Series of lectures: jeu de taquin and asymptotic representation theory

Piotr Śniady

plan for today

Lecture 2A

how to prove *asymptotic determinism of the last box insertion*?

- RSK and (plactic) Littlewood–Richardson rule,
- Jucys–Murphy elements,

Lecture 2B

asymptotic representation theory
of the symmetric groups $\mathfrak{S}_n$ for $n \to \infty$ and $\mathfrak{S}_\infty$

- Thoma characters of $\mathfrak{S}_\infty$,
- random tableaux, random paths in Young graph,
- jeu de taquin,

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Lecture 2A: proof of asymptotic determinism of RSK insertion

Piotr Śniady

Polska Akademia Nauk

RSK is a bijection. . .

Input:

- word $\mathbf{w} = (w_1, \dots, w_n)$

Output:

- semistandard tableau P,
- standard tableau Q,

tableaux P and Q have
the same shape with n boxes

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example:
$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathbf{w})$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

Robinson-Schensted-Knuth algorithm — induction step

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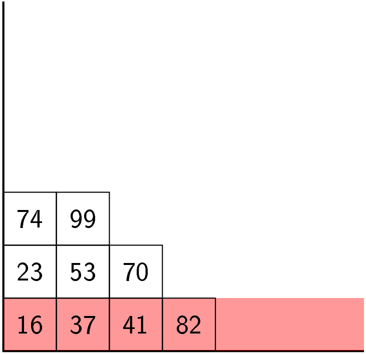
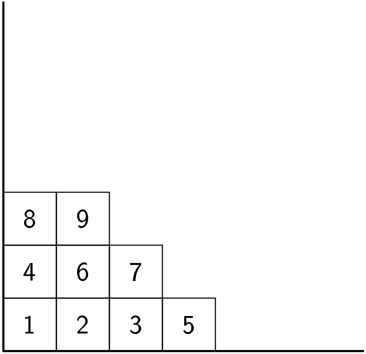


insertion tableau $P(\mathbf{w})$



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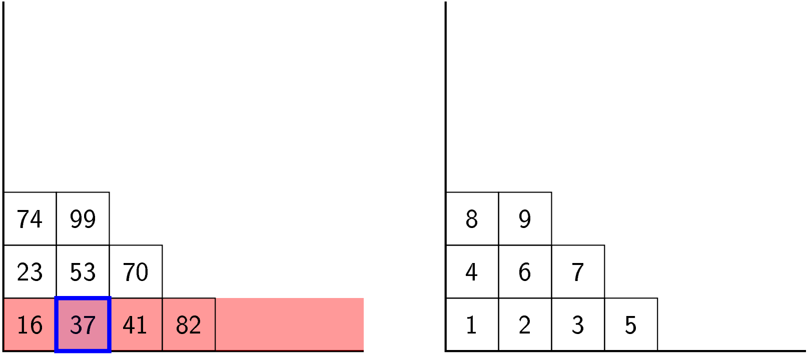

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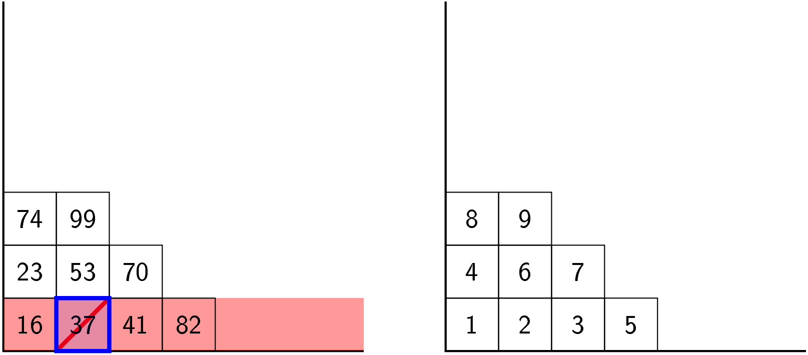

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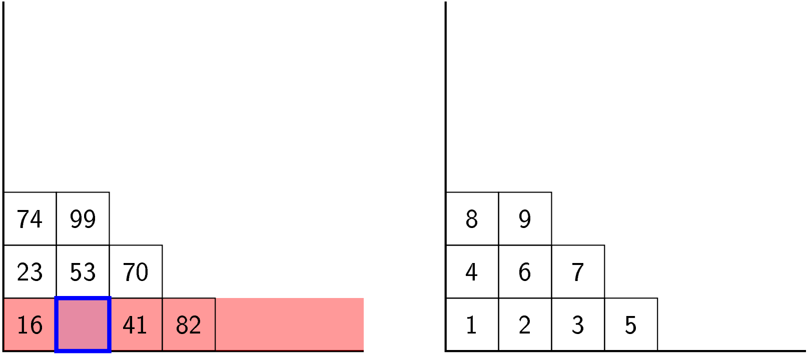

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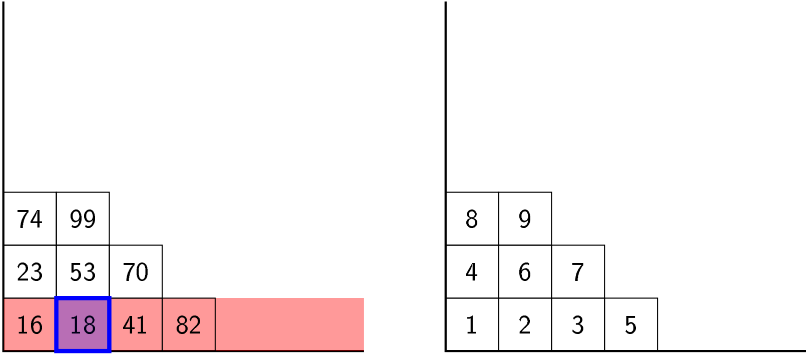

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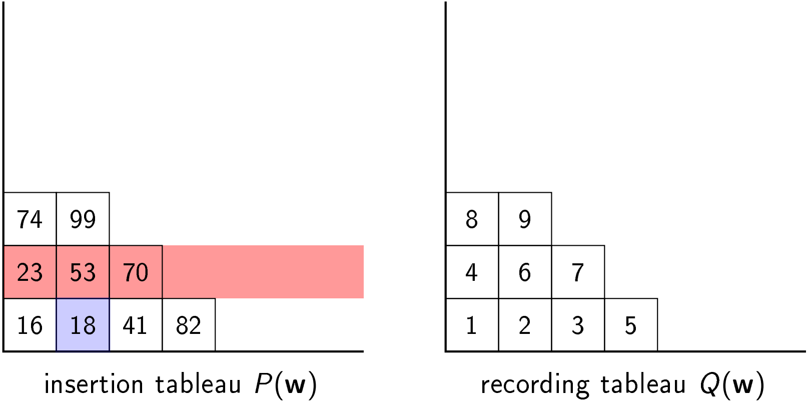

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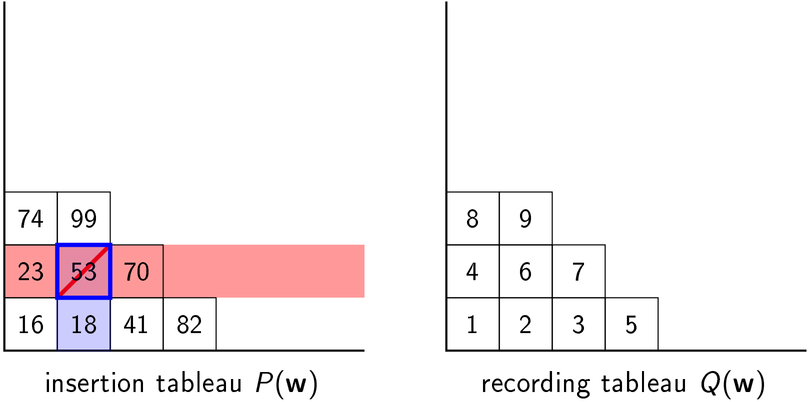

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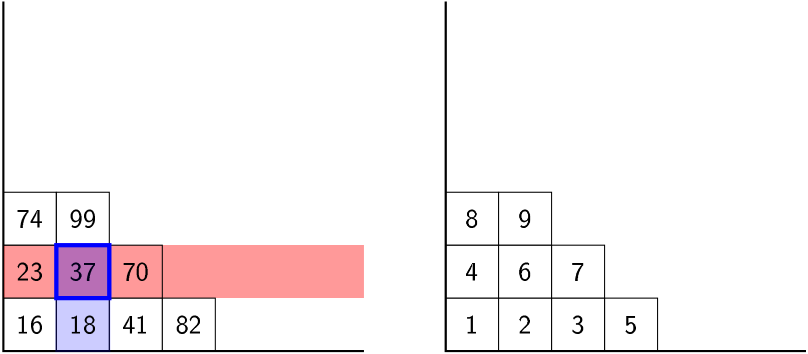

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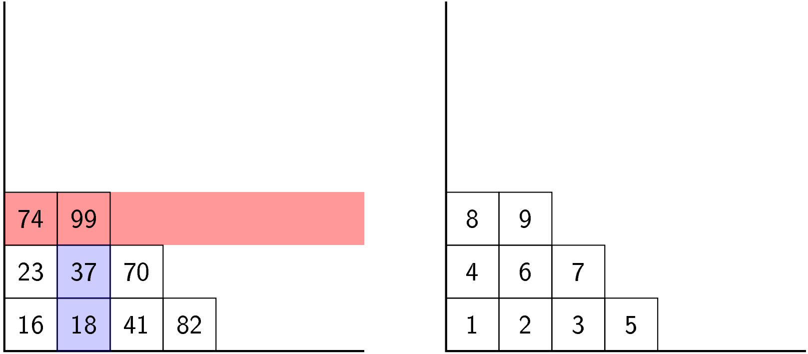

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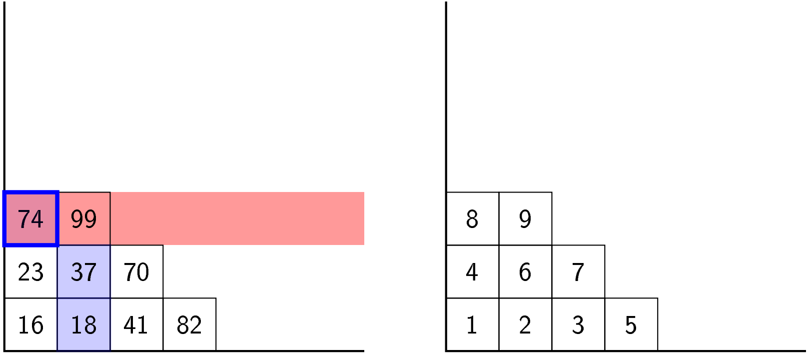

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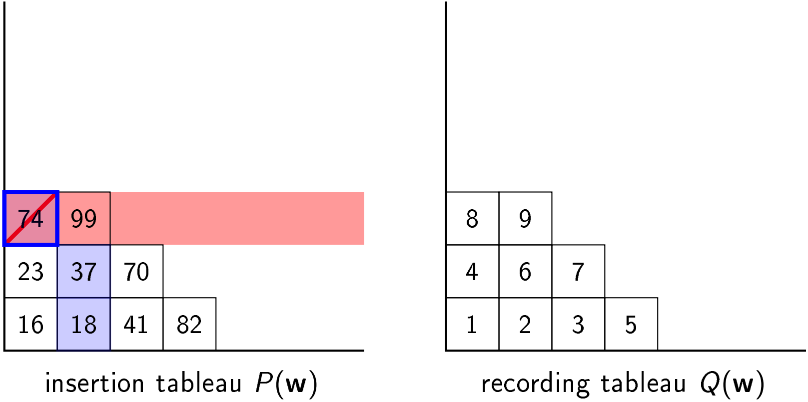

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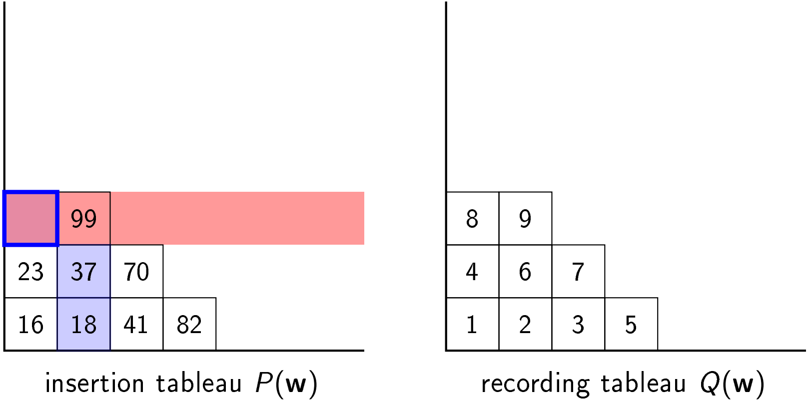

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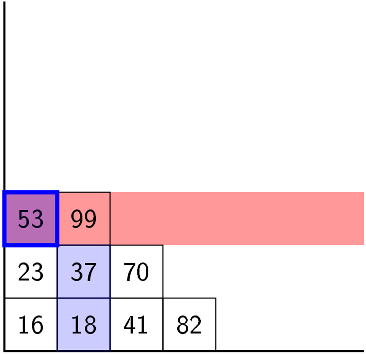
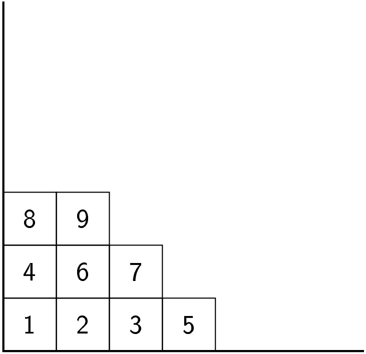


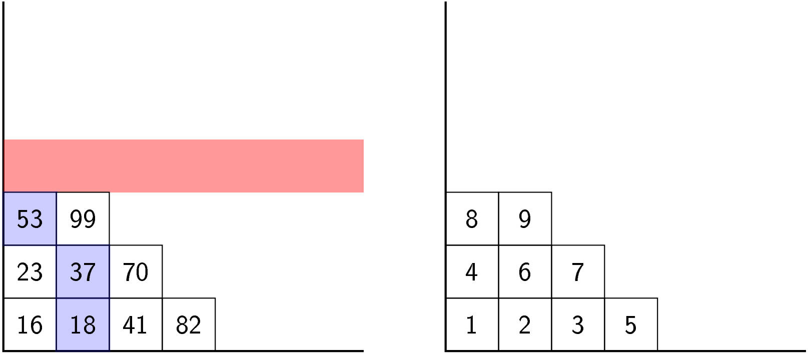
insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

Robinson-Schensted-Knuth algorithm — induction step



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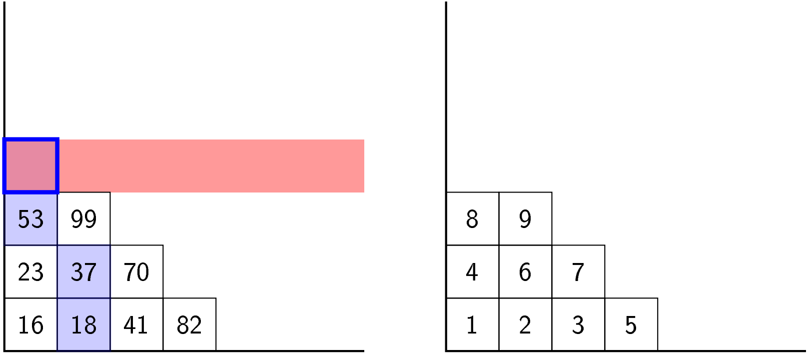

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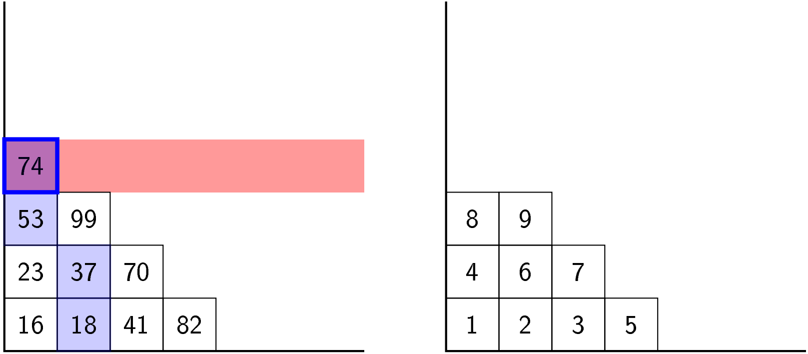

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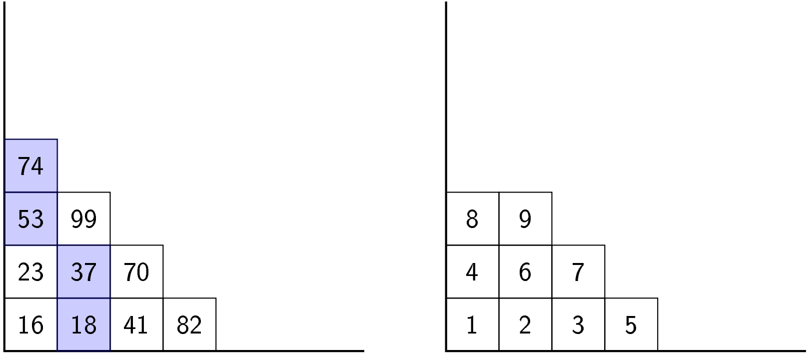

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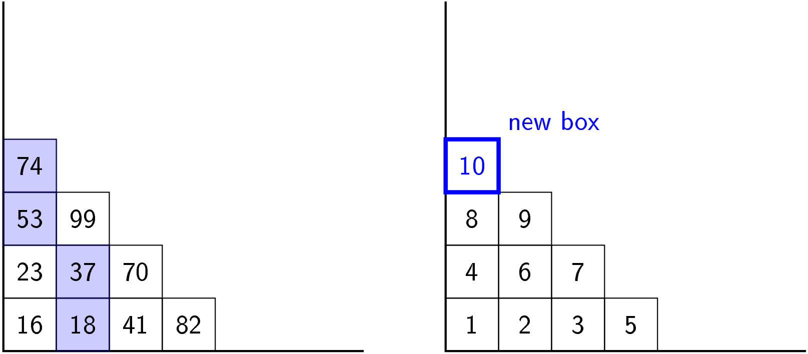

insertion tableau $P(\mathbf{w})$ — recording tableau $Q(\mathbf{w})$

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Robinson-Schensted-Knuth algorithm — induction step

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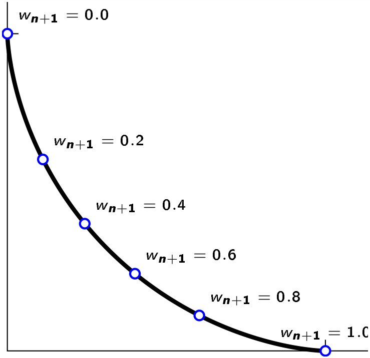
key problem

let $\mathbf{w} = (w_1, \dots, w_n, w_{n+1})$,
with $w_1, \dots, w_n$ random, iid $U(0,1)$
and w_{n+1} deterministic

$\square$ is the box containing $n+1$
in the recording tableau $Q(\mathbf{w})$...

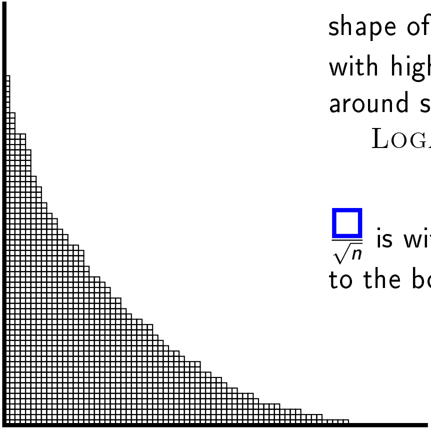


key theorem, Romik&Śniady 2015

$$\left\| \frac{\square}{\sqrt{n}} - (\operatorname{RSKcos} w_{n+1}, \operatorname{RSKsin} w_{n+1}) \right\| \xrightarrow[\text{in probability}]{n\to\infty} 0$$

this lecture = proof of this result

the limit shape



shape of Q_n (scaled by factor $\frac{1}{\sqrt{n}}$) with high probability concentrates around some explicit shape

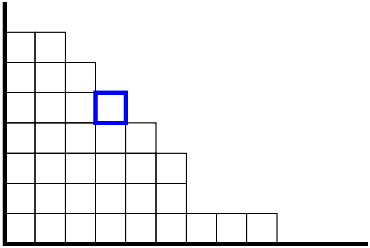
LOGAN, SHEPP, VERSHIK, KEROV

$\frac{\square}{\sqrt{n}}$ is with high probability close to the boundary of this limit shape

reduction of the problem: adding randomness

instead of (for deterministic w_{n+1})...

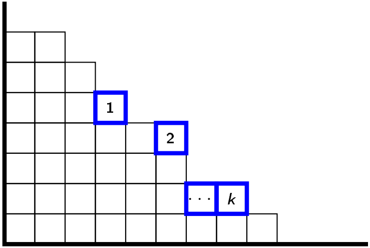
$$Q(w_1, \dots, w_n, \; w_{n+1} \qquad) \setminus Q(w_1, \dots, w_n) = \{ \; \square \; \}$$



reduction of the problem: adding randomness

we study (for random $0 < t_1 < \cdots < t_k < 1$)

$$Q(w_1, \dots, w_n,\ t_1, \dots, t_k) \setminus Q(w_1, \dots, w_n) = \{ \boxed{1}, \dots, \boxed{k} \}$$



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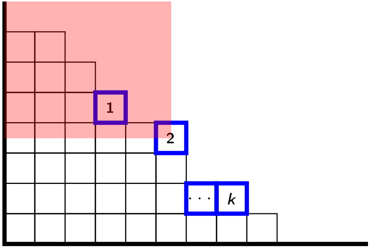

homework

if $w_{n+1} < t_i$ then $\boxed{}$ is north-west from $\boxed{i}$

for $\frac{i}{k} \approx w_{n+1} + \epsilon$, this happens with high probability, as $k \to \infty$

Littlewood–Richardson coefficients

irreducible representation ρ^λ of the symmetric group $\mathfrak{S}_n$ $\longleftrightarrow$ Young diagram λ with n boxes

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Littlewood-Richardson coefficients

$$\left(\rho^\lambda \otimes \rho^\mu\right) \Big\uparrow_{\mathfrak{S}_{|\lambda|}\times\mathfrak{S}_{|\mu|}}^{\mathfrak{S}_{|\lambda|+|\mu|}} = \bigoplus_\nu c^\nu_{\lambda,\mu}\ \rho^\nu$$

---

random irreducible component of reducible representation V:

$$\mathbb{P}(\nu) = \frac{(\text{multiplicity of } \nu \text{ in } V)\cdot(\text{dimension of } \rho^\nu)}{\text{dimension of } V}$$

plactic Littlewood–Richardson rule

if $0 \leq w_1, \dots, w_n \leq 1$ is a random sequence, such that

$$\text{shape of } \operatorname{RSK}(w_1, \dots, w_n) = \lambda;$$

and $0 \leq t_1, \dots, t_k \leq 1$ is a random sequence, such that

$$\text{shape of } \operatorname{RSK}(t_1, \dots, t_k) = \mu$$

then the random Young diagram

$$\text{shape of } \operatorname{RSK}(w_1, \dots, w_n, t_1, \dots, t_k)$$

has the same distribution as random irreducible component of

$$\rho^\lambda \otimes \rho^\mu \uparrow_{\mathfrak{S}_n \times \mathfrak{S}_k}^{\mathfrak{S}_{n+k}}$$

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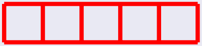

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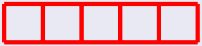
$$\text{shape of } \operatorname{RSK}(t_1, \dots, t_k) = (k) =$$ 

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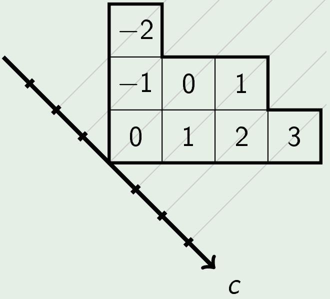
content of the box

$$\text{content}(\square) = (x\text{-coordinate}) - (y\text{-coordinate})$$

Example



content of Young diagram $= (-2, -1, 0, 0, 1, 1, 2, 3)$

Jucys–Murphy elements

$$X_i = (1, i) + (2, i) + \cdots + (i-1, i) \qquad \text{for } i \in \{1, \dots, n\}$$

$X_1, \dots, X_n$ are elements of the symmetric group algebra $\mathbb{C}[\mathfrak{S}_n]$

for any Young diagram λ with contents $(c_1, \dots, c_n)$
and a symmetric polynomial $f(x_1, \dots, x_n)$

$$\chi^\lambda(f(X_1, \dots, X_n)) = \frac{\operatorname{Tr} \rho^\lambda(f(X_1, \dots, X_n))}{\operatorname{Tr} \rho^\lambda(1)} = ?$$

Jucys–Murphy elements

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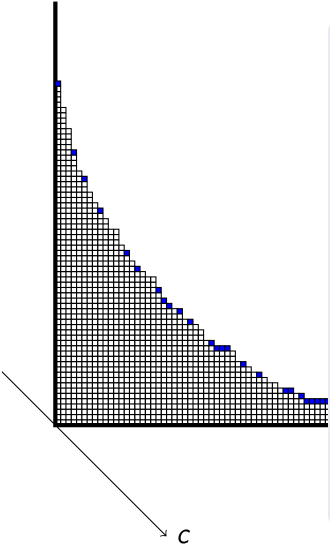
growth of Young diagrams and Jucys-Murphy elements



let $\lambda \vdash n$, $\mu \vdash k$ be fixed Young diagrams

let ν be a random irreducible component of $\rho^\lambda \otimes \rho^\mu \uparrow_{\mathfrak{S}_n \times \mathfrak{S}_k}^{\mathfrak{S}_{n+k}}$

let $c_1, \dots, c_k$ be the contents of boxes of $\nu \setminus \lambda$

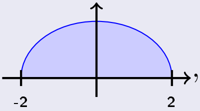
then for any symmetric polynomial $f(x_{n+1}, \dots, x_{n+k})$ we have

$$\left(\chi^\lambda \otimes \chi^\mu\right) \left(f(X_{n+1}, \dots, X_{n+k}) \downarrow_{\mathfrak{S}_n \times \mathfrak{S}_k}^{\mathfrak{S}_{n+k}} \right) = \mathbb{E} f(c_1, \dots, c_k)$$

semicircle law

if $k \approx \sqrt[4]{n}$

$$\mu_k := \frac{1}{k}\left(\delta_{\frac{c_1}{\sqrt{n}}} + \cdots + \delta_{\frac{c_k}{\sqrt{n}}}\right) \xrightarrow[n\to\infty]{\text{in probability}} \mu_{SC} =$$



where $c_j = c(\boxed{\text{j}})$

Hint:

p-th moment of left-hand-side $\frac{1}{k}\sum_j \left(\frac{c_j}{\sqrt{n}}\right)^p$ is a random variable,

show that the mean converges to

p-th moment of μ_{SC} (Catalan numbers!)

show that the variance converges to zero;

use Jucys–Murphy elements

semicircle law

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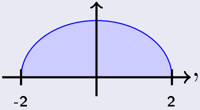

where $c_j = c(\boxed{j})$

since $c_1 < \cdots < c_k$, this implies that

$$w_{n+1} + \epsilon \approx \frac{i}{k} = F_{\mu_k}\left(\frac{c(\boxed{i})}{\sqrt{n}}\right) \xrightarrow{\text{in probability}} F_{\mu_{SC}}\left(\frac{c(\boxed{i})}{\sqrt{n}}\right)$$

$$F^{-1}_{\mu_{SC}}\left(w_{n+1} + \epsilon\right) \approx \frac{c(\boxed{i})}{\sqrt{n}}$$

key problem

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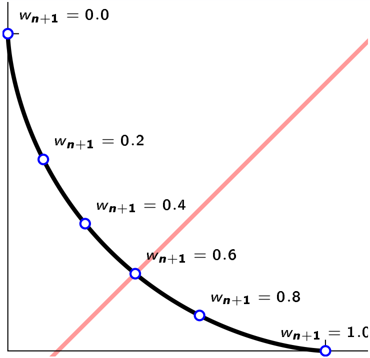
in the recording tableau $Q(\mathbf{w})$...



key theorem, Romik&Śniady 2015

$$\left\| \frac{\square}{\sqrt{n}} - \left(\mathrm{RSKcos}\, w_{n+1}, \mathrm{RSKsin}\, w_{n+1}\right) \right\| \xrightarrow[\text{in probability}]{n\to\infty} 0$$