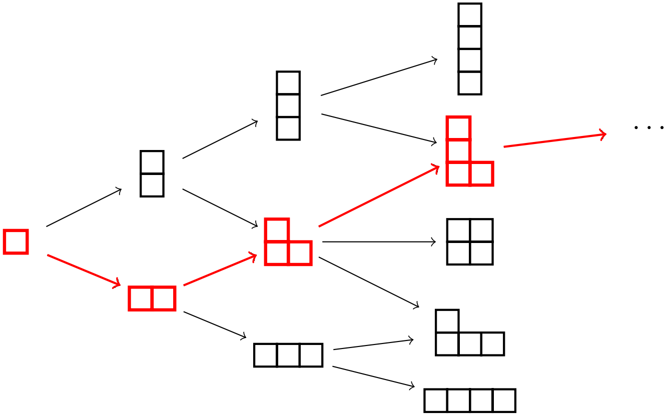
This presentation contains animations which require PDF viewer which accepts JavaScript.

For best results use Acrobat Reader.

Series of lectures:
jeu de taquin and asymptotic representation theory

Piotr Śniady

plan for this series of lectures: representations of the symmetric groups $\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \cdots$ and $\mathfrak{S}_\infty$



plan for this series of lectures:

Lecture 1, August 30

what can we say about RSK
applied to random input?

Lecture 2, September 2

. . . and what does it tell us
about the asymptotic representation theory
of the symmetric groups $\mathfrak{S}_n$ for $n \to \infty$ and $\mathfrak{S}_\infty$?

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Lecture 1A: what can we say about RSK applied to random input?

Piotr Śniady

Polska Akademia Nauk

RSK is a bijection...

Input:

- word $\mathbf{w} = (w_1, \dots, w_n)$

Output:

- semistandard tableau P,
- standard tableau Q,

tableaux P and Q have
the same shape with n boxes

---

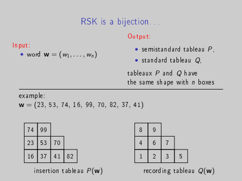
example:
$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathbf{w})$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$



- start with empty tableaux $P := \emptyset$, $Q := \emptyset$;
- read the letters from the word $\mathbf{w}$, one after another;
- for each LETTER:
  - iterate over the rows of the insertion tableau P, start from the first row;
  - insert the LETTER to some box in this row as far to the right as possible, so that the row remains increasing;
  - was this box empty?

    NO the previous tenant must be bumped!
    LETTER:= bumped element;
    proceed to the next row;

    YES update information about the new box into the recording tableau Q,
    proceed to the next letter of the word;

Robinson-Schensted-Knuth algorithm — induction step

74	99		
23	53	70	
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insertion tableau $P(\mathbf{w})$

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recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$$

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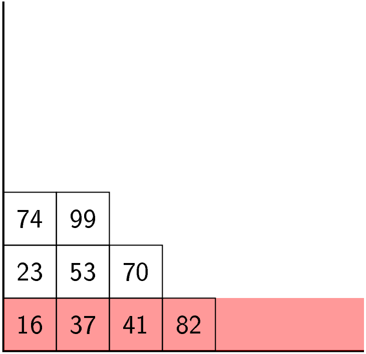
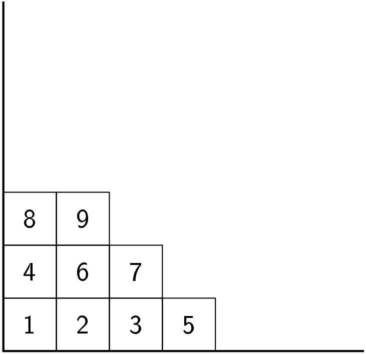
recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

Robinson-Schensted-Knuth algorithm — induction step



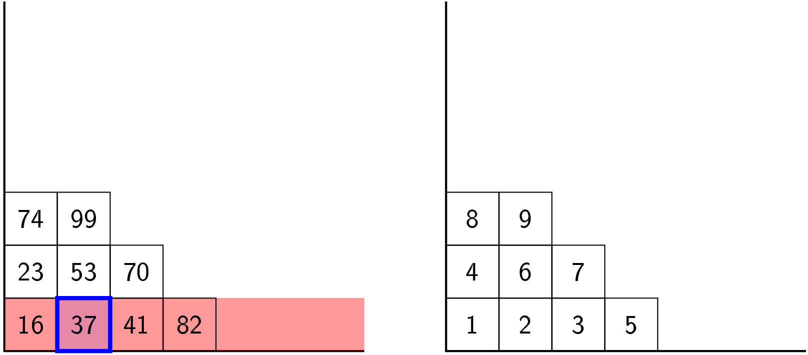
insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step

74	99		
23	53	70	
16	~~37~~	41	82

insertion tableau $P(\mathbf{w})$

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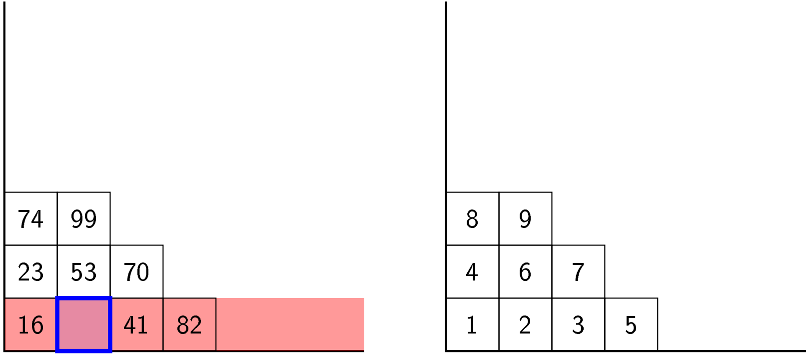

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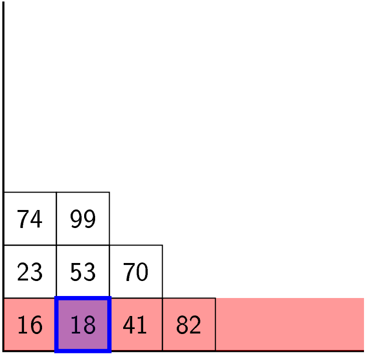
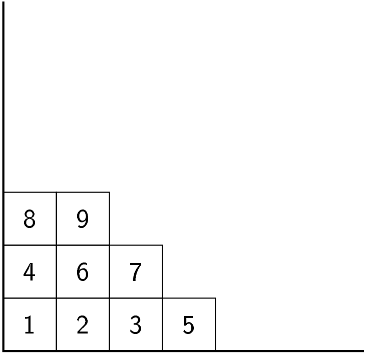


insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

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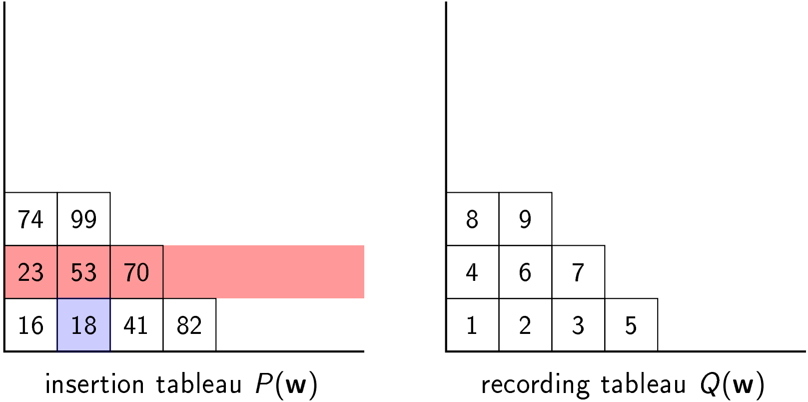
Robinson-Schensted-Knuth algorithm — induction step



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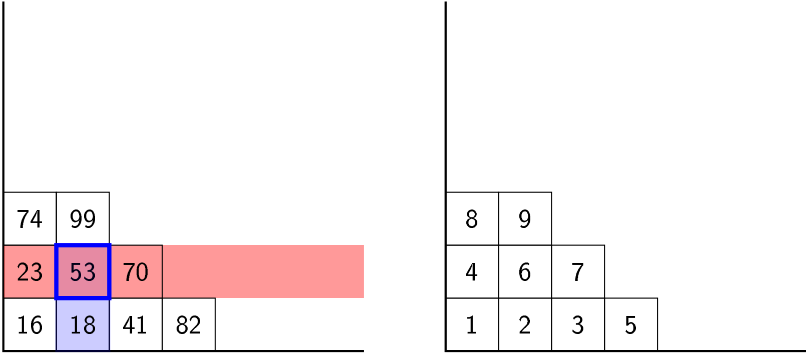
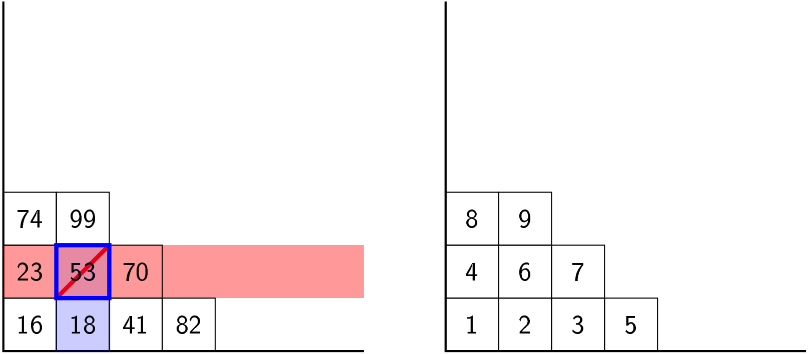
Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

Robinson-Schensted-Knuth algorithm — induction step



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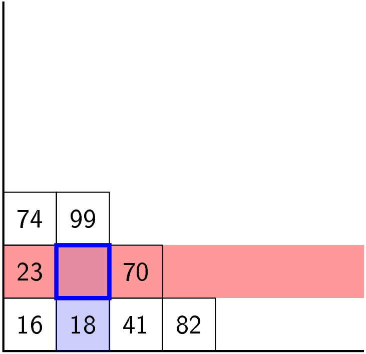
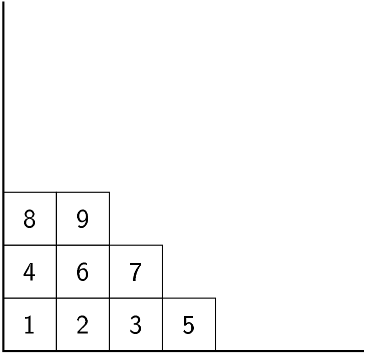


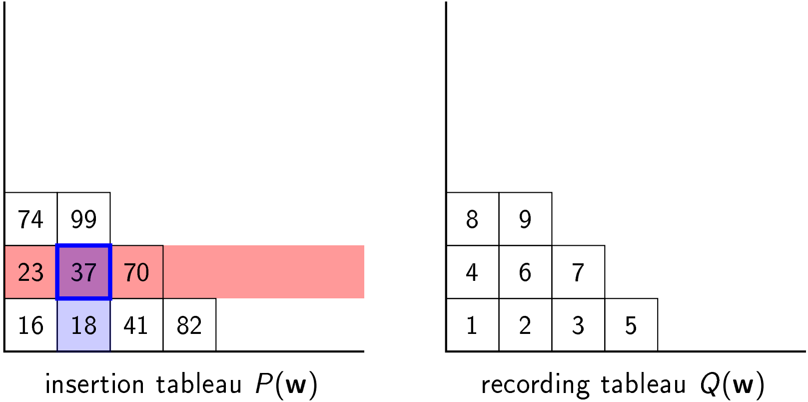
insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

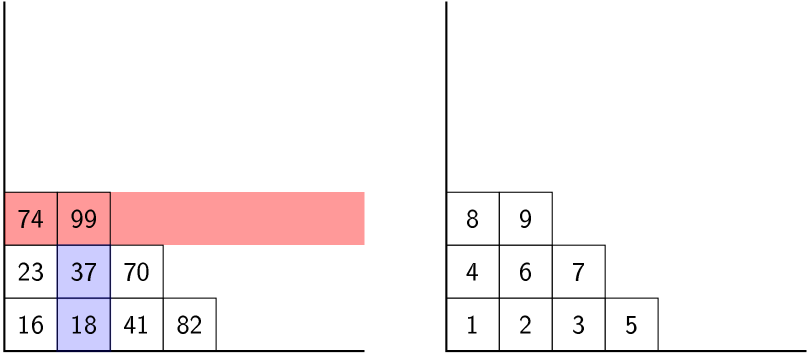
$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

Robinson-Schensted-Knuth algorithm — induction step



$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

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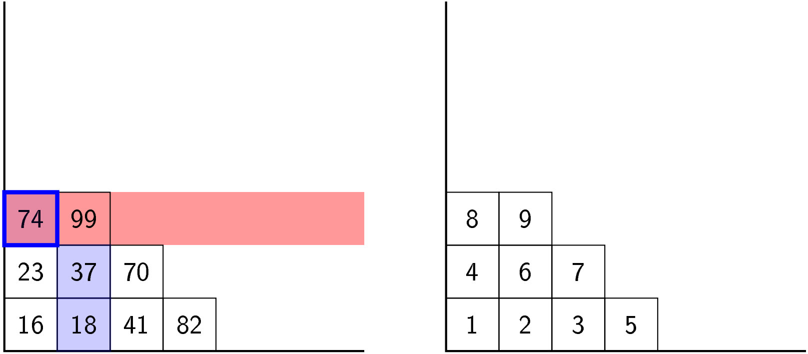

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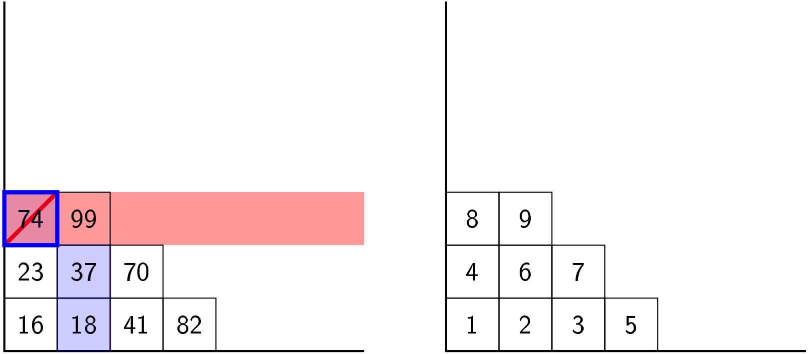

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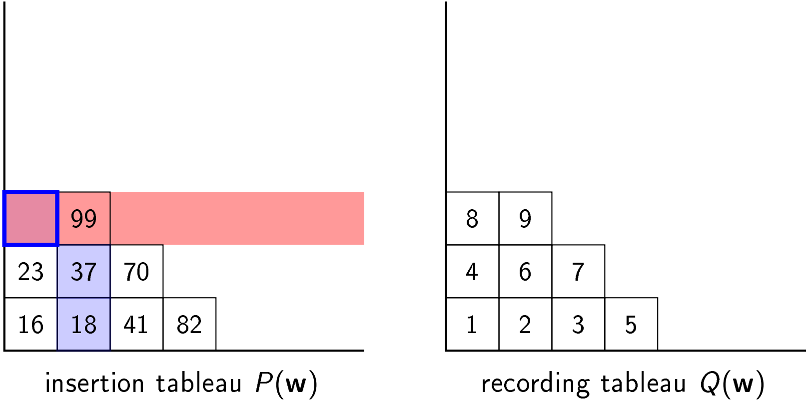

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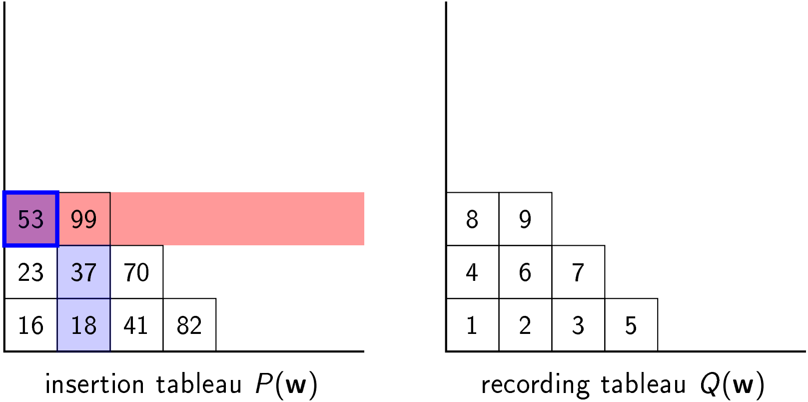

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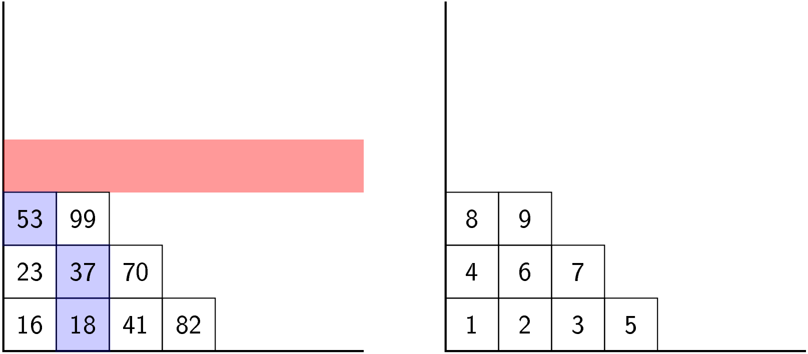

insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

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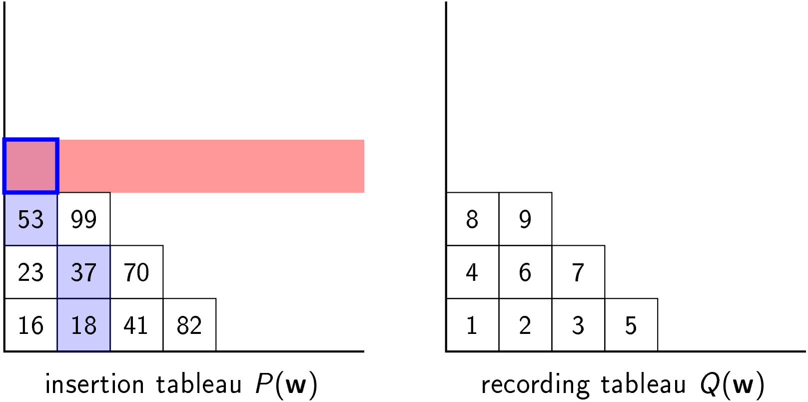
Robinson-Schensted-Knuth algorithm — induction step



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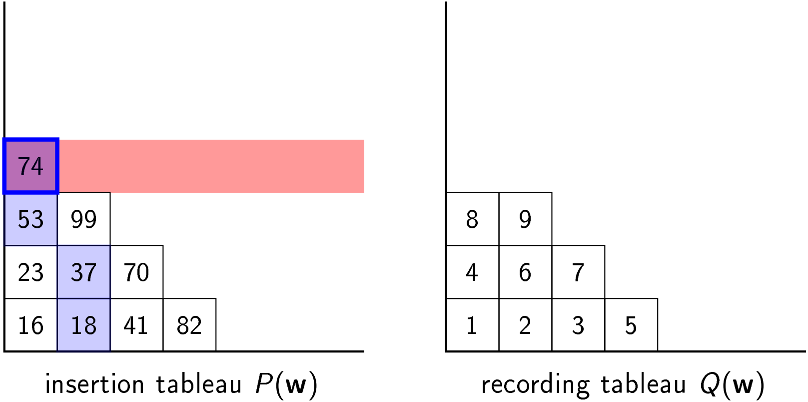

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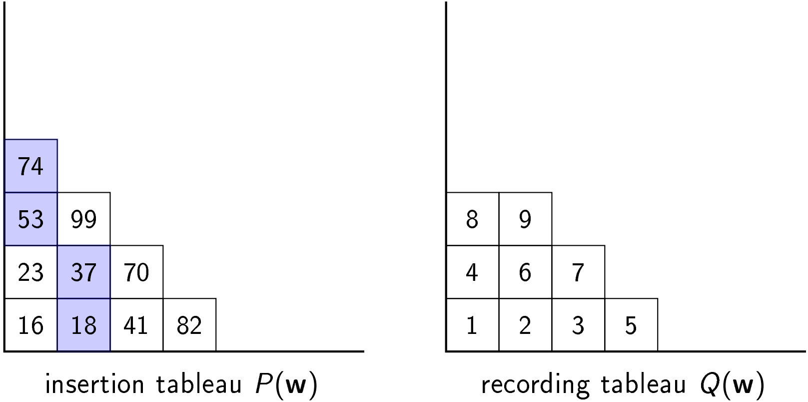

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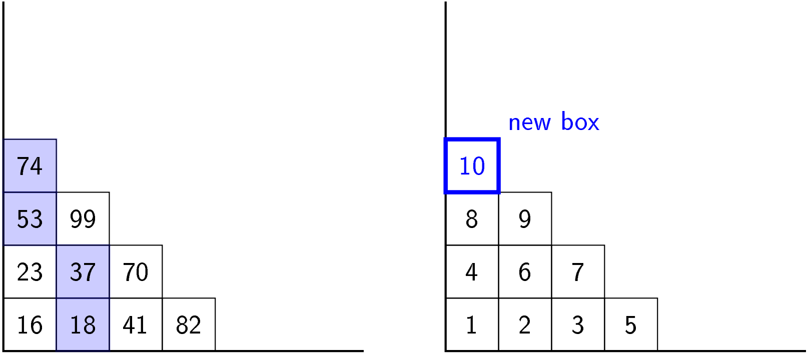

insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

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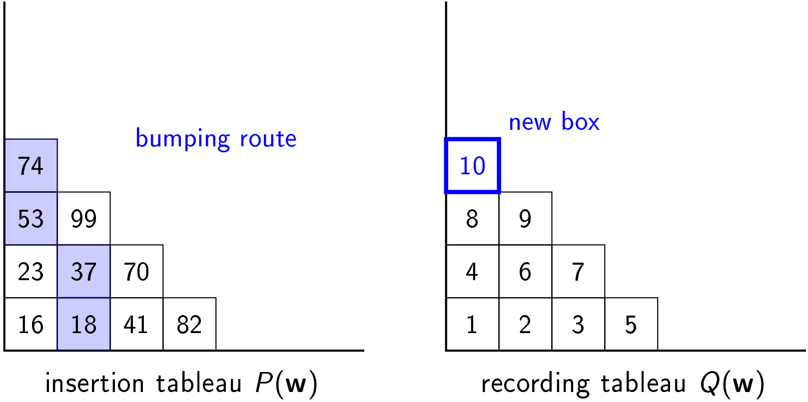
Robinson-Schensted-Knuth algorithm — induction step



$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step

74			
53	99		
23	37	70	
16	18	41	82

insertion tableau $P(\mathbf{w})$

10			
8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

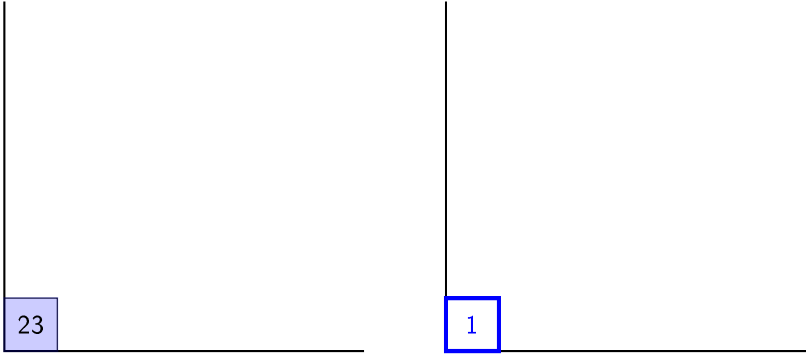
Robinson-Schensted-Knuth algorithm

insertion tableau $P(\mathbf{w})$

recording tableau $Q(\mathbf{w})$

$\mathbf{w} = \emptyset$

Robinson-Schensted-Knuth algorithm

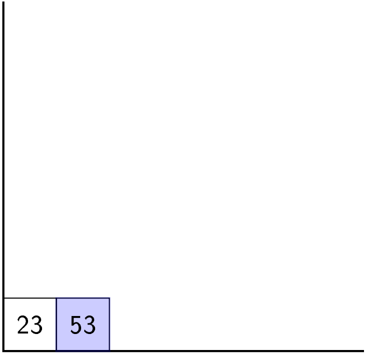


insertion tableau $P(\mathbf{w})$

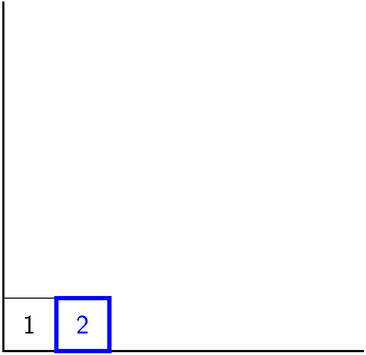
recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23)$

Robinson-Schensted-Knuth algorithm



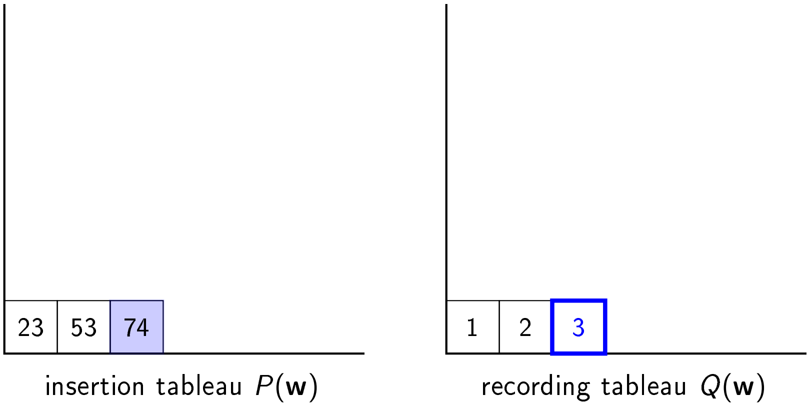
insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

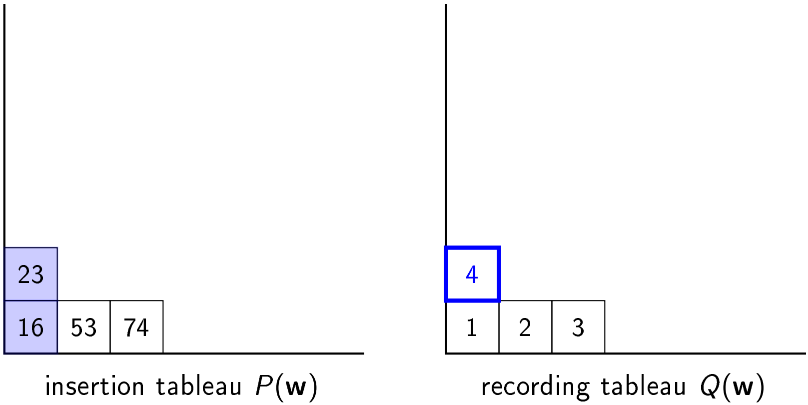
$$\mathbf{w} = (23, 53)$$

Robinson-Schensted-Knuth algorithm



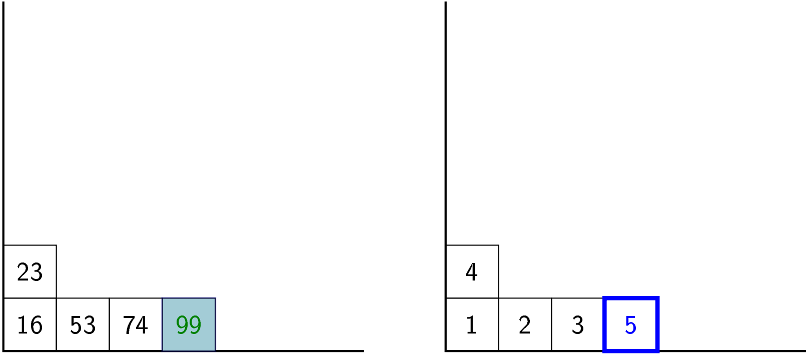
$\mathbf{w} = (23,\ 53,\ 74)$

Robinson-Schensted-Knuth algorithm



$\mathbf{w} = (23, 53, 74, 16)$

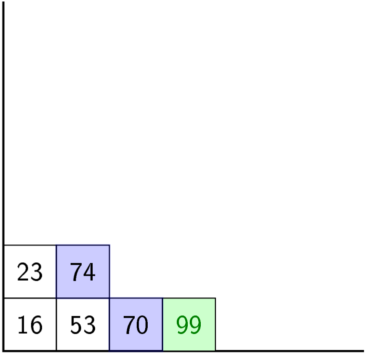
Robinson-Schensted-Knuth algorithm



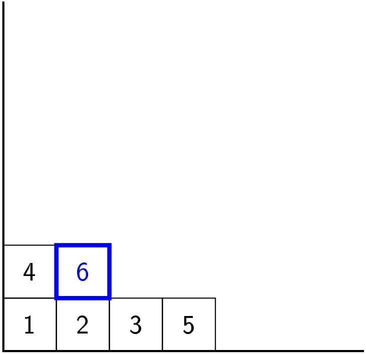
insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99)$$

Robinson-Schensted-Knuth algorithm



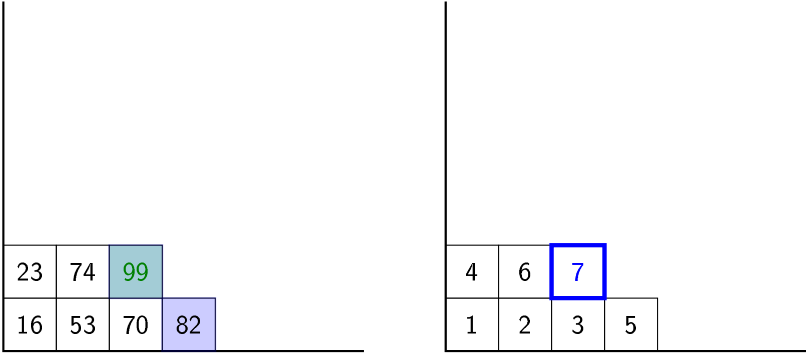
insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23, 53, 74, 16, 99, 70)$

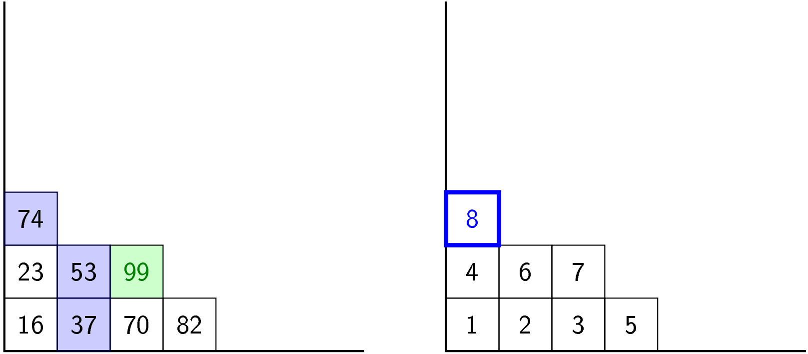
Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82)$$

Robinson-Schensted-Knuth algorithm

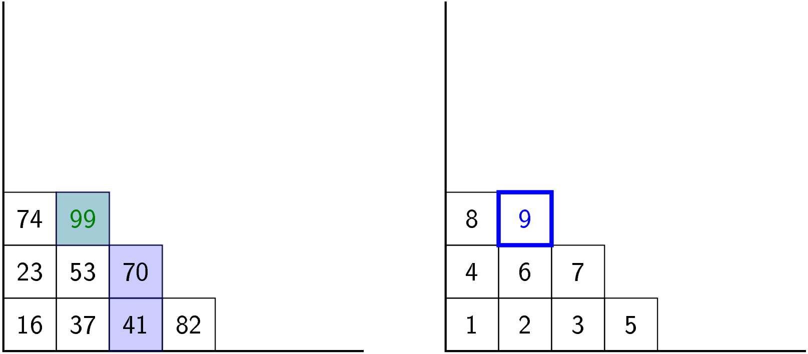


insertion tableau $P(\mathbf{w})$

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37)$$

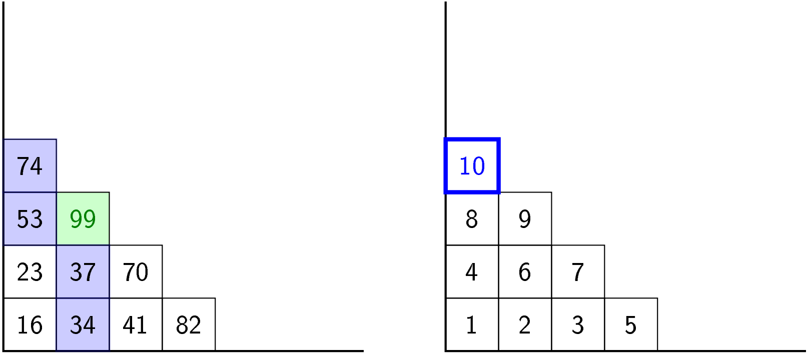
Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{w})$ recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34)$

Robinson-Schensted-Knuth algorithm

74			
53	99		
23	37	70	82
16	34	41	73

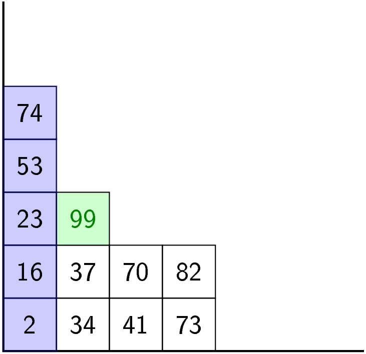
insertion tableau $P(\mathbf{w})$

10			
8	9		
4	6	7	11
1	2	3	5

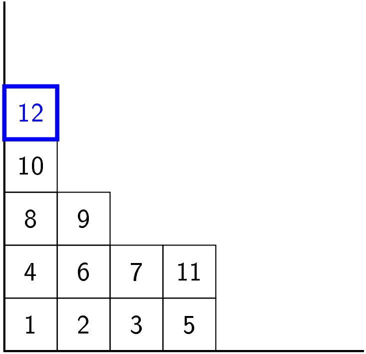
recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73)$$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2)$

Robinson-Schensted-Knuth algorithm

74			
53	99		
23	37		
16	34	70	82
2	24	41	73

insertion tableau $P(\mathbf{w})$

12			
10	13		
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2, 24)$$

Robinson-Schensted-Knuth algorithm

74			
53			
37	99		
23	34		
16	24	70	82
2	17	41	73

insertion tableau $P(\mathbf{w})$

14			
12			
10	13		
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2, 24, 17)$$

the main problem

general problem

what can we say about RSK
applied to random input?

the main problem

general problem

what can we say about RSK
applied to random input?

concrete setup for today, version A

...if the word $\mathbf{w} = (w_1, \dots, w_n)$ is a random permutation from $\mathfrak{S}_n$?

the main problem

general problem

what can we say about RSK
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concrete setup for today, version A

... if the word $\mathbf{w} = (w_1, \ldots, w_n)$ is a random permutation from $\mathfrak{S}_n$?

concrete setup for today, version B

... if $\mathbf{w} = (w_1, \ldots, w_n)$ is a sequence of
iid (independent, identically distributed) random variables
with the uniform distribution $U(0,1)$ on the interval $[0,1]$?

the main problem

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concrete setup for today, version B

...if $\mathbf{w} = (w_1, \dots, w_n)$ is a sequence of
iid (independent, identically distributed) random variables
with the uniform distribution $U(0, 1)$ on the interval $[0, 1]$?

$\longrightarrow$ Ulam 1963

Plancherel measure

exercise

if $\mathbf{w} = (w_1, \dots, w_n)$ is either

- a random permutation, or
- iid sequence $U(0, 1)$

then for each $\lambda \in \mathbb{Y}_n$

$$\mathbb{P}(\overbrace{\text{common shape of } P(\mathbf{w}) \text{ and } Q(\mathbf{w})}^{\text{RSKshape}(\mathbf{w})} \text{ is equal to } \lambda) = \frac{(\text{number of SYT of shape } \lambda)^2}{n!} = \frac{(\dim \rho_\lambda)^2}{n!}$$

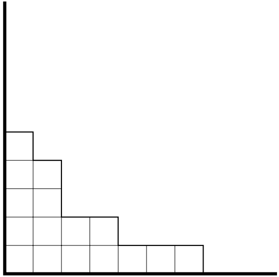
'random irreducible component
of the left regular representation $\ell^2(\mathfrak{S}_n)$
of the symmetric group' $\longrightarrow$ Lecture 2

limit shape for Plancherel measure

problem

what can we say about the common shape of $P(\mathbf{w})$ and $Q(\mathbf{w})$ when $n \to \infty$ and $\mathbf{w} = (w_1, \ldots, w_n)$ is random?

$n = 16$

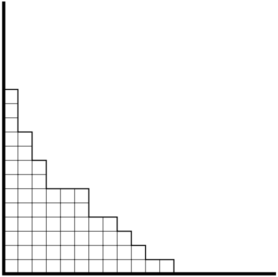


limit shape for Plancherel measure

problem

what can we say about the common shape of $P(\mathbf{w})$ and $Q(\mathbf{w})$ when $n \to \infty$ and $\mathbf{w} = (w_1, \dots, w_n)$ is random?

$n = 64$



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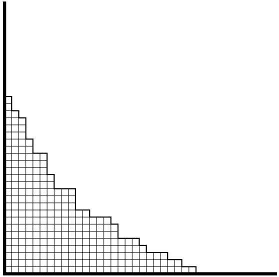
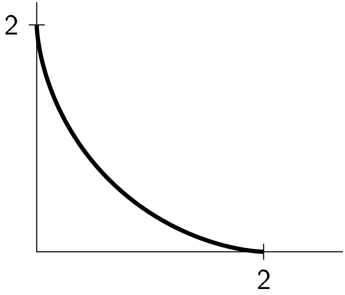
$n = 256$



Theorem (Logan&Shepp, Vershik&Kerov 1977)

in the limit $n \to \infty$
RSKshape($\mathbf{w}$) *(=the common shape of* $P(\mathbf{w})$ *and* $Q(\mathbf{w})$*)*
after rescaling by the factor $\frac{1}{\sqrt{n}}$
becomes (with very high probability)
very close to some concrete limit shape

$\longrightarrow$lectures of Philippe Biane



key problem, sloppy version

where in the recording tableau $Q(\mathbf{w})$ is located our favorite number?

key problem, sloppy version

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key problem, more specific

let $\mathbf{w} = (w_1, \dots, w_{n+1})$,
with $w_1, \dots, w_n$ random, iid $U(0,1)$
and w_{n+1} deterministic

what can we say
about the location of the box
$\square$ containing $n+1$
in the recording tableau $Q(\mathbf{w})$?

key problem, sloppy version

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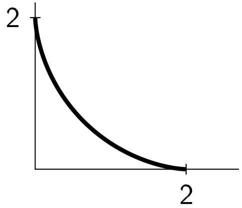

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silly answer:
it is somewhere
at the boundary of RSKshape($\mathbf{w}$)
which is $\approx$LSVK shape

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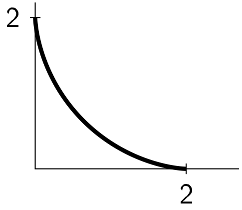

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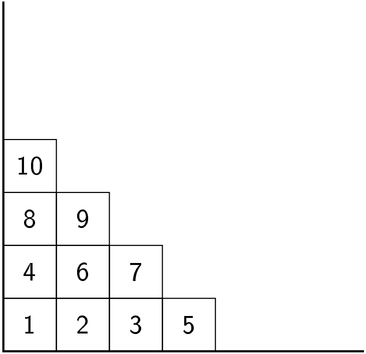
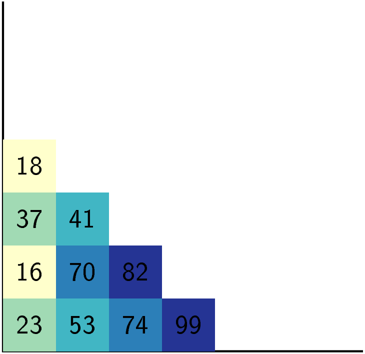
but where exactly?



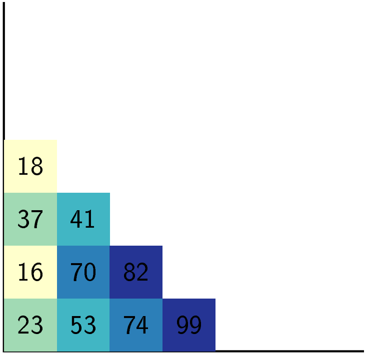
recording tableau $(Q_{ij})_{ij}$



$(w_{Q_{ij}})_{ij}$

$w_1 = 23,\quad w_2 = 53,\quad w_3 = 74,\quad w_4 = 16,\quad w_5 = 99,$

$w_6 = 70,\quad w_7 = 82,\quad w_8 = 37,\quad w_9 = 41$



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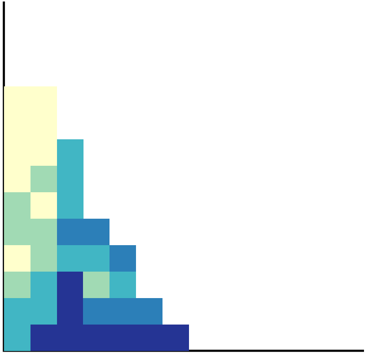

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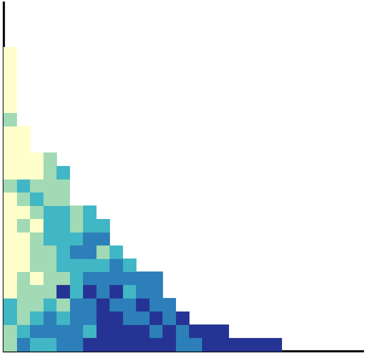

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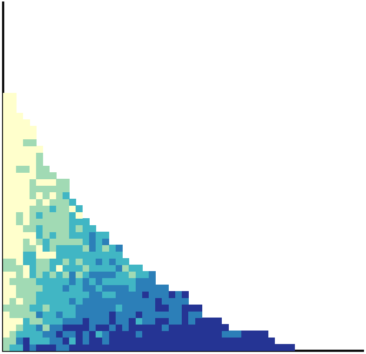

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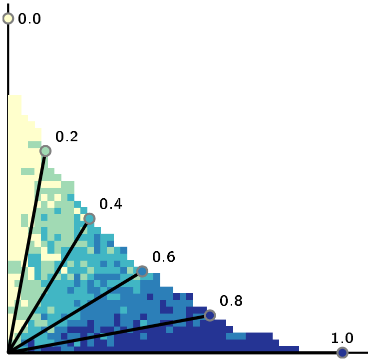

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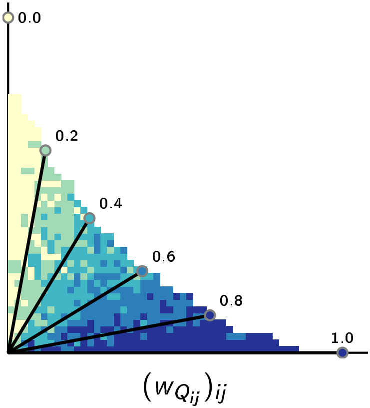
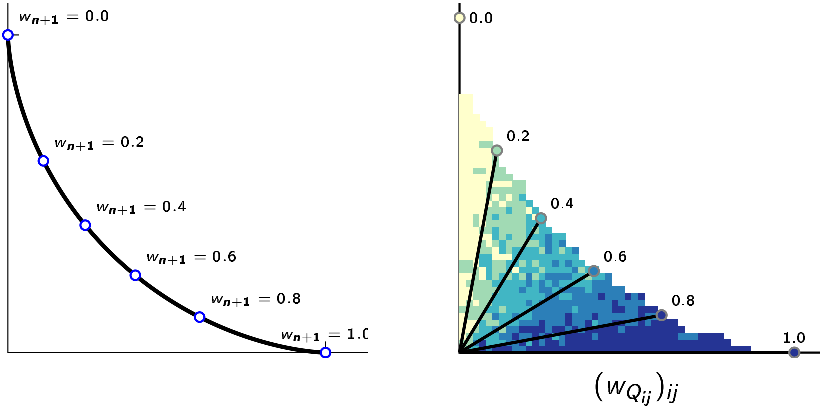


Theorem (Romik&Śniady 2015)

$$\left\| \frac{\square}{\sqrt{n}} - (\mathrm{RSKcos}\, w_{n+1}, \mathrm{RSKsin}\, w_{n+1}) \right\| \xrightarrow[\text{in probability}]{n\to\infty} 0$$



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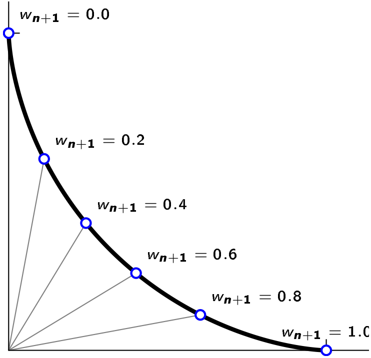

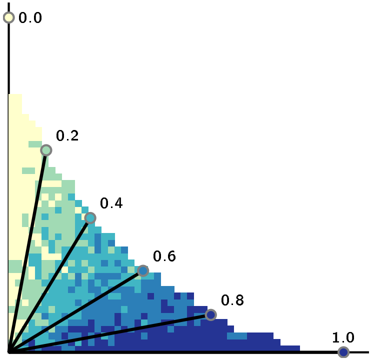

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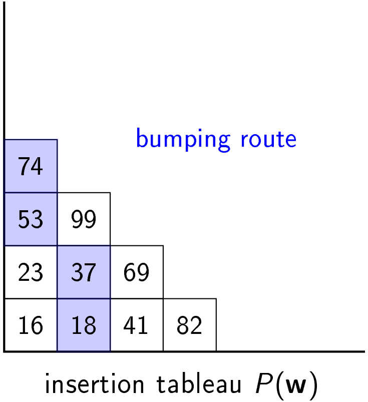
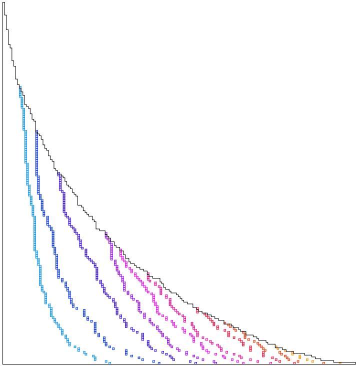
bumping routes



$$\mathbf{w} = (23, 53, 74, 16, 99, 69, 82, 37, 41, \underbrace{18}_{w_n})$$

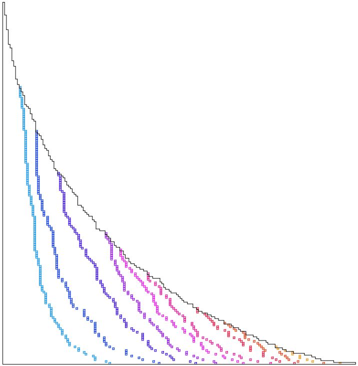
bumping routes



problem ⟶ MOORE 2006

what can we say about the shapes of the bumping routes?

bumping routes



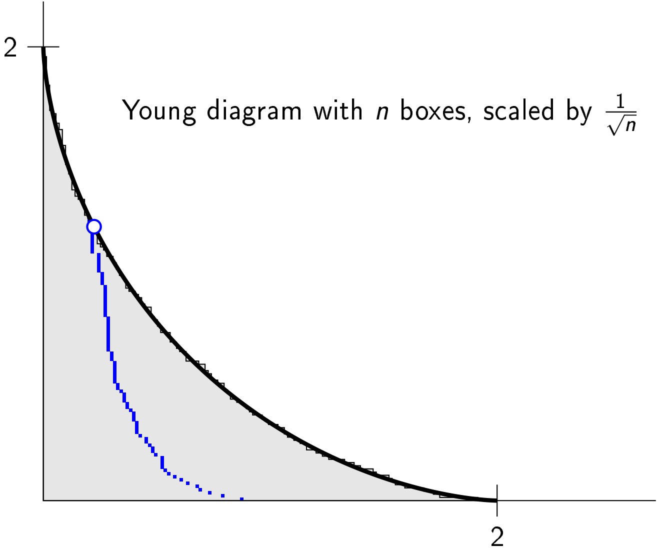
problem ⟶ Moore 2006

what can we say about the shapes of the bumping routes?

Theorem, Romik&Śniady 2014

Bumping route (scaled by factor $\frac{1}{\sqrt{n\ w_n}}$)
obtained by adding entry w_n to the tableau P_{n-1}
converges in probability (as $n \to \infty$) to a deterministic curve G_τ.

the key result explains the behavior of bumping routes



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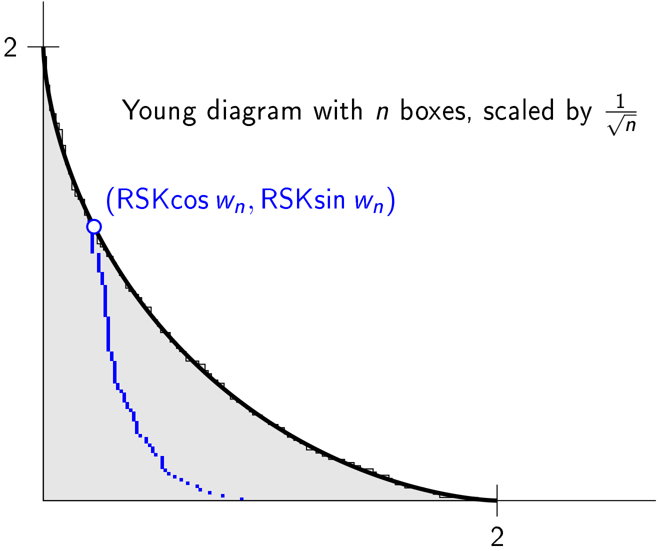

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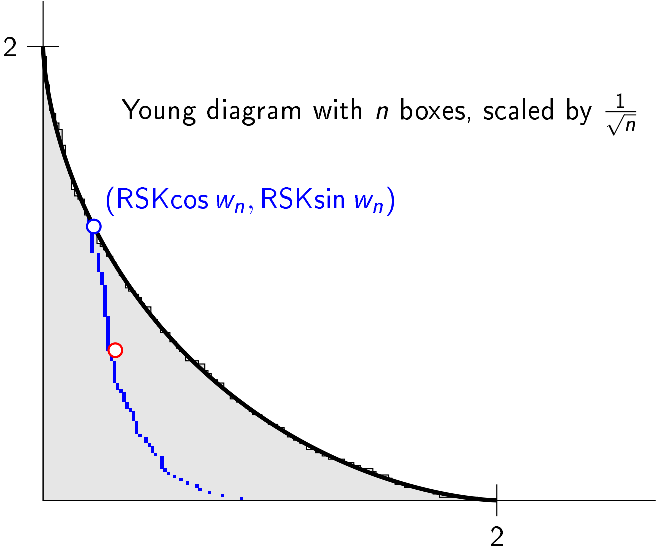

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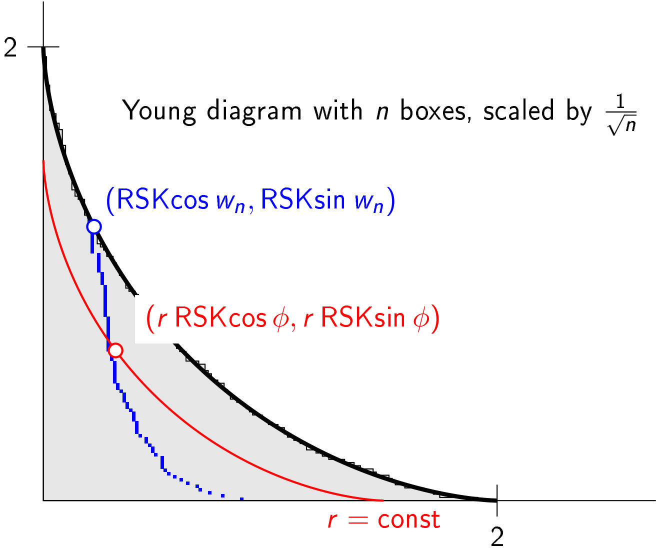

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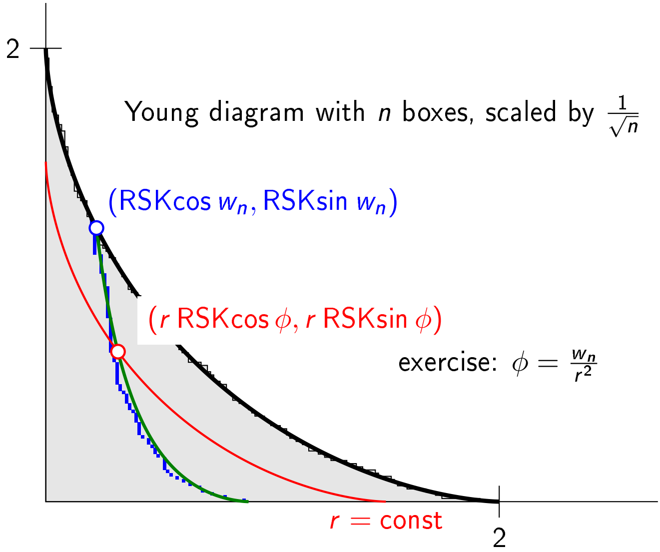


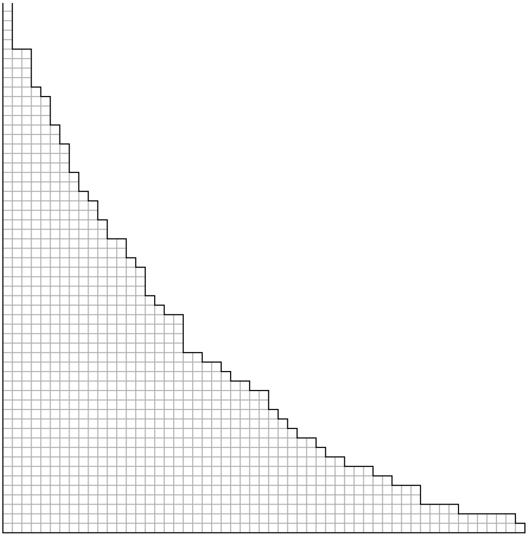
diffusion of a box

- $\boxed{w_n}(P_m)$ denotes the location of the box containing w_n in the insertion tableau $P_m = P(w_1, \dots, w_m)$, for $m \geq n$;

problem

what can we say about the time evolution of $\boxed{w_n}(P_m)$ for $m = n, n+1, \dots$?

diffusion of a box



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Theorem (Śniady, never published)

There exists an explicit function $G : \mathbb{R}_+ \to \mathbb{R}_+^2$ *such that*

$$\frac{\boxed{w_n}\left(P_{\lfloor n e^\tau \rfloor}\right)}{\sqrt{n \ w_n}} \xrightarrow[n\to\infty]{\text{in probability}} G_\tau \qquad \text{for } \tau \geq 0.$$

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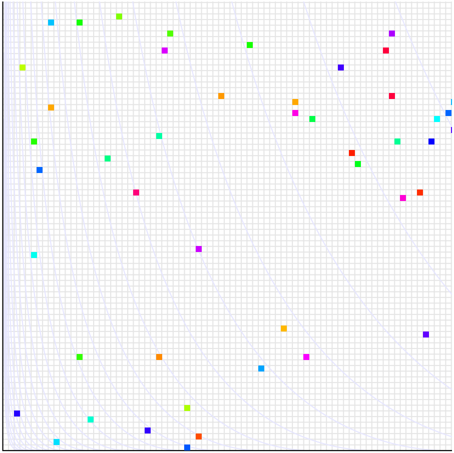
exercise

prove this result using *'asymptotic determinism of last box insertion'*

Hint: if $\mathbf{w}$ is a permutation and $\mathrm{RSK}(\mathbf{w}) = (P, Q)$ then $\mathrm{RSK}(\mathbf{w}^{-1}) = (Q, P)$.

hydrodynamic limit of RSK



$m = 1.6 \cdot 10^4$

hydrodynamic limit of RSK

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exercise

- the above theorem concerns movement of a single particle; what can we say about collective movement of the fluid particles?
  if we consider transformations of the quarterplane describing the time-evolution of the insertion tableau P: in which topology the convergence holds true?
- write a paper about it, add Śniady as coauthor if you like,