1

Normalized characters of symmetric groups.

(usual) character : $\chi^{\lambda}(\pi) = \operatorname{Tr} \underbrace{\rho^{\lambda}}(\pi)$

Young diagram with n boxes

permutation in S_n

irreducible representation corresponding to λ.

Typically, character is viewed as a function on the group.

normalized character:

makes sense because $\pi \in S_k \subset S_n$

Young diagram with n boxes

$$\Sigma_\pi(\lambda) = \underbrace{n\cdot(n-1)\cdots(n-k+1)}_{k \text{ factors}} \; \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\operatorname{Tr} \rho^\lambda(e)}$$

permutation in S_k

"in how many ways S_k can be embedded into S_n?"

dimension of the representation

normalized character: we fix the conjugacy class π and we vary the Young diagram λ.

very smart idea! → Kerov

If $n < k$ we just define $\Sigma_\pi(\lambda) = 0$

2 Stanley formula

Stanley formula will be our favorite tool!

For $\pi \in S_k$

$$\Sigma_\pi(\lambda) = \sum_{\substack{\sigma_1, \sigma_2 \in S_k \\ \sigma_1 \sigma_2 = \pi}} (-1)^{\sigma_1} N_{\sigma_1, \sigma_2}(\lambda)$$

~~later~~ formulated by Stanley in a completely different way
proved by Féray
this formulation: Féray, Śniady

where...

$$N_{\sigma_1,\sigma_2}(\lambda) := N_{\mathcal{G}}(\lambda)$$

bipartite graph associated to σ_1, σ_2

$\mathcal{G}$: Vertices: $\mathcal{V}_1 \sqcup \mathcal{V}_2$

empty vertices

solid vertices

empty ○ vertices correspond to cycles of σ_1

solid ● vertices correspond to cycles of σ_2

(vertices from $\mathcal{V}_i$ correspond to cycles of σ_i)

Vertices c_1 and c_2 are connected by an edge iff cycles c_1 and c_2 have nontrivial intersection.

$N_G(\lambda)$ is the number of pairs (h_1, h_2) s.t.:

- $h_i : V_i \longrightarrow \mathbb{N}$

  we think that

  $h_1 : V_1 \longrightarrow \mathbb{N}$ „columns"

  $h_2 : V_2 \longrightarrow \mathbb{N}$ „rows"

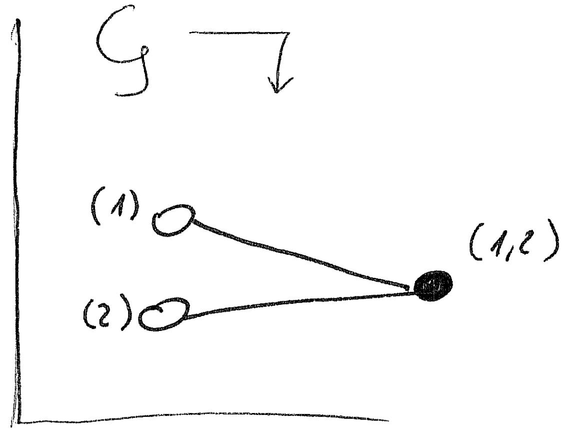
- if vertices $v_1 \in V_1$, $v_2 \in V_2$ are connected by an edge, we require that

$$\underbrace{(h_1(v_1), h_2(v_2))}_{\text{box in } h_1(v_1) \text{ column, } h_2(v_2) \text{ row}} \in \lambda$$

Example

$(12) \in S_2$ transposition.

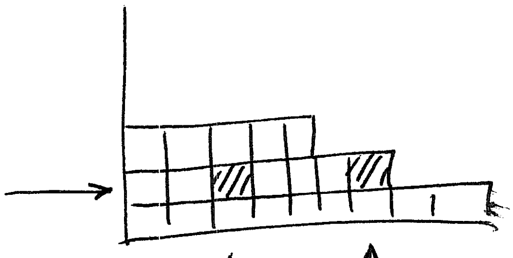
- $(12) = \underbrace{(1)(2)}_{\gamma_1} \cdot \underbrace{(12)}_{\gamma_2}$



$h_2((12))$

h_2 - we chose one of the rows of λ

$j = h_2((12))$



$h_1((1))$

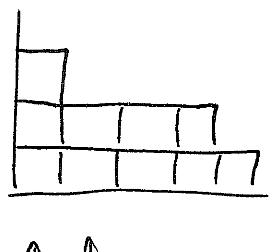
$h_1((2))$

$h_1((1))$, $h_1((2))$ -

- we chose two columns of λ
- there are λ_j^2 choices!

$$N_{\gamma_1,\gamma_2}(\lambda) = \sum_j (\lambda_j)^2$$

We use french convention:



← λ_2 boxes

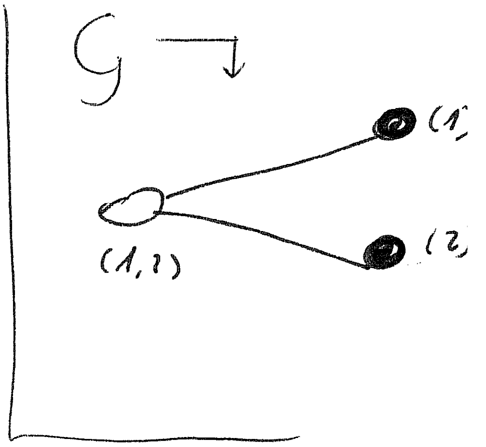
← λ_1 boxes

λ'_1 boxes

λ'_2 boxes

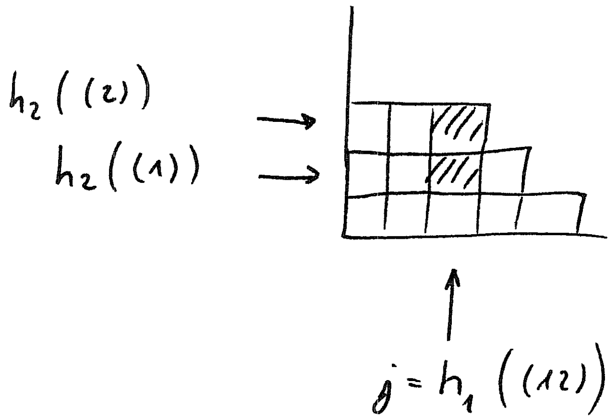
...

- $(12) = \underbrace{(12)}_{g_1} \cdot \underbrace{(1)(2)}_{g_2}$



analogous:

$$N_{g_1, g_2}(\lambda) = \sum_j (\lambda_j')^2$$



Therefore:

$$\underbrace{\sum\nolimits_{(1,2)}(\lambda)} = +\sum_j \lambda_j^2 - \sum_j {\lambda_j'}^2$$

$$= n(n-1)\,\frac{\mathrm{Tr}\, \rho^\lambda((12))}{\mathrm{Tr}\, \rho^\lambda(e)}$$

where $n = |\lambda|$