Forgotten tools of probability theory

Cumulants

Probability space

Kolmogorov probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Conditional expectation

$\mathcal{H} \subseteq \mathcal{F}$ σ-algebra

$\mathbb{E}(X \mid \mathcal{H}) \longleftarrow$ $\mathcal{H}$-measurable function

$\mathcal{F}$-measurable function

s.t. $$\int_H \mathbb{E}(X|\mathcal{H}) \, dP = \int_H X \, dP$$ for each $H \in \mathcal{H}$

Algebraic viewpoint on probability

(Non-) commutative probability space

Example

$\mathcal{A} = L^{\infty-}(\Omega)$, $\quad \mathbb{E} : \mathcal{A} \to \mathbb{C}$

random variables with all moments finite $\quad$ expected value

general commutative unital algebra $\mathcal{A}$

$\mathbb{E} : \mathcal{A} \to \mathbb{C}$ linear map $\quad \mathbb{E}\,1 = 1$

Conditional expectation

<u>Example.</u>

$\mathcal{A} = \mathcal{L}^{\infty-}(\Omega, \mathcal{F})$ $\qquad$ $\mathcal{B} = \mathcal{L}^{\infty-}(\Omega, \mathcal{H})$

$\mathcal{B} \subseteq \mathcal{A}$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathbb{E}_{\mathcal{B}}^{\mathcal{A}}"] \arrow[dr, "\mathbb{E}"'] & \mathcal{B} \arrow[d, "\mathbb{E}"] \\
& \mathbb{C}
\end{tikzcd}$$

$$\mathbb{E}\left(\mathbb{E}_{\mathcal{B}}^{\mathcal{A}}(a)\, b \right) = \mathbb{E}(a\, b)$$

General case:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathbb{E}_{\mathcal{B}}^{\mathcal{A}}"] \arrow[dr, "\mathbb{E} = \mathbb{E}_{\mathbb{C}}^{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "\mathbb{E}_{\mathbb{C}}^{\mathcal{B}}"] \\
& \mathbb{C}
\end{tikzcd}$$

three unital algebras,
three unital maps
no further assumptions.

Key examples.

Starting point:

$\varrho : S_n \longrightarrow \mathrm{End}\, V$ (reducible) representation of S_n

$$V = \bigoplus_{\lambda \vdash n} \underbrace{m^\lambda}_{\in \{0,1,2,\dots\}} V^\lambda$$

multiplicity

Character $\chi : S_n \longrightarrow \mathbb{R}$

$$\chi(\pi) = \mathrm{tr}\, \varrho(\pi) = \frac{\mathrm{Tr}\, \varrho(\pi)}{\dim V}$$

Example 1

$\boxed{\text{Kolmogorov probability space}}$

$\Omega = \mathbb{Y}_n$ Young diagrams with n boxes

$$\mathbb{P}_\varsigma(\lambda) = \frac{m^\lambda \dim V^\lambda}{\dim V}$$

$\boxed{\text{Non-commutative probability space}}$

$\mathcal{A} = C(\mathbb{Y}_n)$ = algebra of functions $\{F : \mathbb{Y}_n \longrightarrow \mathbb{C}\}$

product = pointwise product

$$\mathbb{E} F = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{P}_\varsigma(\lambda)\, F(\lambda)$$

Example 2.

$$\mathcal{A} = Z\,\mathbb{C}[S_n]$$

center of the symmetric group algebra with the convolution product
(= central functions, class functions)

$$\mathbb{E} f = \operatorname{tr} \varrho(f)$$

Example 2 $\cong$ Example 1

isomorphism

$$Z\mathbb{C}[\mathfrak{S}(n)] \ni f \mapsto \widehat{f} \in \mathbb{C}[\mathbb{Y}_n]$$

$$\widehat{f}(\lambda) := \operatorname{tr} \rho^{\lambda}(f)$$

isomorphism of algebras compatible with the expected value

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[rr] \arrow[dr, "\mathbb{E}_1"'] & & \mathcal{A}_2 \arrow[dl, "\mathbb{E}_2"] \\
& \mathbb{C} &
\end{tikzcd}$$

Homework:
the inverse of this isomorphism?

→ Lecture 4a **Example 1, revisited.**

normalized characters of S_n

philosophy: . . . .

$S_1 \subset S_2 \subset S_3 \subset \dots$

how to study all symmetric groups at the same time?

for a permutation $\pi \in S_\ell$

and a Young diagram λ, $|\lambda| = n$

$$Ch_\pi(\lambda) = \begin{cases} \underbrace{n(n-1)\cdots(n-\ell+1)}_{\ell \text{ factors}} \dfrac{\operatorname{Tr} \rho_\lambda(\pi)}{\dim V_\lambda} & \text{if } \ell \le n \\ 0 & \text{if } \ell > n \end{cases}$$

$= n^{\underline{\ell}}$

The usual viewpoint: fix an irreducible representation λ,

character $\pi \longmapsto \operatorname{Tr} \rho^\lambda(\pi)$

is a function on a group

Dual viewpoint: fix conjugacy class π

character

$\lambda \longmapsto Ch_\pi(\lambda)$

is a function on Young diagrams.

not a toy case

with our notations
$A_\pi = \Sigma_\pi$

$$A_2 = \sum_{\substack{1\le a,b\le n,\\ a\neq b}} (a,b) \in \mathbb{C}[\mathfrak{S}(n)]$$

random variable Ch_2
$\mathbb{Y} \ni \lambda \mapsto \mathrm{Ch}_2(\lambda)$

what is its variance?

$$A_2 \xmapsto{\text{isomorphism}} \mathrm{Ch}_2$$

$$\mathrm{Ch}_2(\lambda) = \overbrace{|\lambda|\cdot(|\lambda|-1)}^{\text{in our case } n(n-1)}\ \mathrm{tr}\ \rho^\lambda\big(\overbrace{(1,2)}^{\text{transposition}}\big)$$

$$(A_2)^2 = A_{2,2} + 4A_3 + 2A_{1,1}$$

$$\operatorname{Var} \mathrm{Ch}_2 = \mathbb{E}\left[(\mathrm{Ch}_2)^2\right] - (\mathbb{E}\,\mathrm{Ch}_2)^2 =$$

$$\mathrm{tr}\,\rho\left[(A_2)^2\right] - \left(\mathrm{tr}\,\rho(A_2)\right)^2 =$$

$$n^{\underline{4}}\,\mathrm{tr}\,\rho(2,2) + 4n^{\underline{3}}\,\mathrm{tr}\,\rho(3) + 4n^{\underline{2}}\,\underbrace{\mathrm{tr}\,\rho(\mathrm{id})}_{=1} - \left(n^{\underline{2}}\,\mathrm{tr}\,\rho(2)\right)^2 = ?$$

Moral lesson:
we need more structure.

→ Lecture 4a

Partial permutations

Reading:
Ivanov, Kerov,
„The algebra of conjugacy classes ..."
J. Math. Sciences 2001, 175:5, 4212–4230

Partial permutation of set X is a pair

$$(\pi, A)$$

A – (finite) subset of X

A is called "support"

ALTERNATIVELY

$\pi : X \to X$ is a bijection s.t. $\pi(x) = x$ if $x \notin A$

$\pi : A \to A$ is a bijection

π acts in a non-trivial way only on ~~the support~~ the support

http://goo.gl/N72B9

Reading: Philippe Biane,
„Characters of symmetric groups and free cumulants"
Springer Lecture Notes in Mathematics 1815 (2003)

$$(\pi_1, A_1) \cdot (\pi_2 ; A_2) = (\pi_1 \pi_2 , A_1 \cup A_2)$$

Partial permutations form a semigroup with the unit $(id, \emptyset)$. P_n = semigroup of partial permutations of $\{1, \dots, n\}$.

* Philosophical remark:

~~any permutation can b~~

if we want to turn a permutation into a partial permutation, we need to decide which of its fix-points should belong to the support. This might seem silly but it is a source of major headache if we work with the usual permutations. Partial permutations remember which of its fix-points are „true fix-points" and which are „cycles of length one".

We shall see very soon that partial permutations are much better than permutations!

In particular, a product of two permutations may contain a lot more fixpoints than the original factors.

Inverse system:

$$\mathbb{C}[P_1] \xleftarrow{r_1} \mathbb{C}[P_2] \xleftarrow{r_2} \mathbb{C}[P_3] \longleftarrow \cdots\cdots$$

P_n = semigroup of partial permutations of $\{1,2,\cdots,n\}$

$\mathbb{C}[P_n]$ = partial ~~se~~ permutations algebra

$r_k : \mathbb{C}[P_{k+1}] \longrightarrow \mathbb{C}[P_k]$ is the restriction, r_k is an algebra homomorphism!

$$r_k(\pi, A) = \begin{cases} (\pi, A) & \text{if } A \subseteq \{1,2,\cdots,k\} \\ 0 & \text{otherwise} \end{cases}$$

Exercise: why if we work with symmetric group algebras $\mathbb{C}[S_n]$ this does not work

$$\mathbb{C}[S_1] \leftarrow \mathbb{C}[S_2] \leftarrow \cdots\cdots \quad ?$$

This inverse system is very interesting, but too big for our purposes. We will need only some special elements in it.

$\varprojlim \mathbb{C}[P_n]$ is an algebra where we can study convolution of conjugacy classes in all symmetric groups at the same time

→ lecture 4a

(Normalized) conjugacy classes

For $\pi \in S_k$ we define

usually we will not write this!

$$\Sigma_\pi^{(n)} \in \mathbb{C}[\underbrace{P_n}_{\text{partial permutations}}]$$

as

$$\Sigma_\pi^{(n)} = \sum_{\substack{f:\{1,\dots,k\}\to\{1,\dots,n\} \\ \text{injection}}} \Big(\underbrace{f\circ\pi\circ f^{-1}}_{f\circ\pi\circ f^{-1}:\, A \to A} \,,\, \underbrace{\text{Image of } f}_{\substack{A := \\ \text{(support)}}} \Big)$$

$$\begin{tikzcd}
\mathbb{C}[P_1] & \mathbb{C}[P_2] \arrow[l, "r_1"'] & \cdots \arrow[l] \\
\Sigma_\pi^{(1)} \arrow[u, phantom, "\in" sloped] & \Sigma_\pi^{(2)} \arrow[u, phantom, "\in" sloped] \arrow[l, mapsto, "r_1"'] & \cdots \arrow[l, mapsto]
\end{tikzcd}$$

$$\Sigma_\pi = \varprojlim \Sigma_\pi^{(n)}$$

is a "Platonic object", a heavenly conjugacy class described by π.

Example 1

$\boxed{\text{Non-commutative probability space}}$

~~$\mathcal{A} = C(\mathbb{Y}_n)$ = algebra of functions $\{F : \mathbb{Y}_n \to \mathbb{C}\}$~~

product = pointwise product

$$\mathbb{E} F = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{P}_\xi(\lambda)\ F(\lambda)$$

MORE STRUCTURE!

$$\mathcal{A} = \operatorname{span} \{ Ch_\pi \ : \ \pi\text{-partition} \}$$

⚠ Homework: show that (Ch_π) are linearly independent.

product = pointwise product of functions

⚠ why a product $Ch_{\pi_1} \cdot Ch_{\pi_2}$ is a linear combination of (Ch_π) ?

Hint → Example 2.

remains unchanged.

$$\mathbb{E}\, Ch_\pi = n\,(n-1)\cdots(n-|\pi|+1) \cdot \chi(\pi)$$

Example 2. revisited.

$$\cancel{\mathcal{A} = Z\, \mathbb{C}[S_n]}$$

$$\mathcal{A} = \operatorname{span}\left(\Sigma_\pi : \pi \text{ is a partition} \right)$$

$f = (f^{(1)}, f^{(2)}, \dots)$ element of lim $\longleftarrow$

$$\mathbb{E} f = \operatorname{tr} \varrho(f)$$

$\mathbb{E} f = \operatorname{tr} \varrho (f^{(n)})$

linear combination of partial permutations of $\{1,2,\dots,n\}$.

Hint: ignore <u>support</u> of partial permutations.

$\rightarrow$ Lecture 4

⚠ Check that Σ_π are linear indep.

⚠ Check that a product $\Sigma_{\pi_1} \cdot \Sigma_{\pi_2}$ is a linear combination of (Σ_π).

Example 2 $\cong$ Example 1

$$\Sigma_\pi \longmapsto Ch_\pi$$

disjoint product

with our notations

$A_\pi = \Sigma_\pi$

$\mathbb{C}[\mathfrak{S}(n)] \supseteq Z\mathbb{C}[\mathfrak{S}(n)]$

some calculations are simple

no simple calculations

$(1,2)\cdot(3,4) = (1,2)(3,4)$

$A_2 \cdot A_2 = A_{2,2} + 4A_3 + 2A_{1,1}$

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new disjoint product

$$A_2 \bullet A_2 = A_{2,2},$$
$$A_2 \bullet A_3 = A_{3,2},$$
$$A_{3,2} \bullet A_3 = A_{3,3,2},$$

$\longrightarrow$ partial permutations of Ivanov & Kerov

Disjoint product of partial permutations.

$$(\pi_1, A_1) \cdot (\pi_2; A_2) = (\pi_1 \pi_2 , A_1 \cup A_2)$$ usual product of partial permutations

$$(\pi_1, A_1) \bullet (\pi_2; A_2) = \begin{cases} (\pi_1 \pi_2 , A_1 \cup A_2) & \text{if } A_1 \cap A_2 = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

~~product in $\mathcal{P}_n$~~

product in $\mathbb{C}[\mathcal{P}_n]$

product $\bullet$ works nicely with the inverse limits and gives a well-defined product in $\mathcal{A} = \operatorname{span}(\Sigma_\pi)$

$$\Sigma_{\pi_1} \bullet \Sigma_{\pi_2} = \Sigma_{\pi_1 \cup \pi_2}$$

Example 3. $\mathcal{A} = \operatorname{span}(\Sigma_\pi : \pi \text{ is a partition})$

product = disjoint product $\bullet$

$$\mathbb{E} f = \operatorname{tr} \rho(f)$$

Example 4.

Partitions is a commutative semigroup
product = concatenation of partitions.

$$\mathcal{A} = \mathbb{C}[\text{Partitions}]$$

$$\mathbb{E}\,\lambda = \begin{cases} \operatorname{tr} \varrho(\underbrace{\lambda, 1, \cdots, 1}_{\text{any permutation with this cycle decomposition}}) & \text{if } |\lambda| \leq n \\ 0 & \text{otherwise} \end{cases}$$

four probabilistic structures

example 3

$\mathbb{C}[S_n]$

with disjoint product

example 2

$\mathbb{C}[S_n]$

with convolution product

example 1

$\mathbb{C}[\mathbb{Y}_n]$

$$\mathbb{C}[S_n] \xrightarrow[\text{conditional expectation}]{\mathbb{E} = id} \mathbb{C}[S_n] \underset{\text{easy isomorphism}}{\simeq} \mathbb{C}[\mathbb{Y}_n]$$

$\mathbb{E} = \operatorname{tr} \rho$

$\mathbb{E} = \operatorname{tr} \rho$

"kind-of-easy"

example 4

$\mathbb{C}[\mathcal{P}]$

partitions with concatenation

$$\mathbb{C}[\mathcal{P}] \xrightarrow{\mathbb{E} = \operatorname{tr} \rho} \mathbb{C}$$

four important aspects of a representation

four probabilistic structures: covariances

$$\mathrm{Cov}^{id}(A_2,A_2) = A_2 \bullet A_2 - A_2 \cdot A_2 = A_{2,2} - A_2 \cdot A_2$$

example 2

$$\mathrm{Cov}(A_2,A_2) = \operatorname{tr}\rho(A_2 \cdot A_2) - [\operatorname{tr}\rho(A_2)]^2$$

example 3

$$\mathrm{Cov}^{\bullet}(A_2,A_2) = \operatorname{tr}\rho(A_{2,2}) - [\operatorname{tr}\rho(A_2)]^2$$

example 4

$$\mathrm{Cov}((2),(2)) = \operatorname{tr}\rho(2,2) - [\operatorname{tr}\rho(2)]^2$$

$$\begin{tikzcd}
\mathbb{C}[S_n] \arrow[r, "\mathbb{E}=id"', "\text{conditional expectation}"] \arrow[dr, "\mathbb{E}=\operatorname{tr}\rho"] \arrow[d, dashed, leftrightarrow] & \mathbb{C}[S_n] \arrow[d, "\mathbb{E}=\operatorname{tr}\rho"] \\
\mathbb{C}[\mathcal{P}] \arrow[r, "\mathbb{E}=\operatorname{tr}\rho"] & \mathbb{C}
\end{tikzcd}$$

$\mathbb{C}[S_n]$ with disjoint product

$\mathbb{C}[S_n]$ with convolution product

$\mathbb{C}[\mathcal{P}]$ partitions with concatenation

? RELATIONS

Var Ch_2, part 1

$$\operatorname{Var} \mathrm{Ch}_2 = \mathbb{E}(\mathrm{Ch}_2)^2 - (\mathbb{E}\,\mathrm{Ch}_2)^2 =$$

Cov (A_2, A_2) in $(\mathbb{C}[S_n], \operatorname{tr}\rho)$ with convolution product

$$\operatorname{tr}\rho(A_2^2) - \big(\operatorname{tr}\rho(A_2)\big)^2 =$$

$$\operatorname{tr}\rho(A_{2,2} + 4A_3 + 2A_{1,1}) - \big(\operatorname{tr}\rho(A_2)\big)^2 =$$

$$\underbrace{\operatorname{tr}\rho(A_2 \bullet A_2) - \big(\operatorname{tr}\rho(A_2)\big)^2}_{\operatorname{Cov}^{\bullet}(A_2,A_2)} + \underbrace{\operatorname{tr}\rho(A_2 A_2 - A_2 \bullet A_2)}_{\operatorname{tr}\rho\left(\operatorname{Cov}^{id}(A_2,A_2)\right)}$$

$\operatorname{Cov}^{\bullet}(A_2, A_2)$ in $(\mathbb{C}[S_n], \operatorname{tr}\rho)$ with disjoint product

$\operatorname{tr}\rho\left(\operatorname{Cov}^{id}(A_2, A_2)\right)$

conditional expectation

Var Ch_2, part 2

$$\operatorname{tr}\rho(A_2 \bullet A_2) - \left(\operatorname{tr}\rho(A_2)\right)^2 =$$

$$n^{\underline{4}} \operatorname{tr}\rho(2,2) - \left(n^{\underline{2}}\right)^2 \left(\operatorname{tr}\rho(2)\right)^2 =$$

$$n^{\underline{4}} \underbrace{\left[\operatorname{tr}\rho(2,2) - \left(\operatorname{tr}\rho(2)\right)^2\right]}_{\mathrm{Cov}((2),(2))} + \underbrace{\left[n^{\underline{4}} - \left(n^{\underline{2}}\right)^2\right]}_{\approx n^3} \left(\operatorname{tr}\rho(2)\right)^2$$

Cumulants.

$$k_\ell(A_1,\ldots,A_\ell) = \frac{\partial^\ell}{\partial z_1 \cdots \partial z_\ell} \log \mathbb{E} e^{z_1 A_1 + \cdots + z_\ell A_\ell} \Big|_{z_1=\cdots=z_\ell=0} = \quad (*)$$

multidimensional Laplace transform, works also with Fourier transform.

this definition is ok if we view $z_1, \ldots, z_\ell$ as formal variables ($\to$ formal power series).

$$= [z_1 \cdots z_\ell] \log \mathbb{E}\, e^{z_1 A_1 + \cdots}$$

Small examples:

$$k_1(A_1) = \mathbb{E}A_1,$$

$$k_2(A_1, A_2) = \mathbb{E}A_1A_2 - \mathbb{E}A_1\,\mathbb{E}A_2 = \mathrm{Cov}(A_1, A_2)$$

Information contained in cumulants $\longleftrightarrow$ Information contained in moments.

Cumulants work best for random variables which have all moments finite.
They work even better if the distribution is uniquely determined by its moments.

$\longrightarrow$ Hamburger moment problem

Carleman's condition

Hamburger moment problem

$\longrightarrow$ Wikipedia

For the Hamburger moment problem (the moment problem on the whole real line), the theorem states the following:

Let μ be a measure on $\mathbf{R}$ such that all the moments

$$m_n = \int_{-\infty}^{+\infty} x^n \, d\mu(x) \,, \quad n = 0, 1, 2, \cdots$$

are finite. If

$$\sum_{n=1}^{\infty} m_{2n}^{-\frac{1}{2n}} = +\infty,$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{m_{2n}}} = \infty$$

then the moment problem for m_n is *determinate*; that is, μ is the only measure on $\mathbf{R}$ with

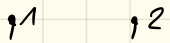
Moment-cumulant formula:

$$\mathbb{E}\, A_1 \cdots A_n = \sum_{\substack{\pi \text{ -} \\ \text{partitions of } \{1,\dots,n\}}} K_\pi (A_1, \dots, A_n) \qquad (**)$$

Examples:

$$\mathbb{E}\, A_1 = K(A_1)$$

$$\mathbb{E}\, A_1 A_2 = K(A_1, A_2) + K(A_1)\, K(A_2)$$



$$\mathbb{E}\, A_1 A_2 A_3 = \quad \dots$$

Moment-cumulant formula gives a triangular system of equations $\Rightarrow$ cumulants in terms of moments.

Homework: show that the definitions of free cumulants via (*) and via (**) are equivalent.

Hint:

$$\mathbb{E}\, X_1 \cdots X_n = [z_1 \cdots z_n]\ \mathbb{E}\, e^{z_1 X_1 + \cdots + z_n X_n}$$

$$= [z_1 \cdots z_n]\, e^{\log \mathbb{E}\, e^{z_1 X_1 + \cdots + z_n X_n}} =$$

we understand its square-free terms via cumulants

$e^a = \sum \frac{1}{k!} a^k$

$$= \sum_k \ \sum_{\substack{\text{ordered partitions of } \{1, \cdots n\} \\ \text{into } k \text{ non-empty parts}}} \frac{1}{k!} \prod_{\substack{b = \{b_1, \cdots, b_i\} \\ b \in \text{partition}}} \underbrace{[z_{b_1} \cdots z_{b_i}]\ \log \mathbb{E}\, e^{z_1 X_1 + \cdots + z_n X_n}}_{\kappa(X_{b_1}, \cdots, X_{b_i})}$$

actually, the order is not important

Fact: joint distribution of random variables $X_1, \dots, X_n$

is (multidimensional) normal

$$\Updownarrow$$

$$\forall \ell \geq 3 \quad \forall i_1, \dots, i_\ell \in \{1, \dots, n\}$$

$$K_\ell (X_{i_1}, \dots, X_{i_\ell}) = 0$$

$\Downarrow$ Hint: Gaussian distribution has a simple explicit Laplace transform

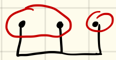
$\Uparrow$ Gaussian distribution is uniquely determined by moments

Formula of Leonov and Shiraev

"how cumulants of products are related to cumulants of individual random variables"

$$\kappa(X_1 X_2, X_3)$$

$$= \kappa(X_1, X_2, X_3)$$



$+$

$$\kappa(X_1, X_3) \quad \kappa(X_2)$$

$+$

$$\kappa(X_2, X_3) \quad \kappa(X_1)$$

$X_1, \dots, X_n$ - random variables

π - partition of $\{1, \dots, n\}$

$$\pi = \Big\{ \underbrace{\{\pi_{i,j} : 1 \le j \le |\pi_i| \}}_{\text{block } \pi_i} : 1 \le i \le \ell(\pi) \Big\}$$

Thm.

$$\kappa\Big(\overbrace{\prod_{1 \le j \le |\pi_i|} X_{\pi_{i,j}}}^{Y_i} : 1 \le i \le \ell(\pi) \Big) = \sum_{\substack{\sigma \text{ - partition of } \{1, \dots, n\} \\ \sigma \vee \pi = \mathbb{1}}} \kappa_\sigma(X_1, \dots, X_n)$$

Hint: take the above formulas ! plural form! as a definition of $\kappa(Y_{i_1}, Y_{i_2}, \dots)$. Check that moment-cumulant formula is fulfilles.

Approximate factorization property.

$$\mathbb{E} : \mathcal{A} \longrightarrow \mathcal{B}$$

graded / filtered algebras

Def.

$\mathbb{E}$ has approximate factorization property if

$$(*) \quad \boxed{\deg_{\mathcal{B}}\left(\kappa(X_1, \dots, X_\ell) \right) \leq \sum \deg_{\mathcal{A}} X_i \quad - 2(\ell-1)} \qquad \forall X_1, \dots, X_\ell \in \mathcal{A}$$

Def. $Z \subseteq \mathcal{A}$ generates nicely $\mathcal{A}$ if each $X \in \mathcal{A}$ can be expressed as a polynomial in the elements of Z AND each monomial of this polynomial is of degree $\leq$ degree of X.

Ex. $1, x, x^2, \dots$ generate nicely $\mathbb{C}[x]$

$1+x, x, x^2, \dots$ DO NOT generate nicely $\mathbb{C}[x]$

Homework: show that if Z generates nicely $\mathcal{A}$ and $(*)$ holds true for all $X_1, \dots, X_\ell \in Z$ then $(*)$ holds true for all $X_1, \dots, X_\ell \in \mathcal{A}$.

HINT: Leonov & Shiraev.

Conditional expectation

$$\begin{tikzcd} \mathcal{A} \arrow[r, "E^{\mathcal{A}}_{\mathcal{B}}"] \arrow[dr, "E = E^{\mathcal{A}}_{\mathcal{C}}"'] & \mathcal{B} \arrow[d, "E^{\mathcal{B}}_{\mathcal{C}}"] \\ & \mathcal{C} \end{tikzcd}$$

three unital algebra,
three unital maps
no further assumptions.

Conditional cumulants

Example.

$$K^{\mathcal{A}}_{\mathcal{C}}(x_1, x_2, x_3) = K^{\mathcal{B}}_{\mathcal{C}}\Big(K^{\mathcal{A}}_{\mathcal{B}}(x_1), K^{\mathcal{A}}_{\mathcal{B}}(x_2), K^{\mathcal{A}}_{\mathcal{B}}(x_3) \Big) + K^{\mathcal{B}}_{\mathcal{C}}\Big(K^{\mathcal{A}}_{\mathcal{B}}(x_1, x_2), K^{\mathcal{A}}_{\mathcal{B}}(x_3) \Big) + \text{(two other similar terms)}$$

$$+ K^{\mathcal{B}}_{\mathcal{C}}\Big(K^{\mathcal{A}}_{\mathcal{B}}(x_1, x_2, x_3) \Big)$$

Brillinger's formula.

$$K^{\mathcal{A}}_{\mathcal{C}}(X_1, \dots, X_n) = \sum_{\substack{\pi\text{-partition of} \\ \{1, \dots, n\}}} K^{\mathcal{B}}_{\mathcal{C}}\Big(K^{\mathcal{A}}_{\mathcal{B}}(X_j : j \in \pi_i) \quad : 1 \le i \le \ell(\pi) \Big)$$

$$\pi = \{\pi_1, \dots, \pi_{\ell(\pi)}\}$$

Hint:

Conditional cumulants.