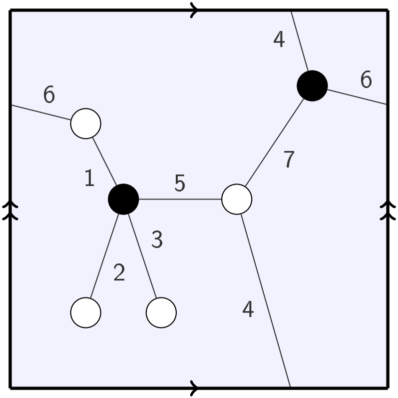
$$\overbrace{\mathrm{Ch}_2}^{\text{character}} = \overbrace{R_3}^{\text{shape}},$$
$$\mathrm{Ch}_3 = R_4 + R_2,$$
$$\mathrm{Ch}_4 = R_5 + 5R_3,$$
$$\mathrm{Ch}_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2,$$
$$\mathrm{Ch}_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3$$

Kerov positivity conjecture:
the coefficients are non-negative integers;
what is their combinatorial meaning?

maps

map

- is a graph drawn on an oriented surface,
- bipartite,
- with one face,
- rooted / with cleverly labeled edges,

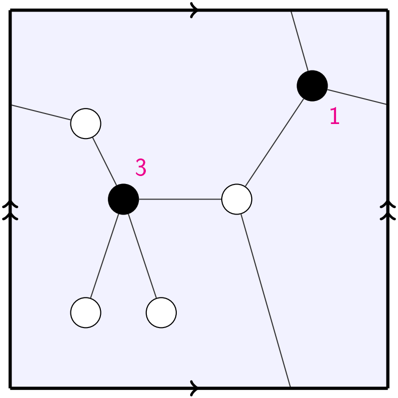


what Kerov polynomials count?

coefficient of $R_{i_1} \cdots R_{i_\ell}$ in Ch_k counts the number of maps with k edges

with black vertices labelled by $i_1 - 1, \dots, i_\ell - 1$,

and such that the map with this labelling is an expander



→ Féray, Dołęga & Śniady

expanders: factories of interesting liquids

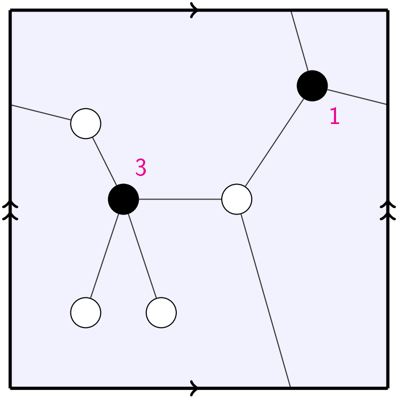
coefficient of $R_{i_1} \cdots R_{i_\ell}$ in Ch_k counts the number of maps with k edges

with black vertices labelled by $i_1 - 1, \dots, i_\ell - 1$,

each black vertex v produces v units of liquid,

each white vertex demands 1 unit of the liquid,

each edge transports strictly positive amout of liquid from black to white vertex



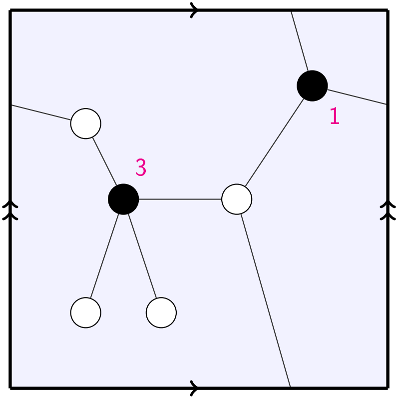
→ Féray, Dołęga & Śniady

expanders: Hall marriage on steroids

expander:

- number of white vertices
  is equal to
  the sum of all labels,

- each non-trivial set of
  black vertices
  has more
  white neighbors than
  the sum of its labels,

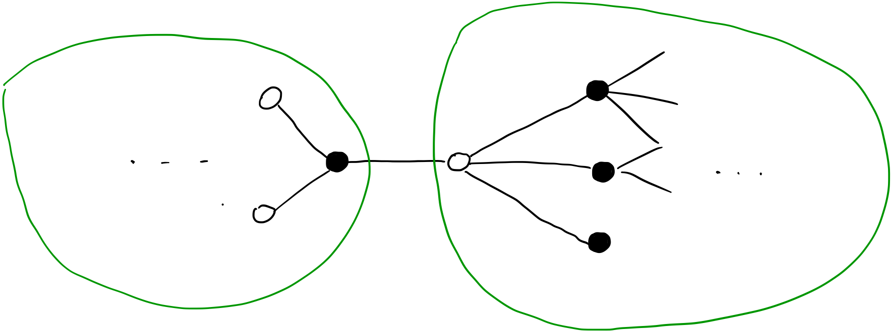


→ Féray, Dołęga & Śniady

no tree-like parts

if a connected graph G
has a disconnecting edge (which is not a leaf)

then G is not an expander (no matter which labeling you try)



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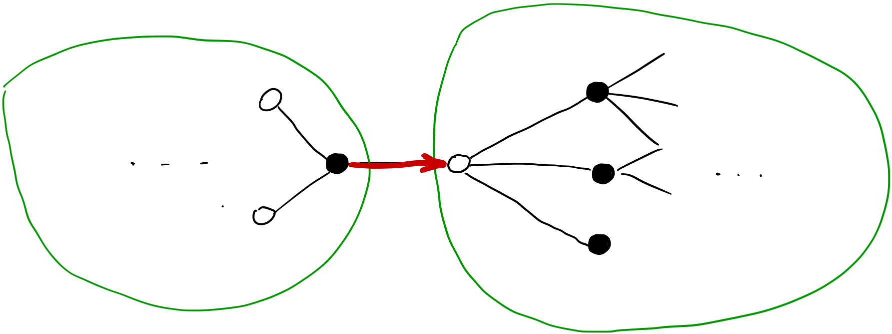

what amount of liquid?

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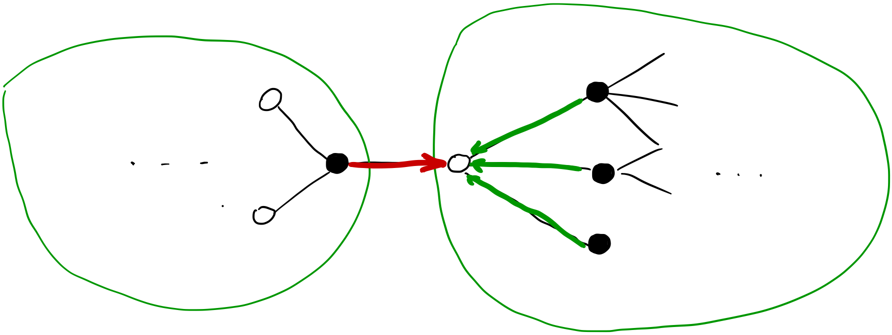
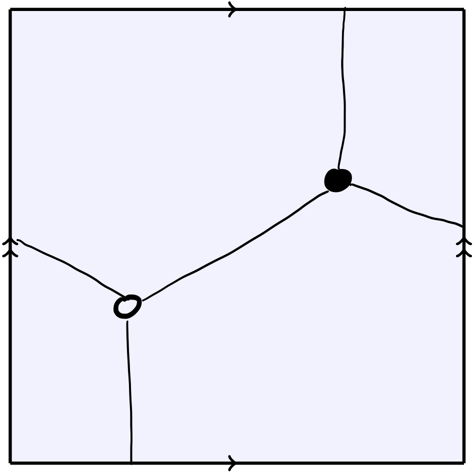


what amount of liquid?

there are not many expanders with fixed genus

with one face



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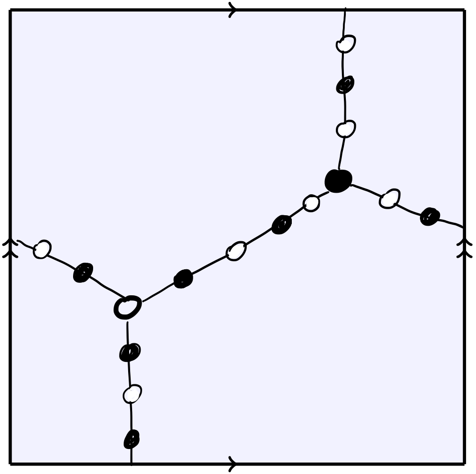

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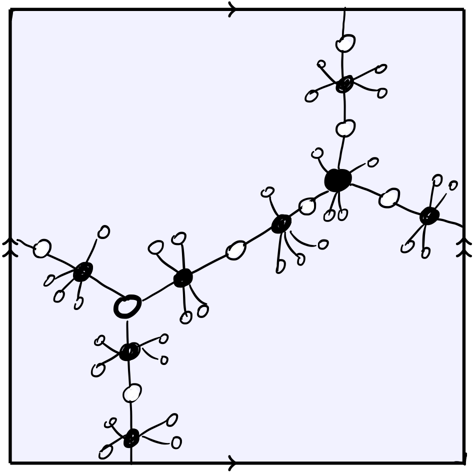

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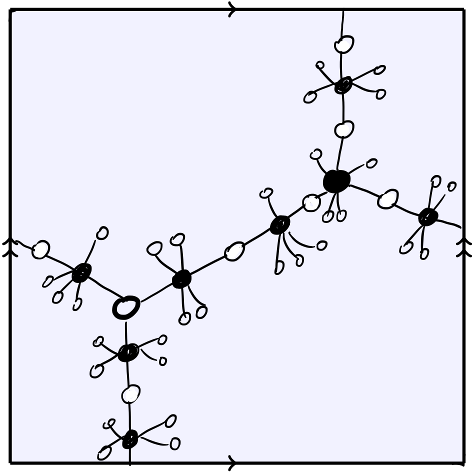
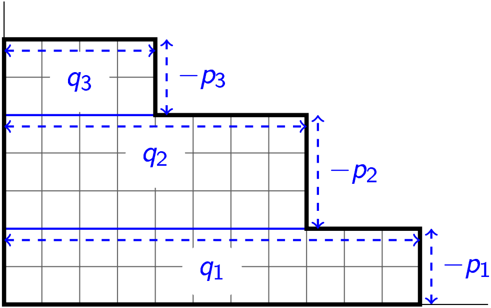


corollary: coefficients of Kerov polynomials are small

Stanley coordinates

$-\mathbf{p} \times \mathbf{q} =$



if $\lambda \mapsto F(\lambda)$ is a nice function on Young diagrams,
it is a good idea to study the polynomial

$$F(-\mathbf{p} \times \mathbf{q})$$

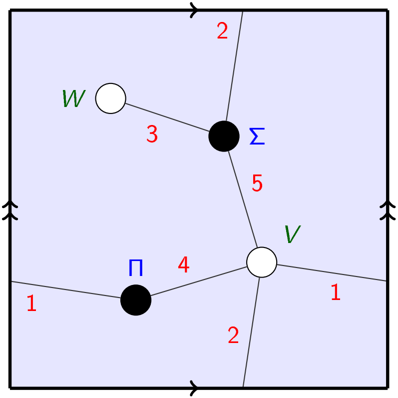
characters in Stanley coordinates

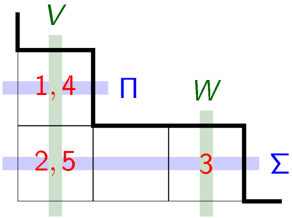
$$-\mathrm{Ch}_1 = p_1q_1 + p_2q_2 + p_3q_3,$$

$$\begin{aligned}-\mathrm{Ch}_2 = {} & p_1^2q_1 + p_1q_1^2 + 2p_1p_2q_2 + p_2^2q_2 + p_2q_2^2 + 2p_1p_3q_3 + \\ & + 2p_2p_3q_3 + p_3^2q_3 + p_3q_3^2,\end{aligned}$$

$$\begin{aligned}-\mathrm{Ch}_3 = {} & p_1^3q_1 + 3p_1^2q_1^2 + p_1q_1^3 + 3p_1^2p_2q_2 + 3p_1p_2^2q_2 + p_2^3q_2 + \\ & + 3p_1p_2q_1q_2 + 3p_1p_2q_2^2 + 3p_2^2q_2^2 + p_2q_2^3 + 3p_1^2p_3q_3 + 6p_1p_2p_3q_3 + \\ & + 3p_2^2p_3q_3 + 3p_1p_3^2q_3 + 3p_2p_3^2q_3 + p_3^3q_3 + 3p_1p_3q_1q_3 + \\ & + 3p_2p_3q_2q_3 + 3p_1p_3q_3^2 + 3p_2p_3q_3^2 + 3p_3^2q_3^2 + p_3q_3^3 + p_1q_1 + \\ & + p_2q_2 + p_3q_3\end{aligned}$$

embedding of a map to a Young diagram

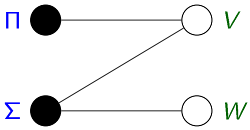


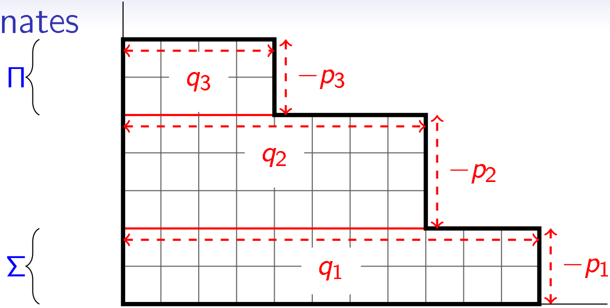


- map edges to boxes;
- map white vertices to columns;
- map black vertices to rows;
- preserve the incidence;

$$\mathfrak{N}_M(\lambda) := (-1)^{\#\text{black vertices}} \, \#\text{embeddings of } M \text{ to } \lambda$$

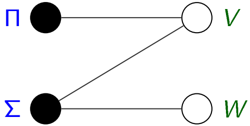
$\mathfrak{N}_M$ in Stanley coordinates

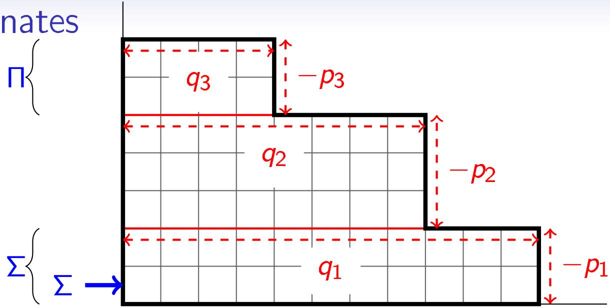




$(-1)^2 \times$

$\mathfrak{N}_M$ in Stanley coordinates





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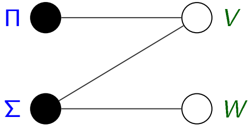

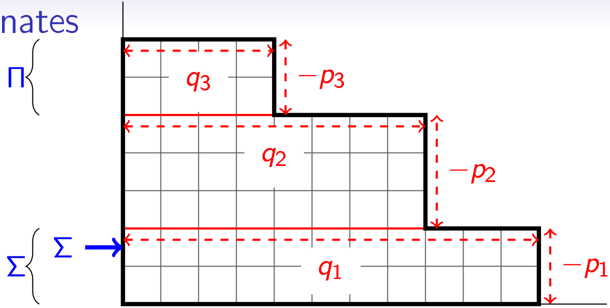

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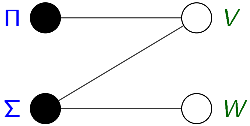
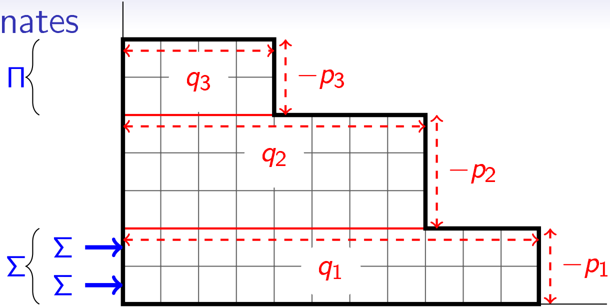
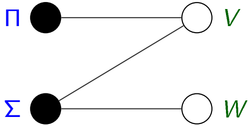
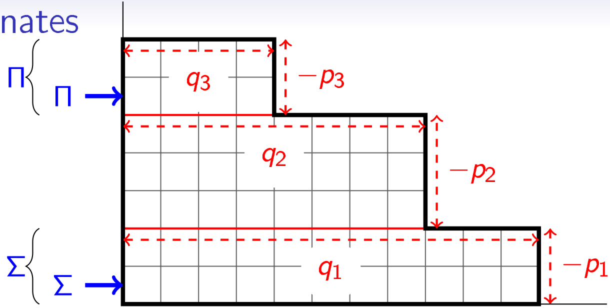




$$(-1)^2 \times (-p_1) \times$$

$\mathfrak{N}_M$ in Stanley coordinates





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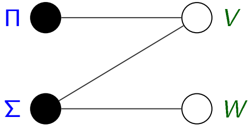

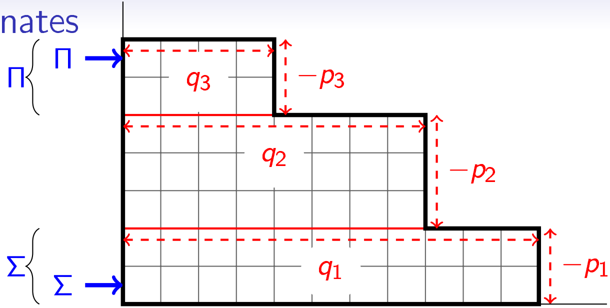

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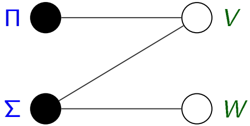
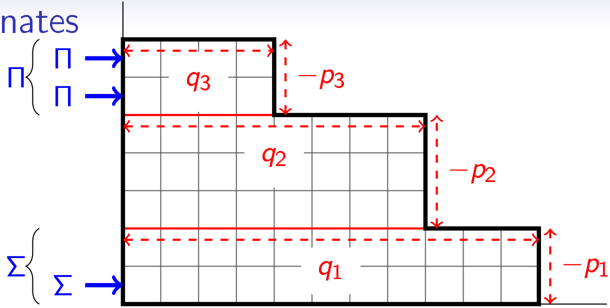
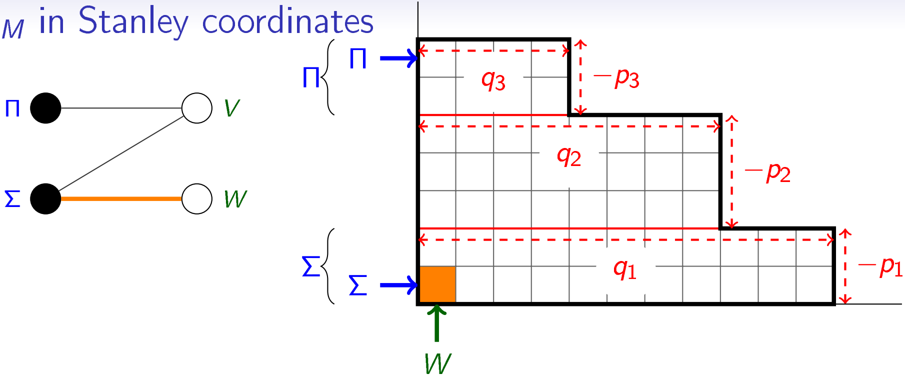




$$(-1)^2 \times (-p_1) \times (-p_3) \times$$

$\mathfrak{N}_M$ in Stanley coordinates



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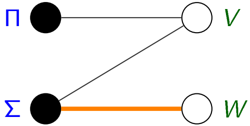

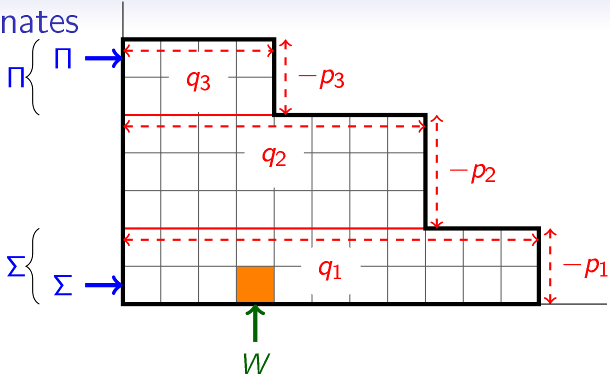

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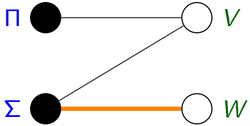

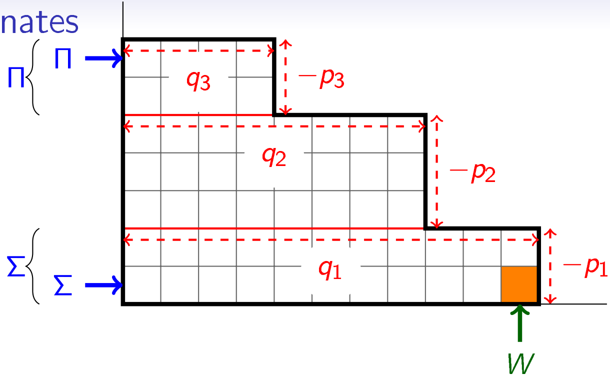

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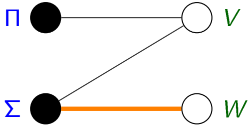
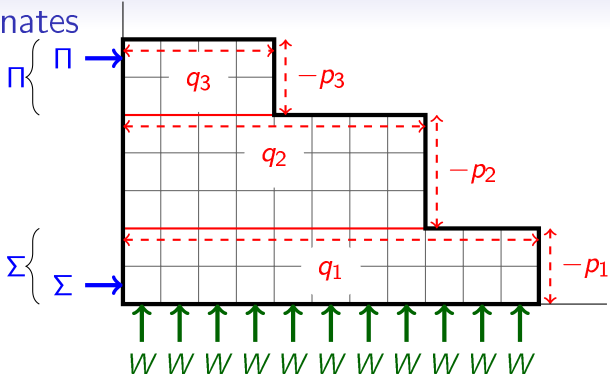
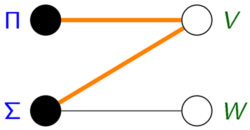
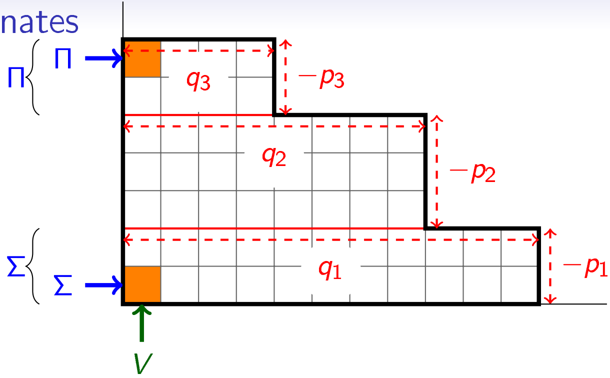




$$(-1)^2 \times (-p_1) \times (-p_3) \times q_1 \times$$

$\mathfrak{N}_M$ in Stanley coordinates





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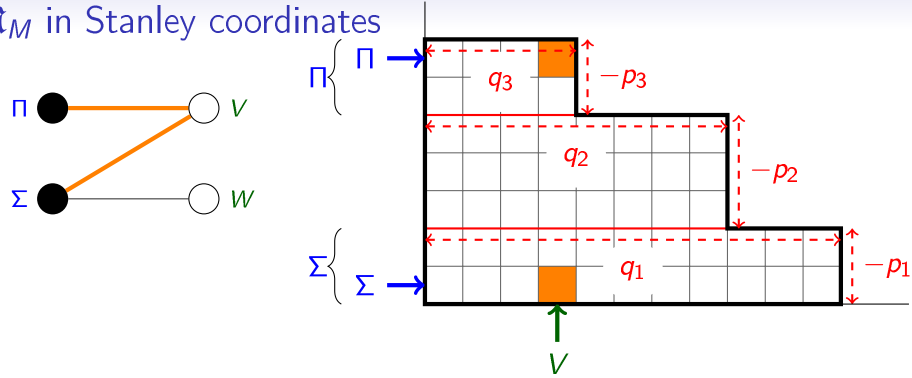

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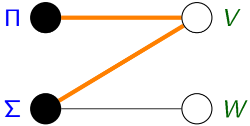

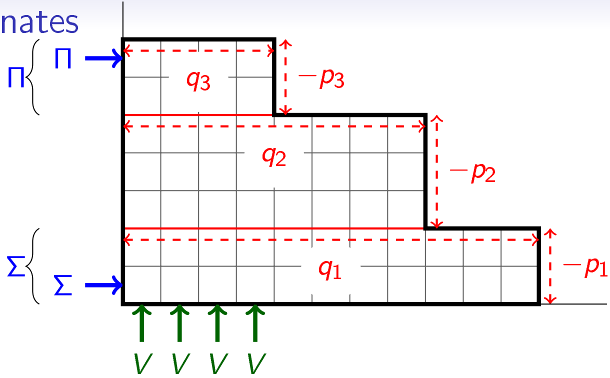

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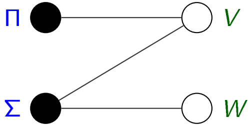
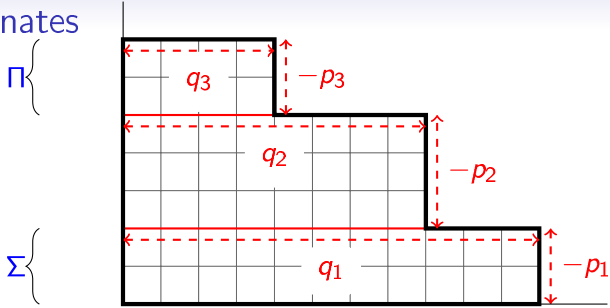
$\mathfrak{N}_M$ in Stanley coordinates





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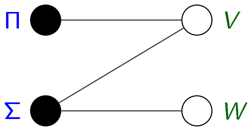
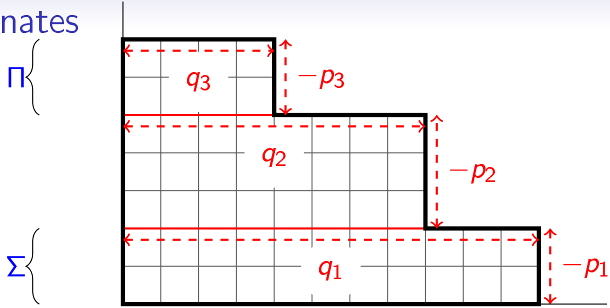
$\mathfrak{N}_M$ in Stanley coordinates

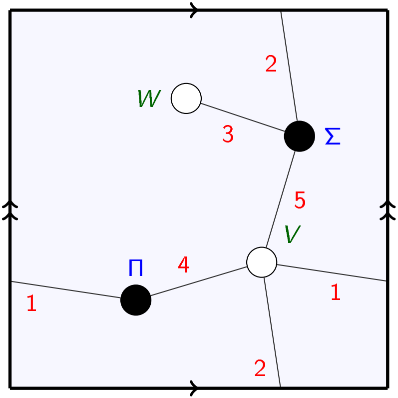


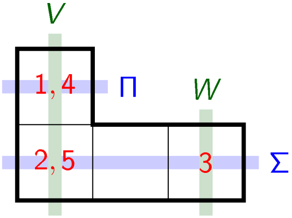


$$\mathfrak{N}_M(-\mathbf{p}\times\mathbf{q}) = \sum_{F:V_\bullet\to\mathbb{N}} \left(\prod_{v\in V_\bullet} p_{F(v)}\right)\left(\prod_{w\in V_\circ} q_{G(w)}\right)$$

$$\text{where } G(w) := \max_{\substack{v\in V_\bullet \\ v \text{ adjacent to } w}} F(v)$$

Stanley's character formula



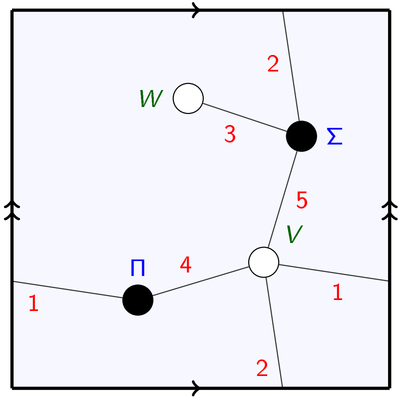


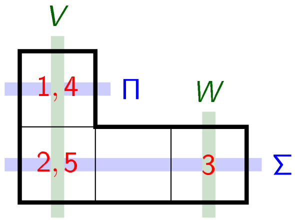
$$\mathfrak{N}_M(\lambda) = (-1)^{|V_\bullet|} \times$$
$$\times \ \# \text{ embeddings of } M \text{ to } \lambda$$

$$-\operatorname{Ch}_k(\lambda) = \sum_M \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with k edges

free cumulants





$$\mathfrak{N}_M(\lambda) = (-1)^{|V_\bullet|} \times$$
$$\times \ \# \text{ embeddings of } M \text{ to } \lambda$$

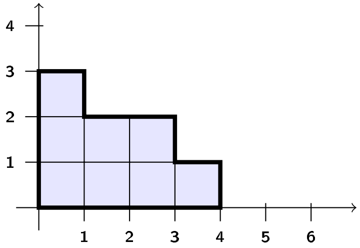
$$-R_{k+1}(\lambda) = \sum_M \mathfrak{N}_M(\lambda),$$

where the sum runs over planar maps M with k edges

functionals of shape

for $k \geq 2$

$$S_k(\lambda) := (k-1) \iint_{(x,y)\in\lambda} (x-y)^{k-2} \, dx \, dy$$



S_k is homogeneous of degree k:

$$S_k(s\lambda) = s^k S_k(\lambda)$$

characters, functionals of shape, free cumulants

$$\mathrm{Ch}_1 = \underbrace{S_2}_{\text{degree 2}},$$

$$\mathrm{Ch}_2 = \underbrace{S_3}_{\text{degree 3}},$$

$$\mathrm{Ch}_3 = \underbrace{S_4 - \frac{3}{2}S_2^2}_{\text{degree 4}} + \underbrace{S_2}_{\text{degree 2}},$$

$$\mathrm{Ch}_4 = \underbrace{S_5 - 4S_2S_3}_{\text{degree 5}} + \underbrace{5S_3}_{\text{degree 3}},$$

$$\mathrm{Ch}_5 = \underbrace{S_6 - 5S_2S_4 - \frac{5}{2}S_3^2 + \frac{25}{6}S_2^3}_{\text{degree 6}} + \underbrace{15S_4 - \frac{35}{2}S_2^2}_{\text{degree 4}} + \underbrace{8S_2}_{\text{degree 2}}.$$

characters, functionals of shape, free cumulants

$$\mathrm{Ch}_1 = \underbrace{S_2}_{R_2},$$

$$\mathrm{Ch}_2 = \underbrace{S_3}_{R_3},$$

$$\mathrm{Ch}_3 = \underbrace{S_4 - \frac{3}{2} S_2^2}_{R_4} + \underbrace{S_2}_{R_2},$$

$$\mathrm{Ch}_4 = \underbrace{S_5 - 4 S_2 S_3}_{R_5} + \underbrace{5 S_3}_{5R_3},$$

$$\mathrm{Ch}_5 = \underbrace{S_6 - 5 S_2 S_4 - \frac{5}{2} S_3^2 + \frac{25}{6} S_2^3}_{R_6} + \underbrace{15 S_4 - \frac{35}{2} S_2^2}_{15R_4 + 5R_2^2} + \underbrace{8 S_2}_{8R_2}.$$

p-square-free terms 1

Theorem

if $F = F(\lambda)$ is a polynomial in $S_2, S_3, \dots$ then for any $k_1, \dots, k_r \geq 2$

$$\frac{\partial}{\partial S_{k_1}} \cdots \frac{\partial}{\partial S_{k_r}} F \Big|_{S_2=S_3=\cdots=0} = (-1)^r [p_1 q_1^{k_1-1} \cdots p_r q^{k_r-1}] F(-\mathbf{p}\times\mathbf{q})$$

Example: for any $k, k_1, k_2 \geq 2$:

$$\begin{aligned} [S_k]F =& (-1)[p_1 q_1^{k-1}] F(-\mathbf{p}\times\mathbf{q}), \\ [S_{k_1} S_{k_2}]F =& [p_1 q_1^{k_1-1} p_2 q_2^{k_2-1}] F(-\mathbf{p}\times\mathbf{q}) \qquad \text{if } k_1 \neq k_2, \\ 2\cdot[S_k^2]F =& [p_1 q_1^k p_2 q_2^k] F(-\mathbf{p}\times\mathbf{q}), \end{aligned}$$

$\mathbf{p}$-square-free terms 1

Theorem

if $F = F(\lambda)$ is a polynomial in $S_2, S_3, \dots$ then for any $k_1, \dots, k_r \geq 2$

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p-square-free terms 1

Theorem

if $F = F(\lambda)$ is a polynomial in $S_2, S_3, \dots$ then for any $k_1, \dots, k_r \geq 2$

$$\frac{\partial}{\partial S_{k_1}} \cdots \frac{\partial}{\partial S_{k_r}} F \bigg|_{S_2=S_3=\cdots=0} = (-1)^r [p_1 q_1^{k_1-1} \cdots p_r q^{k_r-1}] F(-\mathbf{p} \times \mathbf{q})$$

Hint: for $i_1 < \cdots < i_r$ and $r \geq 0$

$$[p_{i_1} \cdots p_{i_r}] \underbrace{S_k(-\mathbf{p} \times \mathbf{q})}_{\text{as polynomial in } \mathbf{p}} = \begin{cases} (-1)\ (k-1)_{r-1}\ \overbrace{q_{i_r}^{k-r}}^{\text{exponent at least 1}} & \text{if } 1 \leq r \leq k-1 \\ 0 & \text{otherwise} \end{cases}$$

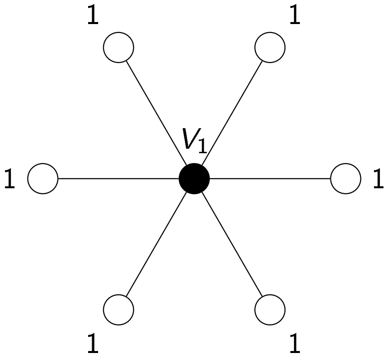
Corollary:

$$(-1)^{\ell-1} \frac{\partial}{\partial S_{i_1}} \cdots \frac{\partial}{\partial S_{i_\ell}} \operatorname{Ch}_k \bigg|_{S_2=S_3=\cdots=0} =$$
$$(-1)[p_1 \cdots p_\ell q_1^{i_1-1} \cdots q_\ell^{i_\ell-1}] \operatorname{Ch}_k(\mathbf{p} \times \mathbf{q}) = \dots$$

number of maps

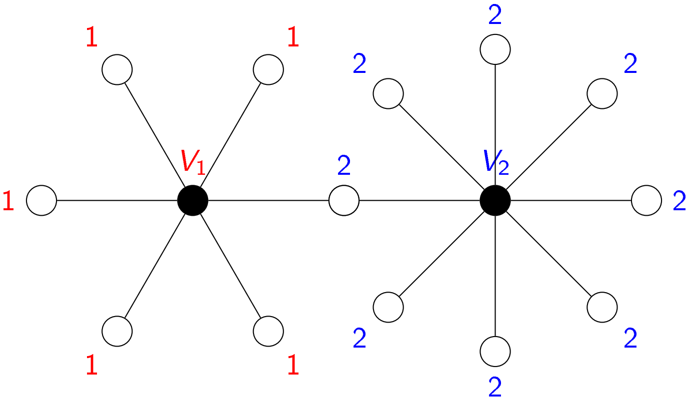
- with k edges,
- which have ℓ black vertices, labeled $V_1, \dots, V_\ell$,
- and there are $i_\ell - 1$ white vertices attached to V_ℓ,
- there are $i_{\ell-1} - 1$ white vertices which are attached to $V_{\ell-1}$ but not attached to V_ℓ,
- there are $i_{\ell-2} - 1$ white vertices which are attached to $V_{\ell-2}$ but not attached to $V_{\ell-1}, V_r$,
- ...
- there are $i_1 - 1$ white vertices which are attached to V_1 but not attached to $V_2, \dots, V_\ell$,

free cumulants in terms of functionals of shape 1



$$R_{k+1} = S_{k+1} + \cdots$$

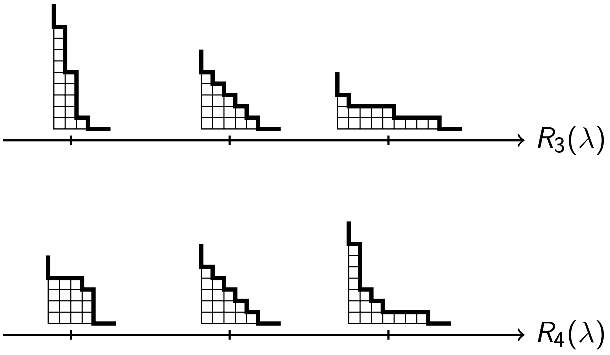
free cumulants in terms of functionals of shape 2



$$R_{k+1} = \cdots - \frac{1}{2} k \sum_{\substack{i_1, i_2 \geq 2, \\ i_1 + i_2 = k+1}} S_{i_1} S_{i_2} + \cdots$$

free cumulants in terms of functionals of shape 3

$$R_{k+1} = S_{k+1} - \frac{1}{2!} k \sum_{\substack{i_1, i_2 \geq 2, \\ i_1+i_2=k+1}} S_{i_1} S_{i_2} + \frac{1}{3!} k^2 \sum_{\substack{i_1, i_2, i_3 \geq 2, \\ i_1+i_2+i_3=k+1}} S_{i_1} S_{i_2} S_{i_3} - \cdots$$



$\mathbf{p}$-square-free terms 2

if $F = F(\lambda)$ is a polynomial in $S_2, S_3, \dots$ then for any $k_1, k_2 \geq 2$

$$[p_1 q_1^{k_1 - 1} p_2 q_2^{k_2 - 1}] F(\mathbf{p} \times \mathbf{q}) =$$
$$\frac{\partial}{\partial S_{k_1}} \frac{\partial}{\partial S_{k_2}} F \bigg|_{S_2 = S_3 = \cdots = 0} =$$
$$\frac{\partial}{\partial S_{k_2}} \frac{\partial}{\partial S_{k_1}} F \bigg|_{S_2 = S_3 = \cdots = 0} =$$
$$[p_1 q_1^{k_2 - 1} p_2 q_2^{k_1 - 1}] F(\mathbf{p} \times \mathbf{q})$$

p-square-free terms 3

if $F = F(\lambda)$ is a polynomial in $S_2, S_3, \dots$ then for any $k \geq 3$

$$[p_1 p_2 q_1^0 q_2^{k-2}]F = -(k-1)\ [S_k]F = (k-1)[p_1 q_1^{k-1}]F$$

Hint: for $i_1 < \cdots < i_r$ and $r \geq 0$

$$[p_{i_1} \cdots p_{i_r}]\ \underbrace{S_k(-\mathbf{p} \times \mathbf{q})}_{\text{as polynomial in } \mathbf{p}} = \begin{cases} (-1)\ (k-1)_{r-1}\ \overbrace{q_{i_r}^{k-r}}^{\text{exponent at least 1}} & \text{if } 1 \leq r \leq k-1 \\ 0 & \text{otherwise} \end{cases}$$

toy example: $[R_{k_1} R_{k_2}]F$

Theorem

if $F = F(\lambda)$ *is a polynomial in* $R_2, R_3, \dots$ *then*

$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} F \bigg|_{R_2 = R_3 = \cdots = 0} = [p_1 p_2 q_1^{k_1 - 1} q_2^{k_2 - 1}] F(\mathbf{p} \times \mathbf{q}) - [p_1 p_2 q_2^{k_1 + k_2 - 2}] F(\mathbf{p} \times \mathbf{q})$$

Hint:

$$\frac{\partial}{\partial R_{k_1}} \frac{\partial}{\partial R_{k_2}} F = \frac{\partial}{\partial S_{k_1}} \frac{\partial}{\partial S_{k_2}} F + (k_1 + k_2 - 1) \frac{\partial}{\partial S_{k_1 + k_2}} F = [p1 p_2 q_1^{k_1 - 1} q_2^{k_2 - 1}] F + (k_1 + k_2 - 1) [p_1 q_1^{k_1 + k_2 - 1}] F$$

toy example: $[R_{k_1} R_{k_2}] \operatorname{Ch}_n$

We are interested in maps with $k_1 + k_2 - 2$ white and two black vertices V_1, V_2.

$$\#(\text{maps such that } V_1 \text{ has} \geq k_1 \text{ friends, } V_2 \text{ has} \geq k_2 \text{ friends}) =$$

$$\#(\text{all maps}) - \#(\text{maps such that } V_1 \text{ has} \leq k_1 - 1 \text{ friends})$$

$$-\#(\text{maps such that } V_2 \text{ has} \leq k_2 - 1 \text{ friends}) =$$

$$(-1) \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq j}} \left[p_1 p_2 q_1^i q_2^j\right] \operatorname{Ch}_k^{\mathbf{p}\times\mathbf{q}} + \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq i\leq k_1-1}} \left[p_1 p_2 q_1^j q_2^i\right] \operatorname{Ch}_k^{\mathbf{p}\times\mathbf{q}}$$

$$+ \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq j\leq k_2-1}} \left[p_1 p_2 q_1^i q_2^j\right] \operatorname{Ch}_k^{\mathbf{p}\times\mathbf{q}} =$$

$$[p_1 p_2 q_1^{k_1-1} q_2^{k_2-1}] \operatorname{Ch}_n^{\mathbf{p}\times\mathbf{q}} - [p_1 p_2 q_2^{k_1+k_2-2}] \operatorname{Ch}_n^{\mathbf{p}\times\mathbf{q}}$$

$$\mathrm{Ch}_k - \underbrace{R_{k+1}}_{\text{degree } k+1} = \frac{(k+1)k(k-1)}{24} \underbrace{C_{k-1}}_{\text{degree } k-1} + \cdots$$

with

$$C_k = \sum_{i_1+\cdots+i_\ell=k} \prod_{1\leq s\leq \ell} (i_s - 1) R_{i_s}$$

$$\vdots$$

$$\mathrm{Ch}_6 - R_7 = \frac{35}{4} C_5 + 42 C_3,$$

$$\mathrm{Ch}_7 - R_8 = 14 C_6 + \frac{469}{3} C_4 + \frac{203}{3} C_2^2 + 180 C_2.$$

→Goulden & Rattan

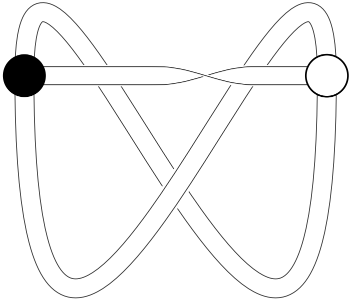
positivity?

Jack deformation

$$\mathrm{Ch}_2 = R_3 + R_2\gamma,$$

$$\mathrm{Ch}_3 = R_4 + 3R_3\gamma + 2R_2\gamma^2 + R_2,$$

$$\mathrm{Ch}_4 = R_5 + 6R_4\gamma + R_2^2\gamma + 11R_3\gamma^2 + 6R_2\gamma^3 + 5R_3 + 7R_2\gamma.$$



- non-negative integer coefficients?
- non-oriented maps?
- partial results:
  $\longrightarrow$ Workshop on Asymptotic representation theory, February 2017

further reading

Maciej Dołega, Valentin Féray, Piotr Śniady
Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations
Adv. Math. 225 (2010), no. 1, 81-120

Piotr Śniady.
Combinatorics of asymptotic representation theory.
European Congress of Mathematics, 531–545, Eur. Math. Soc., Zürich, 2013

Piotr Śniady.
Stanley character polynomials.
Chapter in the book
The Mathematical Legacy of Richard P. Stanley
American Mathematical Society 2017