Lecture 2B: jeu de taquin and asymptotic representation theory

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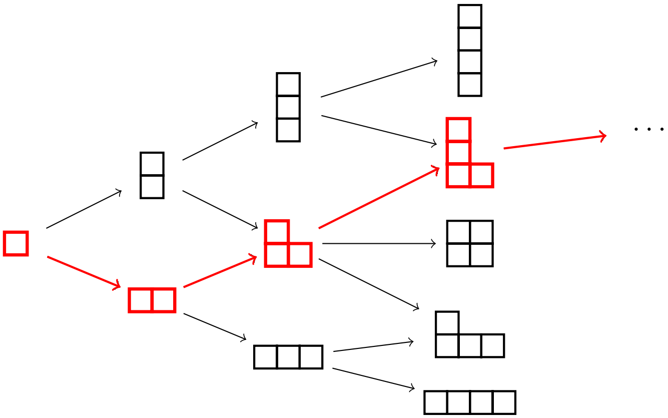
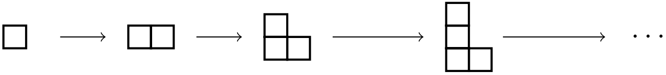
Young graph $\mathbb{Y} = \bigsqcup_{n\geq 1} \mathbb{Y}_n$



what is the Martin boundary of this graph?

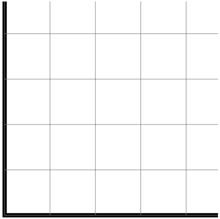
goal: study $\mathfrak{S}_\infty = \bigcup_{n\geq 1} \mathfrak{S}_n$,
the group of finitary permutations $\pi \colon \mathbb{N} \to \mathbb{N}$

paths in Young graph $\longleftrightarrow$ tableaux



Λ^1 Λ^2 Λ^3 Λ^4

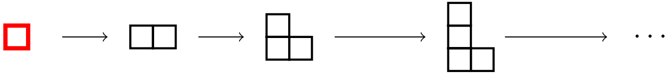
infinite path in Young graph



infinite tableau

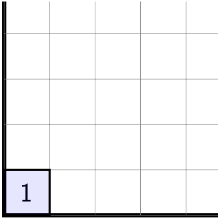
$\mathcal{T}$:= set of infinite tableaux / set of infinite paths

paths in Young graph $\longleftrightarrow$ tableaux



Λ^1 Λ^2 Λ^3 Λ^4

infinite path in Young graph



infinite tableau

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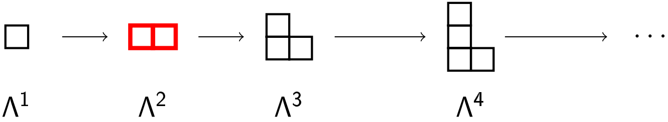
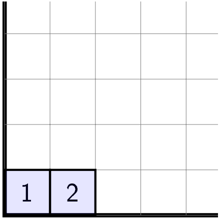
paths in Young graph ⟷ tableaux



infinite path in Young graph



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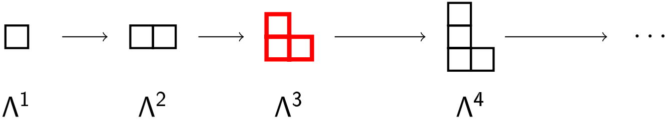
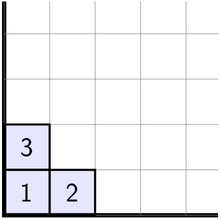


infinite path in Young graph



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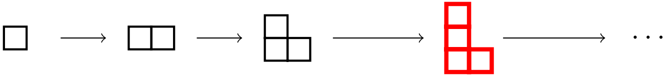

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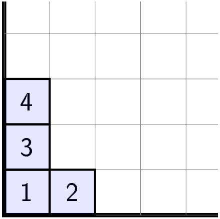
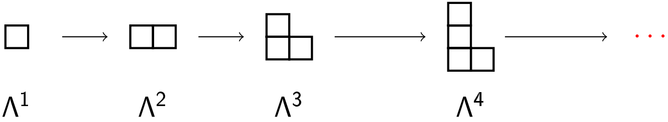
infinite path in Young graph



infinite tableau

$\mathcal{T}$:= set of infinite tableaux / set of infinite paths

paths in Young graph $\longleftrightarrow$ tableaux



infinite path in Young graph

$\vdots$		$\vdots$		
6	15	21	24	
4	12	17	19	$\cdots$
3	5	8	11	
1	2	7	9	$\cdots$

infinite tableau

$\mathcal{T}$:= set of infinite tableaux / set of infinite paths

G is a discrete (infinite) group,

$\chi\colon G \to \mathbb{C}$ is a character if

- χ is normalized: $\chi(e) = 1$,
- χ is constant on each conjugacy class,
- χ is positively definite:

$$\forall_{g_1,\dots,g_n \in G} \quad \forall_{z_1,\dots,z_n \in \mathbb{C}} \quad \sum_{i,j} z_i \, \overline{z_j} \, \chi\left(g_i g_j^{-1}\right) \geq 0$$

example

if G is finite and ρ is a representation,

$$\chi(g) := \frac{1}{\dim \rho} \operatorname{Tr} \rho(g)$$

is a character

goal: find all extremal characters of $\mathfrak{S}_\infty$

character $\mapsto$ random Young diagrams

- given a character $\chi : \mathfrak{S}_\infty \to \mathbb{C}$, restrict it to $\mathfrak{S}_n$,
- the restriction is a convex combination of irreducible characters:

$$\chi\big|_{\mathfrak{S}_n} = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{P}_n(\lambda)\, \chi_\lambda,$$

- coefficients define a probability measure on $\mathbb{Y}_n$

$$\chi \mapsto (\mathbb{P}_1, \mathbb{P}_2, \dots)$$

$$\mathbb{P}_n(\lambda) = \sum_{\mu \nwarrow \lambda} \mathbb{P}_{n+1}(\mu)\, \frac{\dim\, \rho_\lambda}{\dim\, \rho_\mu}$$ *harmonic function*

extremal character $\chi \longleftrightarrow$ extremal $(\mathbb{P}_1, \mathbb{P}_2, \dots)$

character $\mapsto$ random infinite tableau

- given a character $\chi : \mathfrak{S}_\infty \to \mathbb{C}\dots$
- take random Young diagram λ with n boxes with probability distribution $\mathbb{P}_n$,
-
  - take a random standard Young tableau with shape λ, or, equivalently,
  - take a random path in Young graph with endpoint λ,
- take $n \to \infty$,

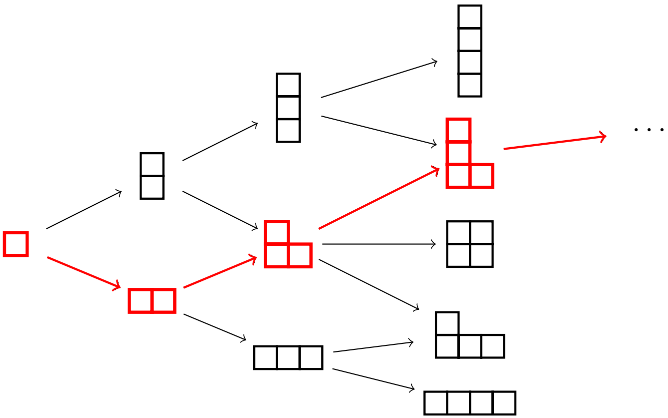
$\chi \mapsto$ harmonic probability measure $\mathbb{P}$ on $\mathcal{T}$

harmonic=probability of each finite path in $\mathbb{Y}$ depends only on its endpoint

extremal character $\longleftrightarrow$ extremal harmonic measure on $\mathcal{T}$

$\longrightarrow$ Vershik&Kerov

Young graph $\mathbb{Y} = \bigsqcup_{n\geq 1} \mathbb{Y}_n$



{extremal characters} $\longleftrightarrow$ Martin boundary of $\mathbb{Y}$

example

extremal character of $\mathfrak{S}_\infty$

$$\chi_{\text{reg}}(g) = \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{P}_n$ is the Plancherel measure on $\mathbb{Y}_n$

$\mathbb{P}$ is the Plancherel measure on $\mathcal{T} \overset{\text{distribution}}{=} Q(w_1, w_1, \dots)$,
where $w_1, w_1, \dots$ are iid $U(0,1)$ random variables

Thoma simplex

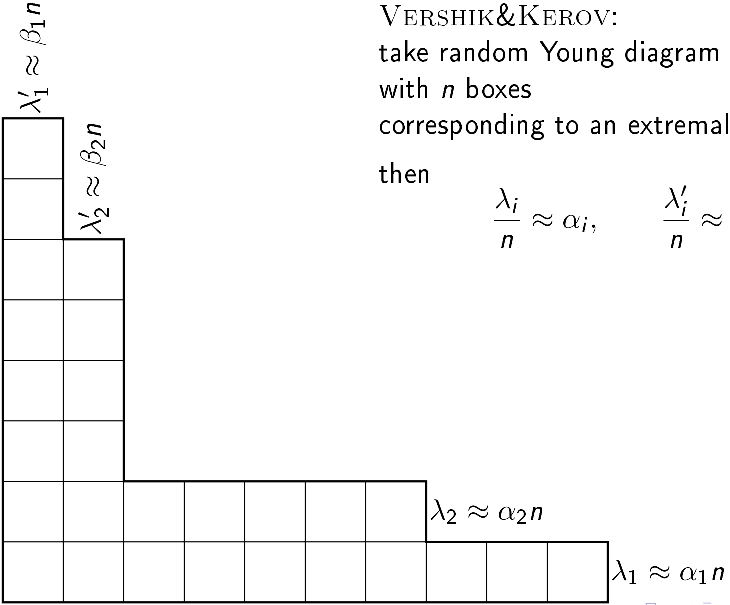
Theorem (Thoma)

$$\{\textit{extremal characters of } \mathfrak{S}_\infty\}$$

$$\updownarrow$$

$$\Big\{(\alpha_1, \alpha_2, \dots, \quad \beta_1, \beta_2, \dots) : \quad \alpha_1 \geq \alpha_2 \geq \dots \geq 0, \qquad \beta_1 \geq \beta_2 \geq \dots \geq 0, \quad \alpha_1 + \alpha_2 + \dots + \beta_1 + \beta_2 + \dots \leq 1\Big\}$$

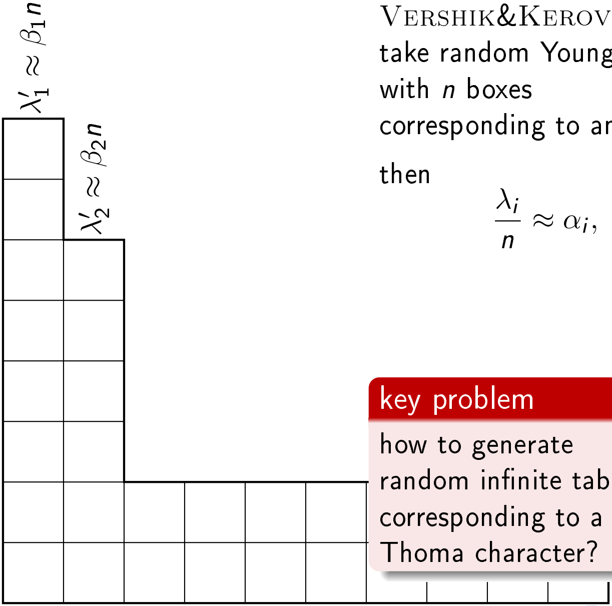
χ_{reg} corresponds to
$(\alpha_1, \alpha_2, \dots) = (0, 0, \dots)$,
$(\beta_1, \beta_2, \dots) = (0, 0, \dots)$,



Vershik&Kerov:
take random Young diagram
with n boxes
corresponding to an extremal character;

then

$$\frac{\lambda_i}{n} \approx \alpha_i, \qquad \frac{\lambda'_i}{n} \approx \beta_i$$



VERSHIK&KEROV:
take random Young diagram
with n boxes
corresponding to an extremal character;

then
$$\frac{\lambda_i}{n} \approx \alpha_i, \qquad \frac{\lambda'_i}{n} \approx \beta_i$$

key problem

how to generate
random infinite tableaux
corresponding to a given
Thoma character?

alphabets: the key example

$$\mathbb{A} := \mathbb{Z}_+ \cup (0,1) \cup \mathbb{Z}_- =$$

$$\underbrace{1 < 2 < 3 < \cdots}_{\text{row letters}} < \cdots <$$

$$\cdots < 0.1 < \cdots < 0.9 < \cdots <$$

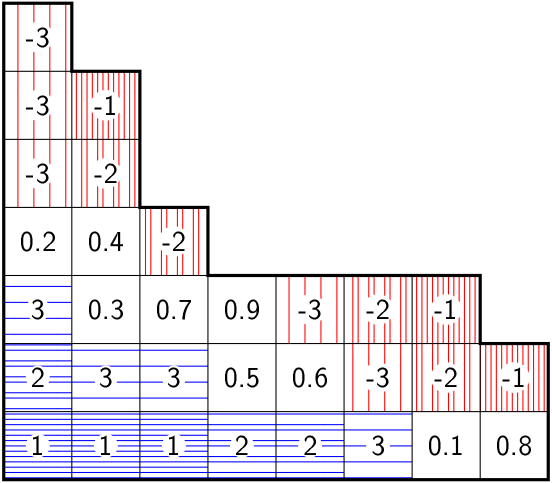
$$< \cdots < \underbrace{\cdots < -3 < -2 < -1}_{\text{column letters}},$$

$$\mathbb{P}(1) = \alpha_1, \quad \mathbb{P}(2) = \alpha_2, \quad \cdots$$

$$\mathbb{P}(-1) = \beta_1, \quad \mathbb{P}(-2) = \beta_2, \quad \cdots$$

$\mathbb{P}$ on $(0,1)$ is equal to $(1 - \alpha_1 - \alpha_2 - \cdots - \beta_1 - \beta_2 - \cdots) \cdot$ Lebesgue

Vershik and Kerov 1985: RSK on steroids



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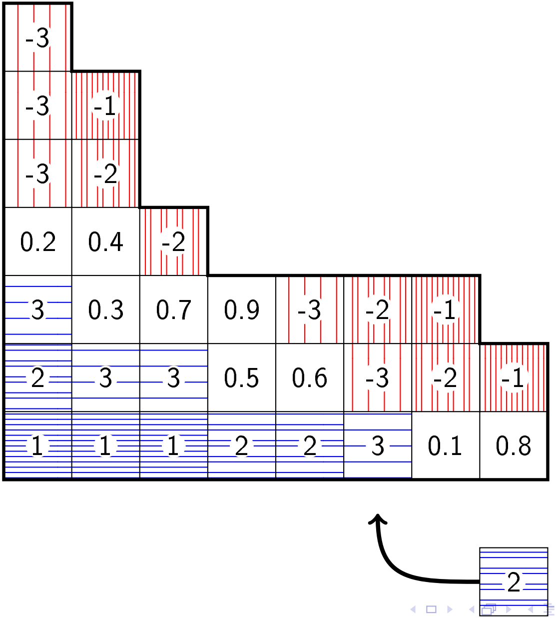

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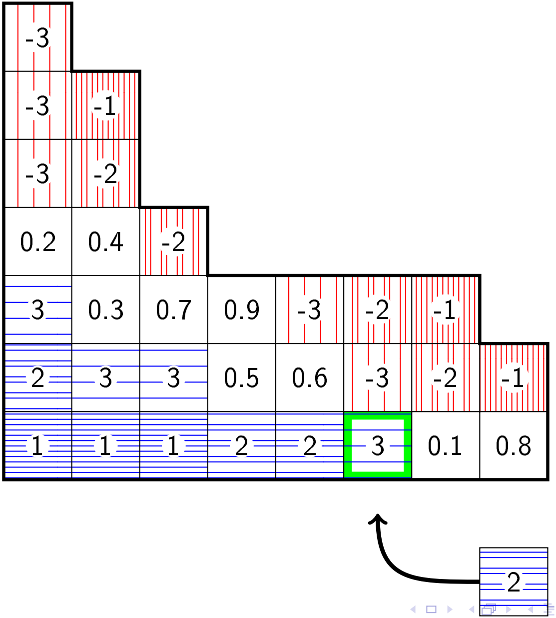

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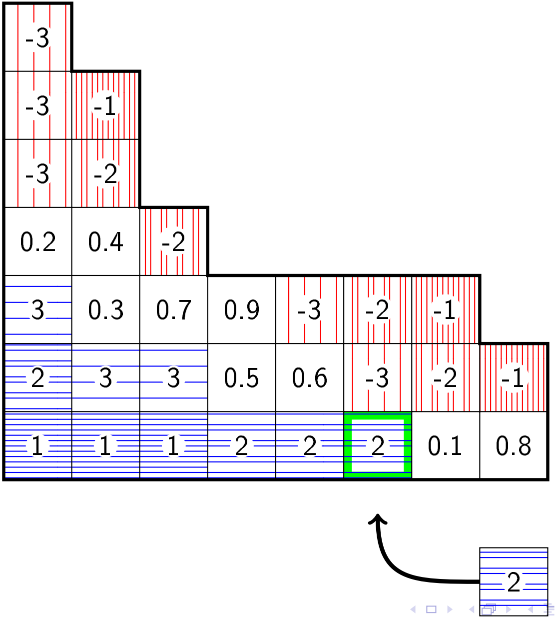

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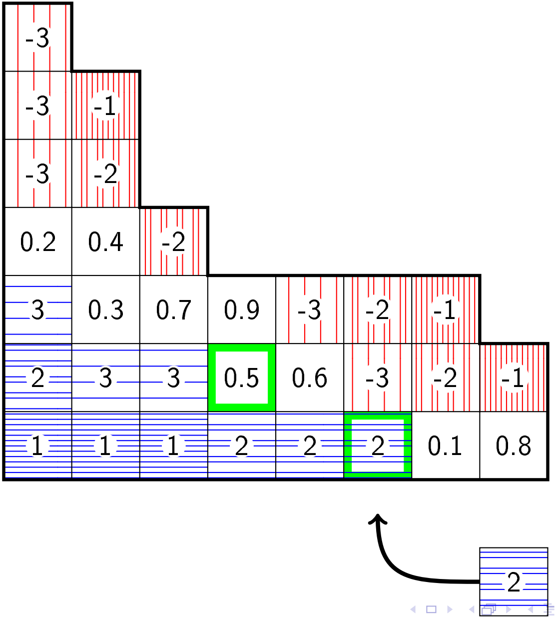

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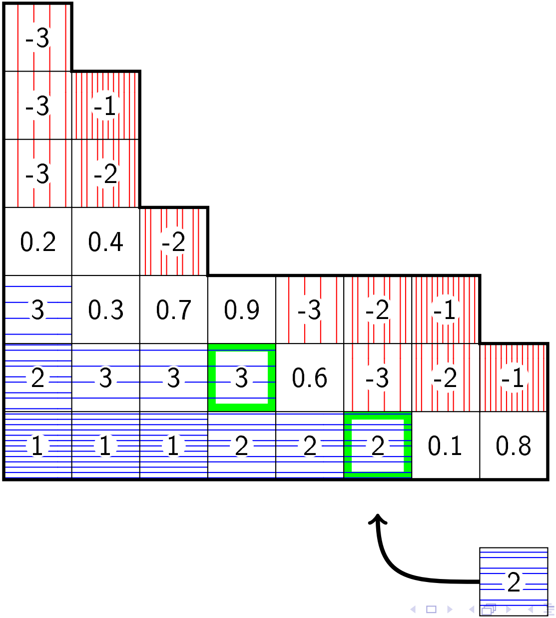

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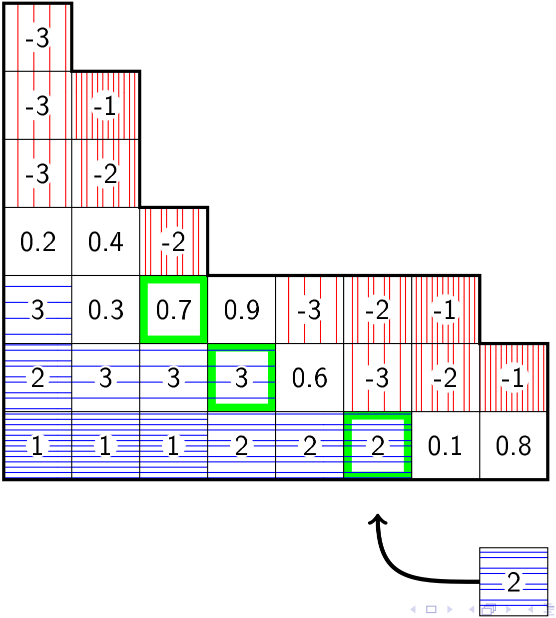

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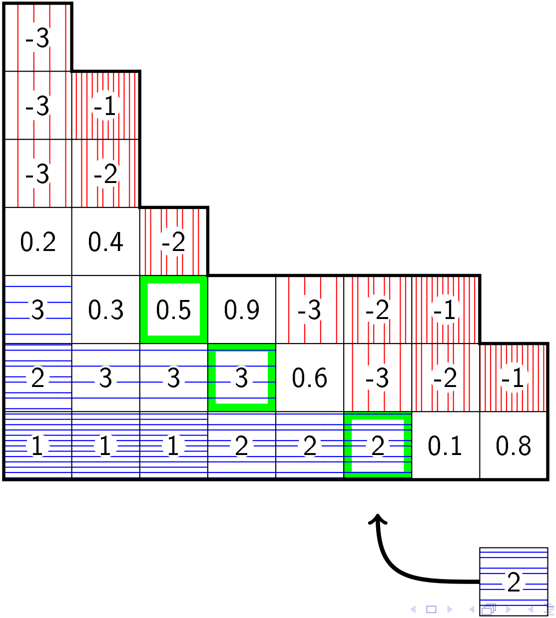

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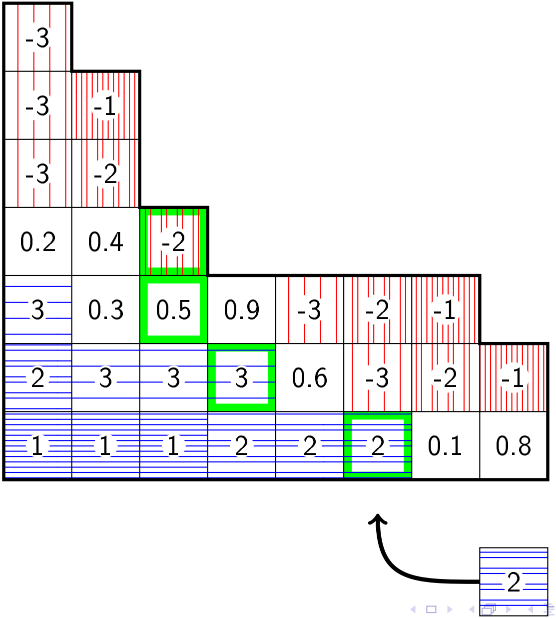

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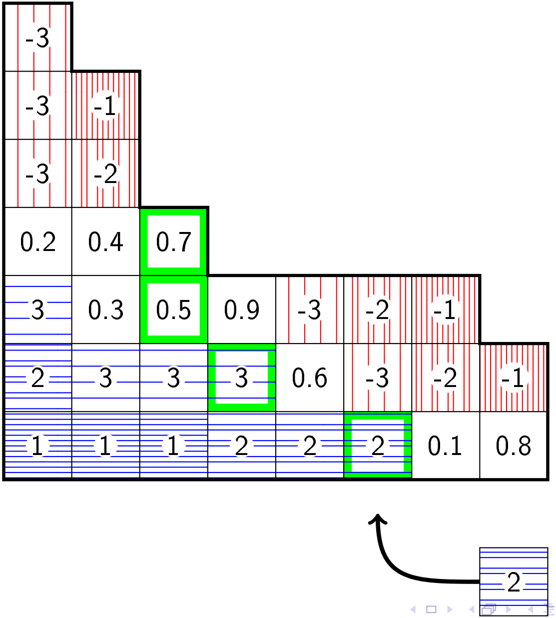

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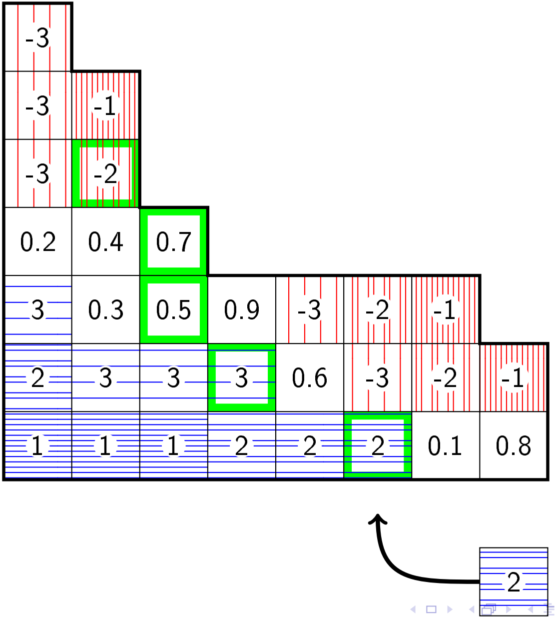

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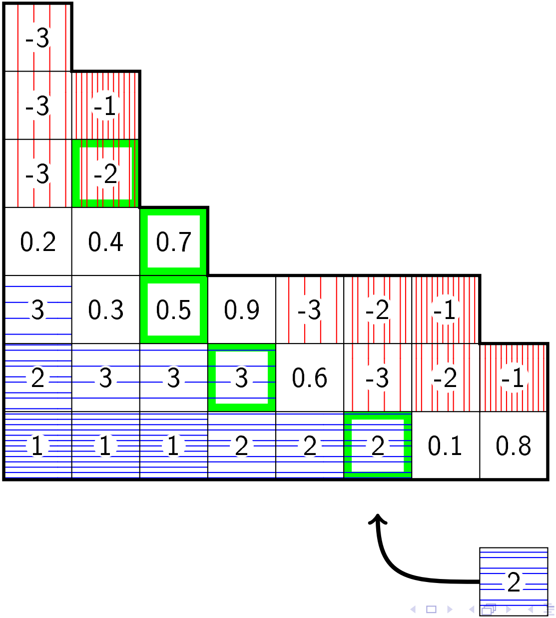

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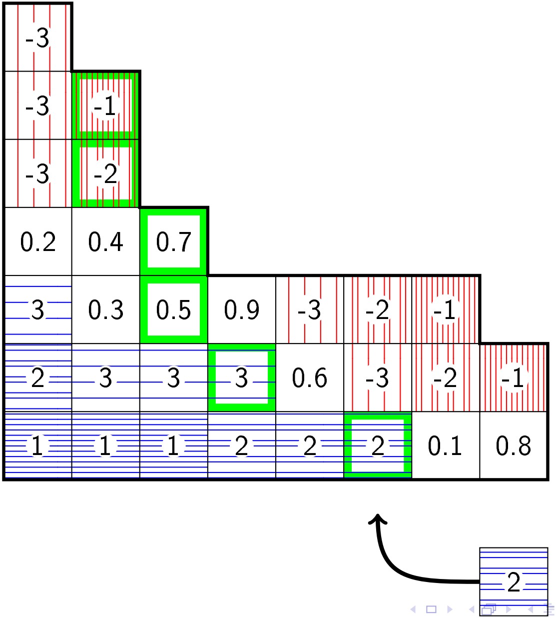

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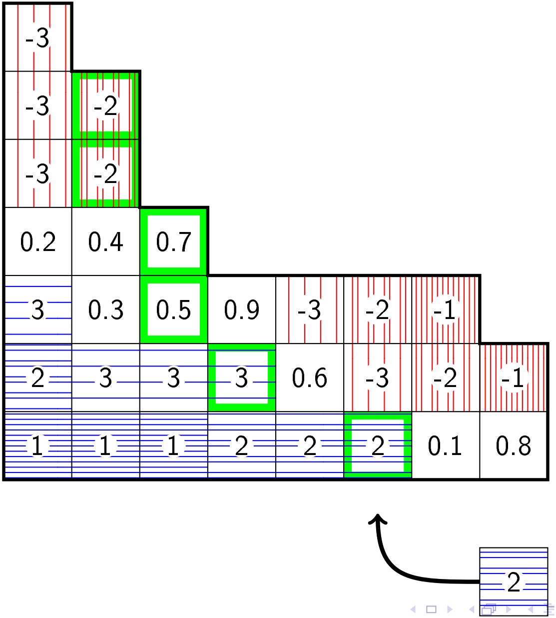

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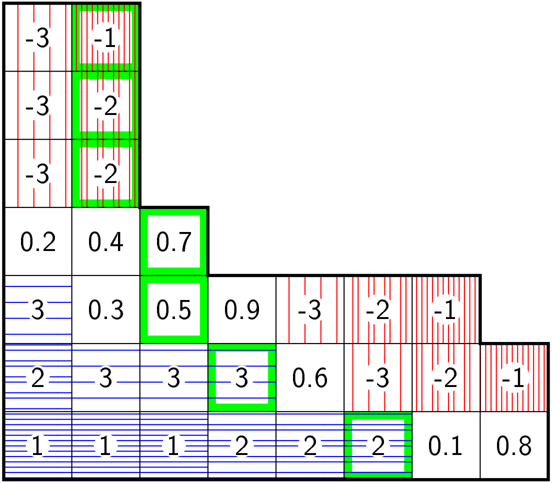

Theorem (Vershik&Kerov 1985)

let $\mathbb{A}$ be an arbitrary alphabet,
with the probabilities of atoms of row letters $\alpha_1 \geq \alpha_2 \geq \cdots$
and the probabilities of atoms of column letters $\beta_1 \geq \beta_2 \geq \cdots$

let χ be the corresponding Thoma character

if $\mathbf{w} = (w_1, w_2, \dots)$ is a sequence of iid random letters from $\mathbb{A}$
then $Q(\mathbf{w})$ is a random infinite tableau
with the distribution corresponding to χ

$$(\mathbb{A}, \mathbb{P})^{\mathbb{N}}$$

$$\Big\downarrow Q$$

$$(\mathcal{T}, \text{harmonic measure corresponding to } \chi)$$

8	13	18	32
6	9	12	23
4	5	7	19
1	2	3	10

jeu de taquin

① start with $t \in \mathcal{T}$,

8	13	18	32
6	9	12	23
4	5	7	19
1	2	3	10

jeu de taquin

① start with $t \in \mathcal{T}$,

② remove corner box,

8	13	18	32
6	9	12	23
4	5	7	19
	2	3	10

jeu de taquin

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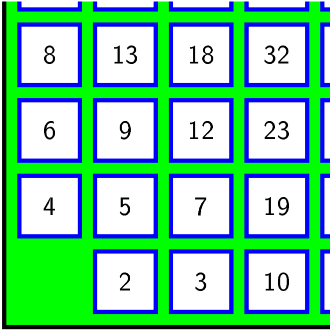

jeu de taquin

① start with $t \in \mathcal{T}$,

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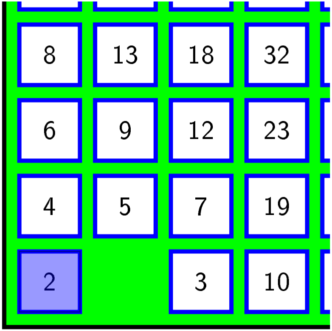
③ sliding,



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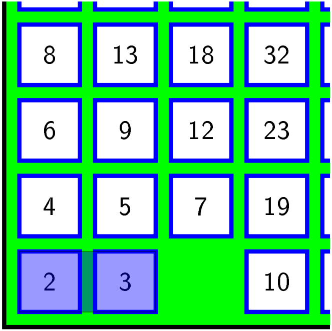

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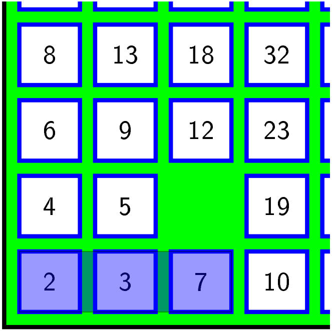

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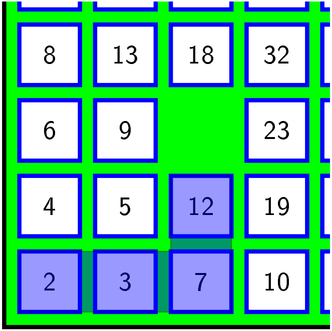

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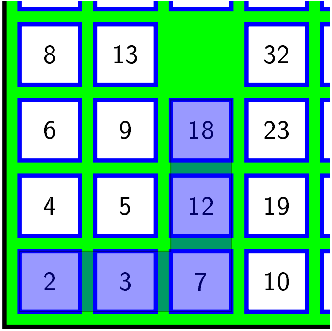

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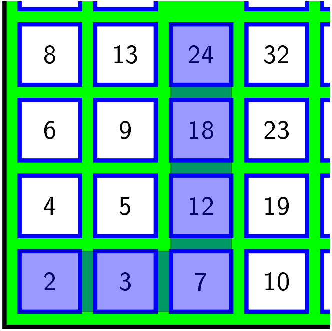

jeu de taquin

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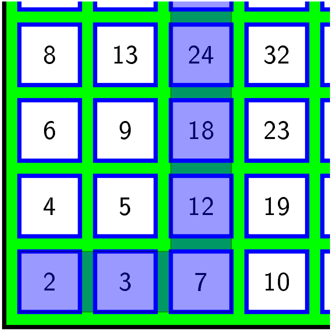

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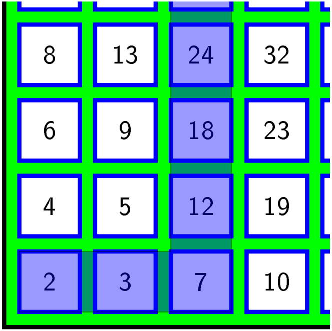
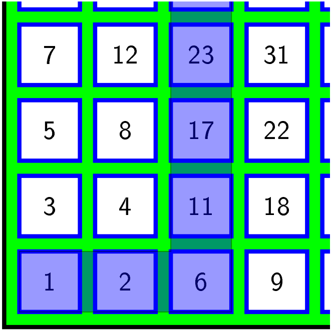


jeu de taquin

① start with $t \in \mathcal{T}$,
② remove corner box,
③ sliding,
④ subtract 1 from all boxes



jeu de taquin

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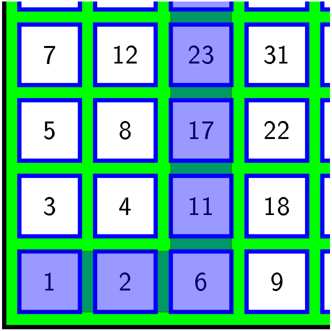
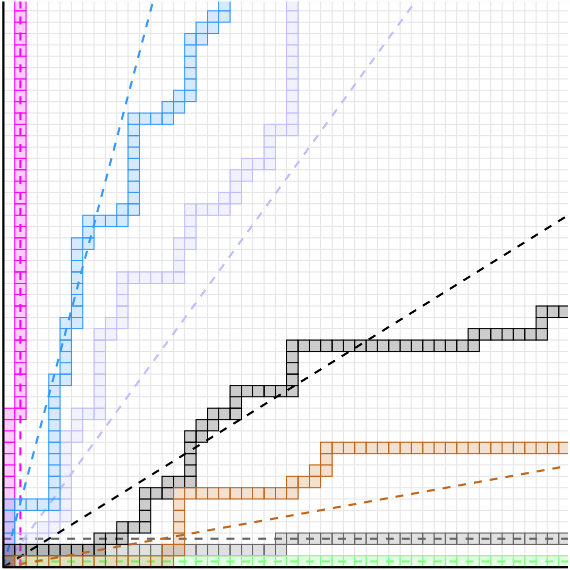


jeu de taquin

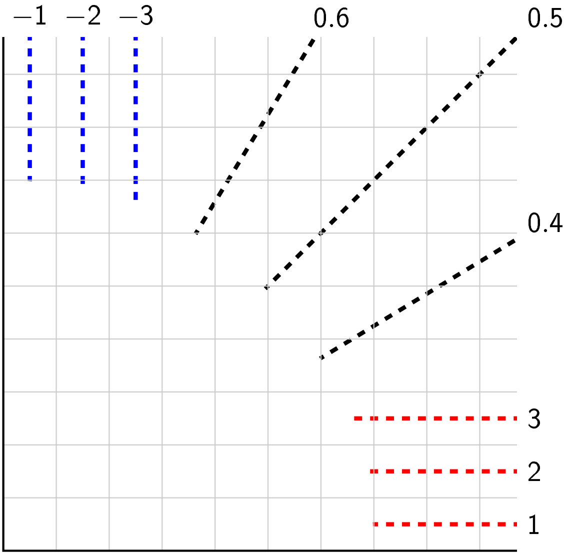
① start with $t \in \mathcal{T}$,
② remove corner box,
③ sliding,
④ subtract 1 from all boxes

output:

- new tableau $J(t)$,
- blue trajectory $\mathbf{c}(t) = (c_1, c_2, \dots)$



possible asymptotes $\Theta(t)$ for jdt trajectory



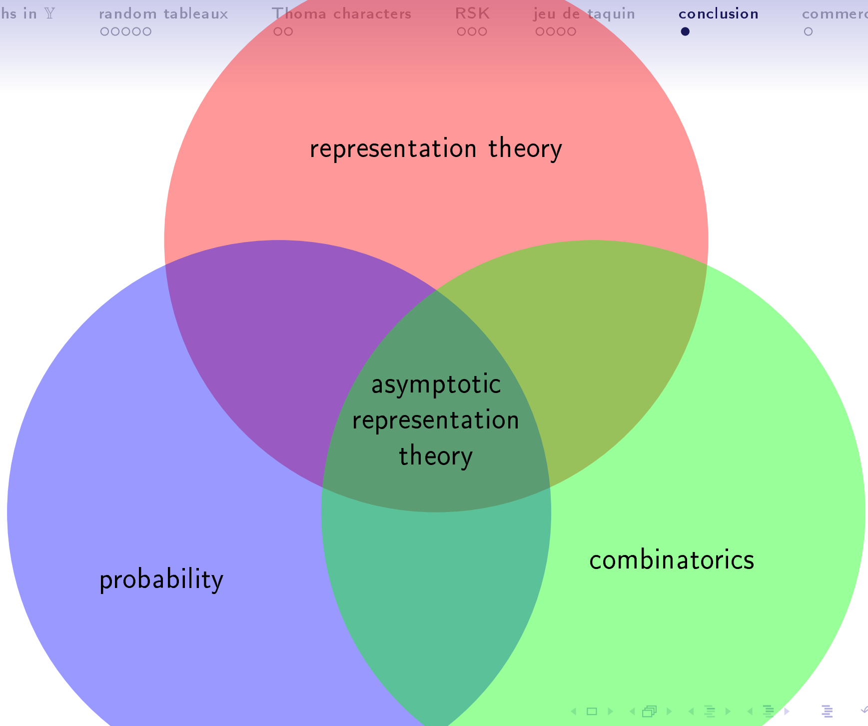
theorem, Śniady 2014

if $\mathbb{A}$ is the key example alphabet

i.i.d. shift dynamical system $(\mathbb{A}^{\mathbb{N}}, S)$

$$
\begin{array}{ccccc}
(w_1, w_2, \dots) & \xmapsto{S} & (w_2, w_3, \dots) & \xmapsto{S} & \cdots \\
\Big\downarrow Q \;\Big\uparrow \text{inverse!} & & \Big\downarrow Q & & \\
t_1 & \xmapsto{J} & t_2 & \xmapsto{J} & \cdots \\
\Big\downarrow \Theta & & \Big\downarrow \Theta & & \\
w_1 & & w_2 & &
\end{array}
$$

jeu de taquin dynamical system $(\mathcal{T}, \mathbb{P}_\chi, J)$


representation theory
asymptotic representation theory
probability
combinatorics

Dan Romik, Piotr Śniady
Jeu de taquin dynamics on infinite Young tableaux and second class particles
Ann. Probab, Volume 43, Number 2 (2015), 682-737

Piotr Śniady
Robinson–Schensted–Knuth algorithm, jeu de taquin and Kerov–Vershik measures on infinite tableaux.
SIAM J. Discrete Math. 28 (2014), no. 2, 598–630.

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preprints of ⟶Cédric Lecouvey, Emmanuel Lesigne, Marc Peigné