SERIES OF LECTURES
CHARACTERS, MAPS, FREE CUMULANTS.
EXERCISES

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List of high priority problems:

- Problem 4.1,
- Problems 4.2 and 4.3,
- Problem 5.8,
- Problems 5.1–5.3,
- Problem 2.1,
- Problem 7.1,
- Problem 3.1.

1. Something concrete

There are three irreducible representations of $\mathfrak{S}(3)$:

- *the trivial representation.* It is a representation on a one-dimensional space $\mathbb{R}$ in which each permutation is mapped to $[1]$ (it is a 1×1 matrix, not a very challenging object). This representation corresponds to the Young diagram (3) which consists of one row with three boxes.
- *the alternating representation.* It is a representation on a one-dimensional space $\mathbb{R}$ in which any permutation π is mapped to $[(-1)^\pi]$ its sign. This representation corresponds to the Young diagram $(1,1,1)$ which consists of one column with three boxes.
- *the fundamental representation.* It is a representation on the plane $\mathbb{R}$ in which permutations from $\mathfrak{S}(3)$ are viewed as permutations of the vertices of an equilateral triangle (the center of the mass in the origin of the coordinate system) and to each permutation we associate the corresponding linear transformation (see Lecture 1). This representation corresponds to the Young diagram $(2,1)$ which looks like a 2×2 square with one box removed.

Problem 1.1. *For each of these Young diagrams λ calculate the corresponding normalized characters $\mathrm{Ch}_\pi(\lambda)$ over all choices of π. If you*

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have too much time, compare these results to the outcome of Stanley character formula.

Problem 1.2. *(time consuming, we will not solve it during the exercise session) How to construct an irreducible* 6*-dimensional representation of* $\mathfrak{S}(5)$ *based on the irreducible* 3*-dimensioinal representation of the alternating group* $\mathfrak{A}(5)$ *presented on the lecture (five cubes inscribed into a dodecahedron)? What you constructed is called* induced representation*; it happens to correspond to the Young diagram* $(3,1,1)$*. Calculate directly its characters.*

2. Characters

For a partition π of k and a Young diagram λ with n boxes the corresponding *normalized character* is defined as

$$\mathrm{Ch}_\pi(\lambda) = \begin{cases} \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr}\rho^\lambda(\pi,1,1,\dots,1)}{\text{dimension of } \rho^\lambda} & \text{if } n \geq k, \\ 0 & \text{if } n < k. \end{cases}$$

Problem 2.1. *Calculate* $\mathrm{Ch}_1(\lambda)$ *and* $\mathrm{Ch}_{1,1}(\lambda)$*.*

What is the relationship between $\mathrm{Ch}_\pi(\lambda)$ *and* $\mathrm{Ch}_{\pi,1}(\lambda)$*?*

3. Maps

Problem 3.1. *Let an oriented map with labeled edges be given. We denote by* σ_1 *(resp.* σ_2*) the permutation which describes the cyclic structure of the white (resp. black) vertices (going counterclockwise). Prove that the permutation* $\sigma_1\sigma_2$ *gives the structure of the faces of the map. Concrete, simplified version: show that the number of cycles of* $\sigma_1\sigma_2$ *is equal to the number of the faces and that the lengths of the cycles correspond (how?) to the sizes (what does it mean?) of the faces. More sophisticated version: if we go along the boundary of a face, touching it with the left hand and read the label of the every send edge which we traverse (hint: read only the labels which we enter at the white endpoint), obtain the cycle structure of the permutation* $\sigma_1\sigma_2$*.*

4. Embeddings

4.1. Liberal embeddings.

Embdeddings can be defined in three equivalent ways, see below.

Viewpoint 1. Suppose that permutations $\sigma_1, \sigma_2 \in \mathfrak{S}(k)$ and a Young diagram λ are given. A *(liberal) embedding* of (σ_1, σ_2) to λ is a function f which maps $[k] = \{1, \dots, k\}$ to the set of boxes of λ and such that:

- for all $a, b \in [k]$ if a, b belong to the same cycle of σ_1 then boxes $f(a), f(b)$ belong to the same column of λ;
- for all $a, b \in [k]$ if a, b belong to the same cycle of σ_2 then boxes $f(a), f(b)$ belong to the same row of λ.

Viewpoint 2. Alternatively, an embedding is a pair (f_1, f_2) such that:

- $f_1 \colon C(\sigma_1) \to \mathbb{N}$ is a function which maps the set of cycles of σ_1 to the set of columns of λ;
- $f_2 \colon C(\sigma_2) \to \mathbb{N}$ is a function which maps the set of cycles of σ_2 to the set of rows of λ;
- for each pair of cycles $c_1 \in C(\sigma_1)$ and $c_2 \in C(\sigma_2)$ which are not disjoint $c_1 \cap c_2 \neq \emptyset$ we have that $\big(f_1(c_1), f_2(c_2)\big) \in \lambda$, i.e. the box in the intersection of column $f_1(c_1)$ and row $f_2(c_2)$ lies within the Young diagram λ.

Viewpoint 3. For a given map M, an embedding of M is a function which maps: white vertices to columns of the Young diagram λ, black vertices to rows of the Young diagram, and edges to the boxes of the Young diagram, and which preserves the notion of the *incidence*, i.e. white (resp. black) vertex and an incident edge are mapped to some column c (resp. row r) and a box which belongs to c (resp. r).

Problem 4.1. *Make sure that the above three viewpoints are equivalent.*

We define the *normalized number of embeddings* as

$$\mathfrak{N}_{\sigma_1,\sigma_2}(\lambda) = (-1)^{|C(\sigma_2)|} \text{ (the number of embeddings of } (\sigma_2, \sigma_2) \text{ to } \lambda).$$

Problem 4.2. *Evaluate $\mathfrak{N}_{\sigma_1,\sigma_2}(\lambda)$ when $k = 2$, permutations σ_1, σ_2 are fixed (there are only 4 possible choices), and λ is arbitrary.*

Problem 4.3. *Evaluate $\mathfrak{N}_{\sigma_1,\sigma_2}(\lambda)$ when $k = 3$, and*

(a) $\sigma_1 = (1)(2)(3)$ is the identity and $\sigma_2 = (1, 2, 3)$ is the full cycle,
(b) $\sigma_1 = (1, 2)(3)$, $\sigma_2 = (1)(2, 3)$ are transpositions.

5. Liberal Stanley formula

The usual (*liberal*) version of Stanley formula says that for a permutation $\pi \in \mathfrak{S}(k)$

$$\mathrm{Ch}_\pi(\lambda) = (-1)^{|C(\pi)|} \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}(k), \\ \sigma_1\sigma_2 = \pi}} \mathfrak{N}_{\sigma_1,\sigma_2}(\lambda).$$

Problem 5.1. *Use the liberal version of Stanley formula in order to calculate* Ch_2.

Problem 5.2. *Use the liberal version of Stanley formula in order to calculate* Ch_3*.*

Problem 5.3. *Use the liberal version of Stanley formula in order to calculate* $\mathrm{Ch}_{2,1}$*. What is relationship between* Ch_2 *and* $\mathrm{Ch}_{2,1}$*?*

5.1. **Injective embeddings.** An embedding is called *injective* if the corresponding function $f\colon [k] \to \lambda$ is an injection. The quantity $\mathfrak{N}^{\text{injective}}_{\sigma_1,\sigma_2}(\lambda)$ is defined analogously to $\mathfrak{N}_{\sigma_1,\sigma_2}(\lambda)$, but we count only the number of *injective* embeddings.

Problem 5.4. *Solve injective versions of Problems 4.2 and 4.3, i.e. calculate* $\mathfrak{N}^{\textit{injective}}_{\sigma_1,\sigma_2}(\lambda)$*.*

5.2. **Injective versus liberal.** The usual (*liberal*) version of Stanley formula says that for a permutation $\pi \in \mathfrak{S}(k)$

$$\mathrm{Ch}_\pi(\lambda) = (-1)^{|C(\pi)|} \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}(k),\\ \sigma_1\sigma_2=\pi}} \mathfrak{N}_{\sigma_1,\sigma_2}(\lambda).$$

Its injective version says that

$$\mathrm{Ch}_\pi(\lambda) = (-1)^{|C(\pi)|} \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}(k),\\ \sigma_1\sigma_2=\pi}} \mathfrak{N}^{\text{injective}}_{\sigma_1,\sigma_2}(\lambda).$$

Problem 5.5. *Use the liberal and the injective version of Stanley formula in order to calculate* Ch_2*. Make sure that they give the same answer.*

Problem 5.6. *Use the liberal and the injective version of Stanley formula in order to calculate* Ch_3*. Make sure that they give the same answer.*

Problem 5.7. *Use the liberal and the injective version of Stanley formula in order to calculate* $\mathrm{Ch}_{2,1}$*. Make sure that they give the same answer. What is relationship between* Ch_2 *and* $\mathrm{Ch}_{2,1}$*?*

5.3. **Rectangular Young diagrams.** if $p < 0$ and $q > 0$ are integers, we denote by $(-p) \times q = \underbrace{(q, \dots, q)}_{(-p) \text{ times}}$ a rectangular Young diagram with $-p$ rows and q columns.

Problem 5.8. *Use the liberal version of Stanley formula to show that*

$$(-1)^{|\ell(\pi)|} \mathrm{Ch}_\pi\big((-p) \times q\big)$$

is a polynomial in p *and* q *and all of its coefficients are non-negative integers. What is the combinatorial interpretation for the coefficient*

$$\big[p^a q^b\big] \ (-1)^{|\ell(\pi)|} \mathrm{Ch}_\pi\big((-p) \times q\big)$$

standing at the monomial $p^a q^b$?

Problem 5.9. *Try to solve the previous problem using the injective version of Stanley formula. Hint: it is really painful. You should not do it.*

5.4. **Small pieces of the proof.**

Problem 5.10. *Assume that one of the faces of a map M is 2-gon (a polygon which consists of two edges) and this face does not form a connected component. Prove that there are no injective embeddings of this map. Reformulate this result in such a way that the map M is replaced by a pair of permutations.*

Problem 5.11. *Assume that a map M' is given. Let M be a map obtained from M' by adding an additional connected component which consists of two vertices connected by a single edge. What is the relationship between $\mathfrak{N}_{M'}(\lambda)$ and $\mathfrak{N}_M(\lambda)$?*

6. Murnaghan–Nakayama rule

Bonus material. We will not do it during the exercise session.

Problem 6.1. *Check the details in the formulation of Murnaghan–Nakayama rule. Use it to calculate the characters which we already calculated in Problem 1.1.*

Problem 6.2. *Use Murnaghan–Nakayama rule in order to calculate the character $\mathrm{Ch}_2(\lambda)$ on the transposition. Try to estimate the computational complexity. Pro tip: hook-length formula speeds up computations.*

7. Normalized conjugacy classes

Normalized conjugacy clases are defined as follows:

$$A_1 = \sum_{a\in[k]} (a),$$

$$A_2 = \sum_{\substack{a,b\in[k],\\ a\neq b}} (a,b),$$

$$A_3 = \sum_{\substack{a,b,c\in[k],\\ a,b,c\text{all different}}} (a,b,c),$$

$$A_{2,1} = \sum_{\substack{a,b,c\in[k],\\ a,b,c\text{all different}}} (a,b)(c).$$

Definition of A_π for an arbitrary partition π is analogous. For simplicity we consider each normalized conjugacy class as the element of the symmetric group algebra $\mathbb{C}[\mathfrak{S}(k)]$ but for best resuts one should rather use *partial permutations* of Ivanov and Kerov (and an inverse limit of those for $k \to \infty$).

Problem 7.1. *Express:*

(a) $A_2 \cdot A_2$,
(b) $A_2 \cdot A_1$,
(c) $A_1 \cdot A_1$

as a linear combination of normalized conjugacy classes (A_π).

8. Jucys–Murphy elements

Biane's matrix is defined as

$$X_{n+1} = \begin{bmatrix} 0 & \rho^\lambda(1,2) & \cdots & \rho^\lambda(1,n) & 1 \\ \rho^\lambda(2,1) & 0 & \cdots & \rho^\lambda(2,n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^\lambda(n,1) & \rho^\lambda(n,2) & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

Problem 8.1. *Calculate* $\operatorname{tr}\left[(X_{n+1})^2\right]$ *and* $\operatorname{tr}\left[(X_{n+1})^3\right]$.

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