Cumulants in action

Toy example: random matrices.

GUE random matrix

$$X = X^{(N)} = [X_{ij}]$$

$(X_{ij})_{1 \le i,j \le N}$ form a N^2-dimensional complex centered Gaussian vector

Covariance:

$$\mathbb{E}\, X_{ij} X_{kl} = \frac{1}{N} [i=l]\,[j=k]$$

$$\overline{X_{ij}} = X_{ji} \qquad (\Leftrightarrow X = X^*)$$

Homework: how to produce such random matrices in a simple and natural way?

Question: joint distribution of the random variables

$$\underbrace{\mathrm{tr}}_{=\frac{1}{N}\mathrm{Tr}}\, X, \quad \mathrm{tr}\, X^2, \quad \mathrm{tr}\, X^3, \dots \qquad ?$$

Example. $\mathbb{E}\, \mathrm{tr}\, X^2 \cdot \mathrm{tr}\, X^2 =$

$$\mathbb{E}\, \mathrm{tr}\, X \cdot X \cdot \mathrm{tr}\, X \cdot X =$$

$$= \frac{1}{N^2} \sum_{1 \leq i_1, \dots, i_4 \leq N} \mathbb{E}\, X_{i_1 i_2} X_{i_2 i_1} X_{i_3 i_4} X_{i_4 i_3} \overset{\text{moment-cumulant formula}}{=}$$

$$= \frac{1}{N^2} \sum_{1 \leq i_1, \dots, i_4 \leq N} X_{i_1 i_2} X_{i_2 i_1} X_{i_3 i_4} X_{i_4 i_3} +$$

cumulants

$\frac{1}{N}$ N^4 $\frac{1}{N}$ 4 loops

$$+ \frac{1}{N^2} \sum_{1 \leq i_1, \dots, i_4 \leq N} X_{i_1 i_2} X_{i_2 i_1} X_{i_3 i_4} X_{i_4 i_3} +$$

$\frac{1}{N}$ $\frac{1}{N}$ N^2 2 loops

+ (the third summand)

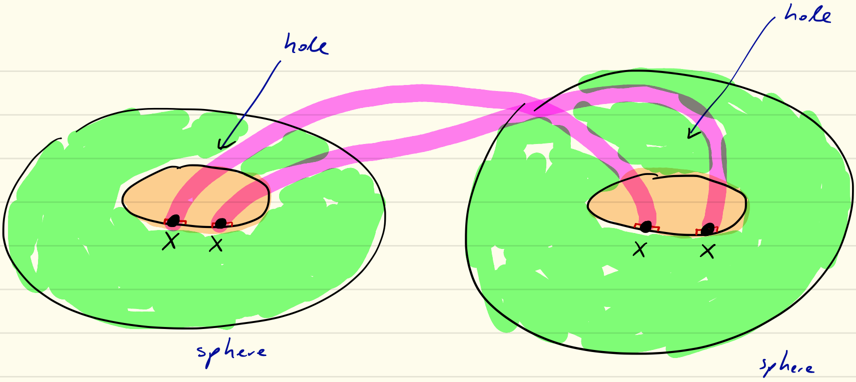
$V = 4$

$E = 4 + \frac{4}{2} =$

$= 6$

$F = 2 + 2$

$\chi = 2$



$\mathrm{tr}\ X^2 \qquad\qquad \mathrm{tr}\ X^2$

pair partition = ribbons which connect vertices on the holes

$$\frac{1}{N^{\text{number of spheres}}} \cdot \frac{1}{N^{\#\text{vertices}/2}} \; N^{\#\text{loops}} = \dots = N^{\chi - 2\,\#\text{traces}}$$

after gluing the ribbons, the collection of spheres with holes becomes ANOTHER collection of spheres with holes:

each new sphere = connected component

each new hole = loop

fixing the holes: to each hole glue a disk.

outcome: surface without boundary.

its topology = ?

Euler characteristic

$\#V = \#\text{factors}$

$\#E = \#\text{factors} + \frac{\#\text{factors}}{2}$

$\#F = \#\text{traces} + \#\text{loops}$

$$\chi = \#V - \#E + \#F = -\frac{\#\text{factors}}{2} + \#\text{traces} + \#\text{loops}$$

Step 1.

$$\mathbb{E}\left[\operatorname{tr} X^{p_1} \cdots \operatorname{tr} X^{p_\ell}\right] =$$

$$\frac{1}{N^\ell} \sum_{1 \leq i_1, \dots, i_L \leq N} \mathbb{E}\left[X_{i_1 i_{\pi(1)}} \cdots X_{i_L i_{\pi(L)}}\right] =$$

$L = p_1 + \dots + p_\ell$

$\pi = (1, 2, \dots, p_1)$
$(p_1 + 1, \dots, p_1 + p_\ell)$
$\vdots$

π encodes the product of traces we are interested in.

$$= \frac{1}{N^{\ell + L/2}} \sum_{1 \leq i_1, \dots, i_L \leq N} \sum_{\substack{\text{matching } (= \text{pair-partition}) \\ \delta}} \prod_{\{a,b\} \in \delta} [i_a = i_{\pi(b)}][i_b = i_{\pi(a)}]$$

↑ Moment-cumulant formula

Covariance

$$= \sum_{\text{matchings}} N^{\chi - 2\ell}$$

Step 2

$$K(\operatorname{tr} X^{p_1}, \dots, \operatorname{tr} X^{p_\ell}) = \sum_{\substack{\text{matchings which} \\ \text{result with a } \underline{\text{connected}} \\ \text{surface}}} N^{\chi - 2\ell}$$

(χ ← 2 - 2genus)

Corollary:

$$K(\operatorname{tr} X^{p_1}, \dots, \operatorname{tr} X^{p_\ell}) = O\left(\frac{1}{N^{2(\ell - 1)}}\right)$$

there are $\ell - 1$ commas here — wow, what a coincidence!

Moral lessons.

- each comma = degree falls by 2
- we consider <u>today</u> only oriented / orientable surfaces
  $\Rightarrow$ $\chi = 2-2g$ is an even number
  $\Rightarrow$ all exponents in N have the same parity
- your combinatorics uses non-orientable surfaces?
  $\Rightarrow$ both parities of exponents in N.
- CLT:

$$\left(\operatorname{Tr} X^{(N)\,k} - \mathrm{Catalan}_{\frac{k}{2}} \right)_{k \geq 1} \xrightarrow{d} \text{centered Gaussian vector}$$

Homework: covariance?

typical phenomena for random matrices and for representations.

Corollary

if $P_i = P_i(\operatorname{tr} X, \operatorname{tr} X^2, \dots)$
$\quad$ ↑ polynomial

then $K(P_1, P_2, \dots, P_\ell) = O\left(\frac{1}{N^{2(\ell-1)}}\right)$

Hint: formula of Leonov & Siraev

'stability of decay of cumulants'
$\longrightarrow$ a lot of CLTs

Approximate factorization property.

$\mathbb{E} : \mathcal{A} \longrightarrow \mathcal{B}$

$\mathbb{Z}$-graded / filtered algebras

How to adapt this to the example of random matrices from the previous page?

$\mathcal{A} = Alg(\operatorname{tr} X, \operatorname{tr} X^2, \dots)$

$\deg \operatorname{tr} X^k = 0$ | very simple gradation

$\mathcal{B} = \mathbb{C}[\frac{1}{N}]$ polynomials in $\frac{1}{N}$

$\deg F(N) \le d \iff F(N) = O(N^d)$

Def.

$\mathbb{E}$ has approximate factorization property if

$$(*) \quad \deg_{\mathcal{B}}\left(\kappa(X_1, \dots, X_\ell)\right) \le \sum \deg_{\mathcal{A}} X_i \quad - 2(\ell-1) \qquad \forall X_1, \dots, X_\ell \in \mathcal{A}$$

Approximate factorization property is a good news.

For applications we also want to have information about

$\kappa(X) = \mathbb{E} X$

and

$\kappa(X_1, X_2) = \operatorname{Cov}(X_1, X_2)$

for most applications it is enough to know only the dominant part (with respect to the filtration on $\mathcal{B}$).

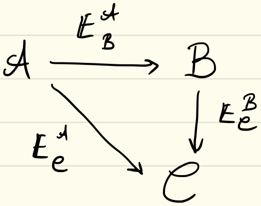
Def. $Z \subseteq \mathcal{A}$ generates nicely $\mathcal{A}$ if
each $X \in \mathcal{A}$ can be expressed as a polynomial in the elements of Z
AND each monomial of this polynomial is of degree $\leq$ degree of X.

Ex. $1, x, x^2, \dots$ generate nicely $\mathbb{C}[x]$
$1+x, x, x^2, \dots$ DO NOT generate nicely $\mathbb{C}[x]$

Homework: show that if Z generates nicely $\mathcal{A}$ and $(*)$ holds true for all $X_1, \dots, X_\ell \in Z$ then $(*)$ holds true for all $X_1, \dots, X_\ell \in \mathcal{A}$.

HINT: Leonov & Shiraev.

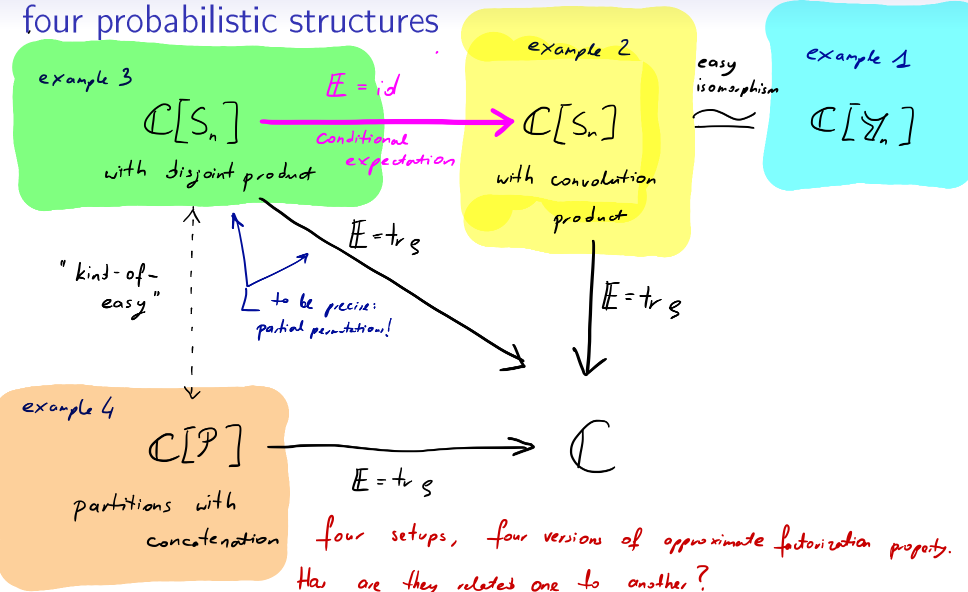
moral lesson: it is enough to prove approximate factorization property on some (special) basis !



Thm if $\mathbb{E}^{\mathcal{A}}_{\mathcal{B}}$ and $\mathbb{E}^{\mathcal{B}}_{\mathcal{C}}$ have approximate factorization property then $\mathbb{E}^{\mathcal{A}}_{\mathcal{C}} = \mathbb{E}^{\mathcal{B}}_{\mathcal{C}} \circ \mathbb{E}^{\mathcal{A}}_{\mathcal{B}}$ also has approximate factorization property.

Hint: Brillinger's formula + count commas

four probabilistic structures



four setups, four versions of approximate factorization property.

How are they related one to another?

HINT: they are equivalent and we know how $\mathbb{E}$ and Cov are related.

Representation theory and approximate factorization.

Common setup:

(ρ_n) is a sequence of (reducible) representations

$\rho_n : S_n \longrightarrow \operatorname{End} V_n$

pointwise product of functions

this example has a probabilistic meaning

Example 1.

$\mathcal{A}$ = algebra of <u>certain</u> functions on $\mathbb{Y}$

free cumulant

$\mathcal{A} = \operatorname{Alg}(Ch_\pi : \pi \text{ is a partition}) = \operatorname{Alg}(R_k : k \geq 2) =$

$= \operatorname{Span}(Ch_\pi : \pi \text{ is a partition})$

$= \operatorname{Alg}(Ch_k : k \geq 1)$

$= \operatorname{Alg}(S_k : k \geq 2)$ ← functional of shape

filtration: $\deg Ch_\pi = |\pi| + \ell(\pi)$

gradation: $\deg R_k = \deg S_k = k$

it takes some time and effort to check that all above definitions of $\mathcal{A}$ and its filtration are equivalent

$B = \{ (a_n) : |a_n| \text{ grows at most like a polynomial} \}$

filtration:

$\deg(a_n) \leq d \iff a_n = O(\sqrt{n}^{\,d}) = O(n^{d/2})$

!

$\mathbb{E}^{\mathcal{A}}_{B}(F) = \mathbb{E}\, F(\lambda_n)$

where λ_k - random Young diagram with k boxes

$\mathbb{P}_n(\lambda) = \frac{m_\lambda \cdot \dim V^\lambda}{\dim V_n}$

$V_n = \bigoplus_\lambda m_\lambda \cdot V^\lambda$

if $\mathbb{E}^{\mathcal{A}}_{B} : \mathcal{A} \to B$ has approximate factorization property, we can view law of large numbers and CLT.

Example 1 ≃ Example 2.

$\mathcal{A} = \operatorname{span}(\Sigma_\pi : \pi \text{ is a partition})$

$\operatorname{Alg}(\Sigma_\pi : \pi \text{ is a partition})$

$\operatorname{Alg}(\Sigma_k : k \geqslant 1)$

elements of $\mathcal{A}$ are certain elements in $\mathbb{C}[\mathcal{P}] = \varprojlim \mathbb{C}[\mathcal{P}_n]$

product = the usual "convolution" product

filtration: $\deg \Sigma_\pi = |\pi| + \ell(\pi)$

Example 1 ≃ Example 2.

so approximate factorization property in Example 1 $\Longleftrightarrow$ approximate factorization property in Example 2.

$$\mathrm{Ch}_\pi \longmapsto \Sigma_\pi$$

so it is trivial to relate $\mathbb{E}$ and Cov in both contexts.

The only challenge: find relation between:

- free cumulants (R_k)
- functionals of shape (S_k)
- characters (Ch_π)

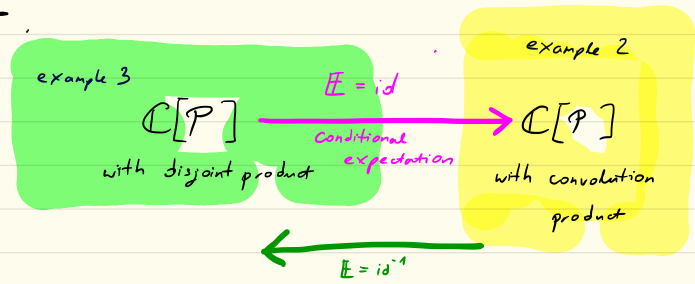
Example 3.

$$\mathcal{A} = \operatorname{span}\left(\Sigma_{1,\pi} : \pi \text{ is a partition}\right)$$
$$\phantom{\mathcal{A}} = \operatorname{Alg}\left(\Sigma_{1,\pi} : \pi \text{ is a partition}\right)$$
$$\phantom{\mathcal{A}} = \operatorname{Alg}\left(\Sigma_{1,k} : k \geq 1\right)$$

product = disjoint product.

filtration: $\deg \Sigma_{\pi} = |\pi| + \ell(\pi)$

Thm

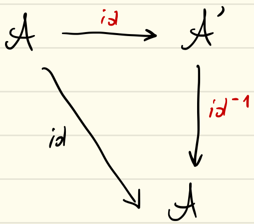


this identity map is not as trivial as it might seem! different multiplication structures: cumulants of $\mathbb{E} = id$ measure the difference between the convolution product and disjoint product.

both $\xrightarrow{id}$ and $\xleftarrow{id^{-1}}$ have approximate factorization property.

Exercise. It is enough to prove this result for <u>one</u> arrow.

HINT:



this is true identity.

cumulants have VERY simple form (which?)

$\mathcal{A}$ and $\mathcal{A}'$ are equal as vector spaces but NOT as algebras.

id is an isomorphism of VECTOR SPACES and not of ALGEBRAS.

HINT: use Brillinger's formula to find relationship between cumulants for id and id^{-1}!

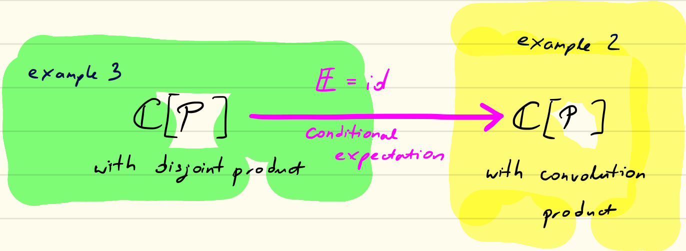
Example. $0 = K^{\text{true id}}(x,y) = K^{id^{-1}}\big(\underbrace{K^{id}(x)}_{x}, \underbrace{K^{id}(y)}_{y}\big) +$

$$\underbrace{K^{id^{-1}}\big(K^{id}(x,y)\big)}_{K^{id}(x,y)} =$$

$$= K^{id^{-1}}(x,y) + K^{id}(x,y)$$

$\longrightarrow$ USE INDUCTION!

Proof for $\xrightarrow{id}$



Enough to show that: "cycles $(\Sigma_k : k \geq 1)$ generate the algebra in a nice way that is compatible with filtration":

$$\deg\big(\underbrace{K^{id}(\Sigma_{k_1}, \dots, \Sigma_{k_l})}_{= ?}\big) \leq \sum \deg \Sigma_{k_i} - 2(l-1)$$

due to the isomorphism between Example 1 and Example 2 we can identify Σ_π with Ch_π

with a small abuse of notation it makes sense to speak about $K^{id}(Ch_{k_1}, \dots, Ch_{k_l})$

$$\deg\big(\underbrace{K^{id}(Ch_{k_1}, \dots, Ch_{k_l})}_{= ?}\big) \leq \sum \deg Ch_{k_i} - 2(l-1)$$

Example. $K^{id}(\Sigma_{k_1}, \Sigma_{k_2}) = \mathbb{E}(\Sigma_{k_1} \cdot \Sigma_{k_2}) - \mathbb{E}(\Sigma_{k_1}) \cdot \mathbb{E}(\Sigma_{k_2}) =$

$$\Sigma_{k_1,k_2} - \Sigma_{k_1} \cdot \Sigma_{k_2}$$

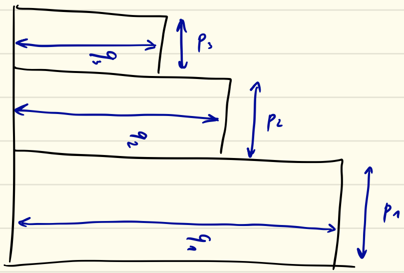
$$\stackrel{!}{=} Ch_{k_1,k_2} - Ch_{k_1} \, Ch_{k_2}$$

Lemma. Let $F \in \mathrm{span}(Ch_\pi)$

then $\deg F =$ degree of Stanley polynomial $F(\mathbf{p} \times \mathbf{q})$

— today we need only $\leq$

! carefully, we need $\mathbf{p} = (p_1, \dots, p_\ell)$, $\mathbf{q} = (q_1, \dots, q_\ell)$ of arbitrary length



the filtration on $\mathrm{span}(Ch_\pi)$ was constructed from the very start so that this condition is fulfilled.

degree of $F \in \mathrm{span}(Ch_\pi)$ = information on asymptotics of $F(\lambda)$ on large Young diagrams.

$\mathfrak{N}^{\langle k \rangle}_{\sigma_1,\sigma_2}(\lambda) := (-1)^{\#\text{cycles of } \sigma_2} \#\text{embeddings of } (\sigma_1,\sigma_2) \text{ to } \lambda$

Stanley formula:

$$Ch_{k_1,\dots,k_\ell}(\lambda) = (-1)^{\ell(\pi)} \sum_{\substack{\sigma_1,\sigma_2 \in S(k_1+\dots+k_\ell) \\ \sigma_1\sigma_2 = (1,2,\dots,k_1)(k_1+1,\dots,k_1+k_2)\cdots}} \mathfrak{N}_{\sigma_1,\sigma_2}(\lambda)$$

$$(Ch_{k_1,\dots,k_\ell} \cdot Ch_{k'_1,\dots,k'_{\ell'}})(\lambda) = (-1)^{\ell+\ell'} \sum_{\substack{\sigma_1,\sigma_2 \in S(k_1+\dots+k_\ell)\times S(k'_1+\dots+k'_{\ell'}) \\ \sigma_1\sigma_2 = \cdots}}$$

Corollary:

$$\left[K^{id}(Ch_{k_1},\dots,Ch_{k_\ell})\right](\lambda) =$$

$$(-1)^{\ell} \sum_{\substack{\sigma_1,\sigma_2 \in S(k_1+\dots+k_\ell) \\ \sigma_1\sigma_2 = (1,2,\dots,k_1)(k_1+1,\dots,k_1+k_2)\cdots \\ \langle \sigma_1,\sigma_2\rangle \text{ is TRANSITIVE}}} \mathfrak{N}_{\sigma_1,\sigma_2}(\lambda)$$

(= map is CONNECTED)

degree = #cycles (σ_1) + #cycles (σ_2) =

#Vertices of the map $\leq$

$k_1+\dots+k_\ell - \ell + 2$ =

$(k_1+1)+\dots+(k_\ell+1) - 2\ell + 2$ =

= deg $Ch_{k_1}+\dots+$ deg $Ch_{k_\ell} - 2(\ell-1)$

the map is connected!

$\chi = 2 - 2\,\text{genus} \leq 2$

"

$\#V - \#E + \#F =$

$\#V - (k_1+\dots+k_\ell) + \ell$

Proof for $\xleftarrow{id^{-1}}$

Example. $K^{id^{-1}}(\Sigma_{k_1}, \Sigma_{k_2}) = \mathbb{E}(\Sigma_{k_1} \cdot \Sigma_{k_2}) - \mathbb{E}(\Sigma_{k_1}) \bullet \mathbb{E}(\Sigma_{k_2}) =$

$$\Sigma_{k_1} \cdot \Sigma_{k_2} - \Sigma_{k_1, k_2}$$

$$K^{id}\Big(\underbrace{(\pi_1, A_1)}_{\text{partial permutation}}, \dots, (\pi_\ell, A_\ell)\Big) = \begin{cases} 0 & \text{if it is possible to divide } A_1, \dots, A_\ell \text{ to two non-empty classes, such that sets from one are disjoint with the sets from the other} \\ \Big(\pi_1 \pi_2 \cdots \pi_\ell, \underbrace{A_1 \cup A_2 \cup \dots \cup A_\ell}_{\substack{\text{size of support} \leq \\ |A_1| + \dots + |A_\ell| - (\ell - 1)}}\Big) & \text{otherwise} \end{cases}$$

this is not enough and a topological argument is necessary.

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Gaussian fluctuations of characters of symmetric groups and of Young diagrams

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Abstract. We study asymptotics of reducible representations of the symmetric groups S_q for large q. We decompose such a representation as a sum of irreducible components (or, alternatively, Young diagrams) and we ask what is the character of a randomly chosen component (or, what is the shape of a randomly chosen Young diagram). Our main result is that for a large class of representations the fluctuations of characters (and fluctuations of the shape of the Young diagrams) are asymptotically Gaussian; in this way we generalize Kerov's central limit theorem. The considered class consists of representations for which the characters almost factorize and this class includes, for example, the left-regular representation (Plancherel measure), irreducible representations and tensor representations. This class is also closed under induction, restriction, outer product and tensor product of representations. Our main tool in the proof is the method of genus expansion, well known from the random matrix theory.

1. Introduction

1.1. Representations of large symmetric groups

Irreducible representations of the symmetric groups S_q are indexed by Young diagrams and nearly all questions about them, such as values of the characters or decomposition into irreducible components of a restriction, induction, tensor product or outer product of representations can be answered by combinatorial algorithms such as Murnaghan-Nakayama formula or the Littlewood-Richardson rule. Unfortunately, these exact combinatorial tools become very complicated and cumbersome when the size of the symmetric group S_q tends to infinity. For example, a restriction of an irreducible representation consists typically of a very large number of Young diagrams and listing them all does not give much insight into their structure. In order to deal with such questions in the asymptotic region when $q \to \infty$ we should be more modest and ask questions of a more statistical flavor: what is the typical shape of a Young diagram contributing to a given representation? what are the fluctuations of the Young diagrams around the most probable shape?

In this article we are interested in the situation when—speaking informally—a typical Young diagram contributing to the considered representation of S_q has

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