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IMPAN lectures 2017/2018

Free probability and random matrices

Lecture 2. October 18, 2017
Freeness

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* asymptotic freeness

* abstract framework
  freeness

* distribution, convergence in distribution
  $\to$ S. 2.1.

* free CLT

freeness

→ MS, Section 1.12

non-commutative probability space

→ philosophical insight:

☆ Tao, Section 2.5.
pages 183–191, 194, 201

RECOMMENDED.

TODO

* non-commutative distribution

Q:
can we do better than just moments?

* convergence in distribution.

"there is no weak convergence"

philosophy:
what is
and what is NOT
captured by the limit.

Definition 12. In general we refer to a pair $(\mathcal{A}, \varphi)$, consisting of a unital algebra $\mathcal{A}$ and a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(1) = 1$, as a *non-commutative probability space*. If $\mathcal{A}$ is a $*$-algebra and φ is a *state*, i.e., in addition to $\varphi(1) = 1$ also positive (which means: $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$), then we call $(\mathcal{A}, \varphi)$ a $*$-probability space. If $\mathcal{A}$ is a C^*-algebra and φ a state, $(\mathcal{A}, \varphi)$ is a C^**-probability space*. Elements of $\mathcal{A}$ are called *non-commutative random variables* or just random variables.

If $(\mathcal{A}, \varphi)$ is a $*$-probability space and $\varphi(x^*x) = 0$ only when $x = 0$ we say that φ is *faithful*. If $(\mathcal{A}, \varphi)$ is a non-commutative probability space, we say that φ is *non-degenerate* if we have: $\varphi(yx) = 0$ for all $y \in \mathcal{A}$ implies that $x = 0$; and $\varphi(xy) = 0$ for all $y \in \mathcal{A}$ implies that $x = 0$. By the Cauchy-Schwarz inequality, for a state on a $*$-probability space "non-degenerate" and "faithful" are equivalent. If $\mathcal{A}$ is a von Neumann algebra and φ is a faithful normal state, i.e. continuous with respect to the weak-* topology, $(\mathcal{A}, \varphi)$ is called a W^**-probability space*. If φ is also a trace, i.e., $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$, then it is a *tracial* W^**-probability space*. For a tracial W^*-probability space we will usually write (M, τ) instead of $(\mathcal{A}, \varphi)$.

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Example? random matrices

$\longrightarrow$ SOON

10-minutes summary of lecture 1.

* Ginibre ensemble:

$X = (f_{ij})$ is $N \times N$ random matrix

$(\operatorname{Re} f_{ij}, \operatorname{Im} f_{ij})$ - iid $N(0, \sigma^2)$ random variables

* GUE random matrix

$Y = (g_{ij})$ is $N \times N$ random matrix

$$Y = \underbrace{X}_{\text{Ginibre.}} + X^*$$

our favorite normalization: $\mathbb{E}\, |g_{ij}|^2 = \frac{1}{N}$

* $$\lim_{N \to \infty} \mathbb{E}\, \underbrace{\frac{1}{N} \operatorname{Tr}}_{\substack{tr \,=\\ \text{normalized}\\ \text{trace}}} Y^k = \sum_{\substack{\pi \in NC_2(k) \\ \text{"noncrossing 2-partitions}\\ \text{of } [k] = \{1, \dots, k\}\text{"}}} 1$$

IMPORTANT TODAY!

PLAN FOR TODAY: use GUE as a motivating example for (asymptotic) freeness.

One good example is a good thing ☺

$\to$ MS, Section 1.11

NON-COMMUTATIVE PROBABILITY SPACE – THE KEY EXAMPLE

let $Y_{N,1}, \ldots, Y_{N,s}$ be independent $N \times N$ GUE random matrices

(we often skip it)

$\mathcal{A}_{N,i} = \mathbb{C}[Y_{N,i}]$ polynomials in $Y_{N,i}$

$\mathcal{A}_N = \mathbb{C}[Y_{N,1}, \ldots]$ (non-commutative) polynomials in $Y_{N,1}, \ldots$

$\varphi_N : \mathcal{A}_N \to \mathbb{C}$ functional

$\varphi_N(A) := \mathbb{E} \operatorname{tr} A$

"each N is a separate world".

$\boxed{\text{the limiting object.}}$

$\mathcal{A}$ = algebra of non-commutative polynomials in (abstract) variables $Y_1, \ldots, Y_s$

(✡)

$$\varphi\big(p(Y_1, \ldots, Y_s)\big) := \lim_{N \to \infty} \varphi_N\big(p(Y_{N,1}, \ldots, Y_{N,s})\big)$$

THE LIMIT EXISTS

$\to$ NEXT PAGE!

Many GUE random matrices.

M&S for long time denoted GUE random matrices by the symbol Y. At page 23 they start to use the symbol X. We have to live with this.

Assume $Y_1, \ldots, Y_s$ are independent $N \times N$ GUE matrices.

We proved that $\rightarrow$ [MS, Sect. 1, Lemma 9]

(kind of, one has to revisit the proof)

$$\lim_{N\to\infty} \mathbb{E}\, \mathrm{tr}\, Y_{i_1} \cdots Y_{i_k} = \sum_{\pi - NC_2} \underbrace{\prod_j \left[i_{\pi_{j,1}} = i_{\pi_{j,2}} \right]}_{\text{M\&S say that } \pi \text{ RESPECTS the COLORING } (i_1, \ldots, i_k)}$$

$\pi = \{ \{\pi_{1,1}, \pi_{1,2}\},$
$\vdots$

almost* like Wick formula for independent $N(0,1)$ Gaussian random variables

only non-crossing pairings.

→ MS, Section 1.10, Theorem 10

Asymptotic freeness of GUE random matrices.

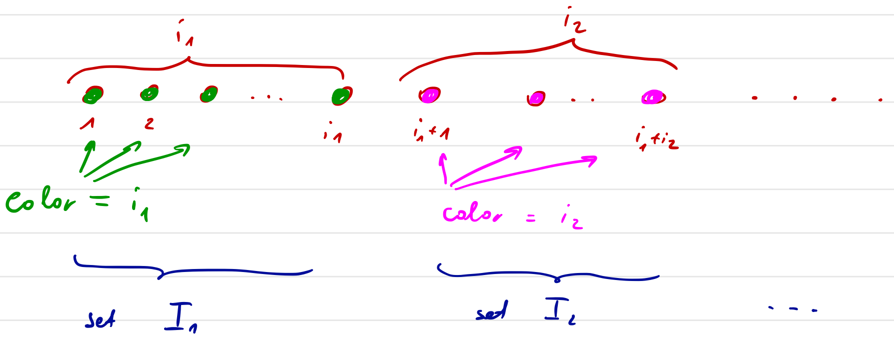
$r > 0$!

$\boxed{\text{Thm}}$ if $i_1 \neq i_2,\ i_2 \neq i_3, \dots, i_{r-1} \neq i_r$ "NEIGHBORS DIFFERENT" "ALTERNATING PRODUCT"

$$\lim_{N\to\infty} \underbrace{E\ tr}_{\varphi_N} \Big[\underbrace{\left(Y_{i_1}^{m_1} - c_{m_1}\right)}_{\text{each factor is 'centered' (at least in the limit)}} \cdots \underbrace{\left(Y_{i_r}^{m_r} - c_{m_r}\right)}\Big] = 0$$

where $c_m = \lim_{N\to\infty} E\ tr\ Y^m$

Proof DECLARATION.* WE ACCEPT ONLY 2-PARTITIONS OF $[m_1 + \dots + m_r]$ WHICH RESPECT THE COLORING GIVEN BY $i_1, \dots, i_r$.
"π respects i".



$\to$ MS, Section 1.10, Theorem 10

Asymptotic freeness of GUE random matrices.

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"NEIGHBORS DIFFERENT"

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Proof DECLARATION.* WE ACCEPT ONLY 2-PARTITIONS OF $[m_1 + \cdots + m_r]$ WHICH RESPECT THE COLORING GIVEN BY $i_1, \ldots, i_r$.

"π respects i":

$I_1 = \{1, \ldots, m_1\}$
$I_2 = \{m_1+1, \ldots, m_1+m_2\}$
$\vdots$

E_j = Non-crossing pair partitions* of $[m_1 + \cdots + m_r]$ which connect I_j only to itself

$$\ldots = \sum_{M \in [r]} (-1)^{|M|} \underbrace{\#NC_2^*\left(\bigcup_{j \notin M} I_j\right) \cdot \prod_{j \in M} \underbrace{\#NC_2(I_j)}_{c_j}}_{\# \bigcap_{j\in M} E_j} =$$

IDEA: to a NC partition of $[m_1 + \cdots + m_r]$ we associate a NC partition of $[r]$. "Connections between countries".

inclusion/exclusion

$$= \#\left(NC_2^* \setminus (E_1 \cup \cdots \cup E_r)\right) = 0$$

(1) "each country connected to another country." (3) non-crossing.

(2) $* \Rightarrow$ "cannot connect to a neighbor"

this argument will appear again soon

Hint:

each non-crossing partition contains at least one block which is an INTERVAL $\{a, a+1, \dots, b\}$

$\boxed{\text{freeness}}$

abstract framework inspired by GUE random matrices.

→ [MS] Section 1.11

DEFINITION

Let $(\mathcal{A}, \varphi)$ be a unital algebra with a unital linear functional.

Suppose $\mathcal{A}_1, \mathcal{A}_2, \dots$ are <u>unital</u> subalgebras.

We say that $\mathcal{A}_1, \mathcal{A}_2$ are freely independent (or, shortly, free) with respect to φ if for each $r \geq 2$ and $a_1, \dots, a_r \in \mathcal{A}$ st:

- $\varphi(a_i) = 0$ for $i \in [r]$
- $a_i \in \mathcal{A}_{j_i}$
- $j_1 \neq j_2, j_2 \neq j_3, \dots, j_{r-1} \neq j_r$

we must have

$$\varphi(a_1 \cdots a_r) = 0$$

"alternating product of centered elements is centered"

THIS IS JUST A DEFINITION.
WHETHER IT IS USEFUL OR NOT DEPENDS ON <u>EXAMPLES</u>

Note to myself:
sometimes we use indices $i_1, \dots, i_r$
sometimes $j_1, \dots, j_r$

DEFINITION

elements $y_1, \dots, y_r \in \mathcal{A}$ are free if

the unital algebras which they generate are free
(in the above sense).

$= \mathbb{C}[y_1], \mathbb{C}[y_2], \dots$ algebras of polynomials!

DEFINITION

sets $\subseteq \mathcal{A}$ are free if

the unital algebras which they generate are free

asymptotic freeness

Our previous results on GUE random matrices can be reformulated as follows:

the joint distribution of $Y_{N,1}, \dots, Y_{N,s}$ converges for $N \to \infty$ to the joint distribution of some elements $y_1, \dots, y_s$ which are <u>free</u>.

Hint: we have to define algebra $\mathcal{A}$ and linear unital map $\varphi : \mathcal{A} \to \mathbb{C}$.

$\mathcal{A}$ = polynomials in non-commuting variables $y_1, \dots, y_s$.

φ = ?

don't worry if φ is faithful, etc. !

RECALL:

$\longrightarrow$ MS, Section 1.10, Theorem 10

Asymptotic freeness of GUE random matrices.

$\boxed{\text{Thm}}$ if $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{r-1} \neq i_r$

"NEIGHBORS DIFFERENT"

"ALTERNATING PRODUCT"

$$\lim_{N \to \infty} \underbrace{\mathbb{E}\, \mathrm{tr}}_{\varphi_N} \Big[\underbrace{\Big(Y_{i_1}^{m_1} - c_{m_1} \Big)}_{} \cdots \underbrace{\Big(Y_{i_r}^{m_r} - c_{m_r} \Big)}_{} \Big] = 0$$

each factor is 'centered' (at least in the limit)

where $c_m = \lim \mathbb{E}\, \mathrm{tr}\, Y^m$

Theorem ⊚

any noncommutative polynomial in variables $y_1, y_2, \dots, y_r$
can be written as a
linear combination of a finite number of summands of the form

- a complex number in $\mathbb{C}$
- a product of the form $r > 0$!

$$p_1(y_{i_1})\, p_2(y_{i_2}) \cdots p_r(y_{i_r})$$

where:
- $i_1 \neq i_2,\ i_2 \neq i_3,\ \dots,\ i_{r-1} \neq i_r$
- $\varphi_{i_k}\left(p_k(y_{i_k})\right) = 0$

it is beneficial to have this decomposition in an explicit form. but it is not easy to find it.

PROOF $\longrightarrow$ SOON!

Algorithmic (inductive?) proof of ⊚

see [MS], Section 1.12 Proposition 13

→ we have to write a generic polynomial

$$(\star) \qquad p_1(y_{i_1})\, p_2(y_{i_2}) \cdots p_r(y_{i_r})$$

in this form.

HINT: we use secretly induction over r

SANITY CHECK: is $r=0$ ok?

→ without loss of generality we may assume $i_1 \neq i_2$, $i_2 \neq i_3$, ...

(otherwise we can redefine polynomials $p_1, \dots, p_r$. AND get a SHORTER product!)

→ each factor can be written as

$$p_k(y_{i_k}) = \underbrace{\varphi_k\left(p_k(y_{i_k})\right)}_{\in \mathbb{C}} + \underbrace{\left(p_k(y_{i_k}) - \varphi_k\left(p_k(y_{i_k})\right)\right)}_{\text{centered}}$$

→ multiplication is distributive! we get 2^r summands.

ONE of them is an alternating product of r factors, each centered.

a complex number does not count as a 'factor'

THE REMAINING $2^r - 1$ summands: each is a product of at most $r-1$ factors.

apply INDUCTIVE HYPOTHESIS

can iterate this algorithm.

CONSEQUENCE

if unital algebras $\mathcal{A}_1, \mathcal{A}_2, \dots$ are free

then φ is uniquely determined on $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \dots$

by $\varphi|_{\mathcal{A}_1}, \varphi|_{\mathcal{A}_2}, \dots$

Example. for ⊚

$$a_1^\circ = a_1 - \varphi(a_1)$$
$$b^\circ = b - \varphi(b)$$
$$a_2^\circ = a_2 - \varphi(a_2)$$

$$a_1 b a_2 =$$

$$\left[\varphi(a_1) + a_1^\circ\right]\left[\varphi(b) + b^\circ\right]\left[\varphi(a_2) + a_2^\circ\right] =$$

$$= \varphi(a_1)\,\varphi(b)\,\varphi(a_2) +$$

$$+ \varphi(a_1)\,\varphi(b)\,a_2^\circ +$$

$$+ \varphi(a_1)\,b^\circ\,\varphi(a_2) +$$

NICE: in the multiplication by complex numbers THE ORDER IS NOT IMPORTANT.

$$+ \varphi(a_1)\,b^\circ\,a_2^\circ +$$

$$+ a_1^\circ\,\varphi(b)\,\varphi(a_2) +$$

$$+ a_1^\circ\,\varphi(b)\,a_2^\circ +$$

SOME THINGS CAN GO WRONG:

$= \varphi(b)\,a_1^\circ a_2^\circ$ NOT ALTERNATING!

THE ALGORITHM HAS TO BE APPLIED AGAIN FOR SUCH TERMS

$$+ a_1^\circ\,b^\circ\,\varphi(a_2) +$$

$$+ a_1^\circ\,b^\circ\,a_2^\circ$$

we like this summand most.

REALLY INTERESTING THINGS START TO HAPPEN FOR A PRODUCT OF 4 FACTORS

= 16 complicated summands!

Example.

assume $\left.\begin{array}{l} a_1, a_2 \in \mathcal{A}_1 \\ b \in \mathcal{A}_2 \end{array}\right\}$ free

APPLY FUNCTIONAL φ

$$\varphi(a_1 b a_2) =$$

$$\varphi\Big(\big[\varphi(a_1) + a_1^\circ\big]\big[\varphi(b) + b^\circ\big]\big[\varphi(a_2) + a_2^\circ\big]\Big) =$$

$$= \varphi(a_1)\,\varphi(b)\,\varphi(a_2) +$$

$$+ \cancel{\varphi(a_1)\,\varphi(b)\,\varphi(a_2^\circ)} +$$

ZERO!

$$+ \cancel{\varphi(a_1)\,\varphi(b^\circ)\,\varphi(a_2)} +$$

$$+ \cancel{\varphi(a_1)\,\varphi(b^\circ\ a_2^\circ)} +$$

$$+ \cancel{\varphi(a_1^\circ)\,\varphi(b)\ \varphi(a_2)} +$$

TROUBLESOME SUMMAND.

$$+ a_1^\circ\ \varphi(b)\ a_2^\circ +$$

$$+ \cancel{\varphi(a^\circ\ b^\circ\ \varphi(a))} +$$

$$+ \cancel{\varphi(a^\circ\ b^\circ\ a^\circ)} =$$

ZERO!

the final answer is simple; the intermediate calculations complicated. Really interesting things happen for 4 factors

$$= \varphi(a_1)\,\varphi(a_2)\ \varphi(b) +$$

$$+ \varphi\Big(\big[a_1 - \varphi(a_1)\big]\big[a_2 - \varphi(a_2)\big]\Big)\ \varphi(b) =$$

$$= \varphi(a_1 a_2)\ \varphi(b)$$

we will need this calculation later!

Exercise.

$$\varphi(a_1 b_1 a_2 b_2) = ?$$

Exercise

"freeness is associative" → do [MS] prove this result?

suppose unital algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are free.

prove that the algebras

$\mathcal{A}_1$ and $\mathrm{Alg}(\mathcal{A}_2, \mathcal{A}_3)$ are free.

$\mathcal{A}_2 \vee \mathcal{A}_3$

"if y_1, y_2, y_3 are free then

y_1 and $\{y_2, y_3\}$ are free as well"

HINT: recycle Theorem @

$\boxed{\text{free CLT}}$

→ [MS], Section 2.1

→ [NS], Lecture 8

= Nica & Speicher

Hint: [MS] and [NS] prove **BOTH** classical CLT AND free CLT using the same machinery.
Nice: shows parallel ideas in these worlds.

$a_1, a_2, \dots$

— sequence of ~~independent~~ freely independent identically distributed random variables

which are centered: $\varphi(a_1) = \varphi(a_2) = \cdots = 0.$

which have all moments finite.

$\varphi(a_i^2) = 1$ NORMALIZATION

$$S_k = \frac{a_1 + \dots + a_k}{\sqrt{k}}$$

¿ distribution of S_k in the limit $k \to \infty$?

$$\varphi(S_k^{\,n}) = \frac{1}{k^{n/2}} \sum_{i:[n]\to[k]} \varphi(a_{i_1} a_{i_2} \cdots a_{i_n})$$

Def.

kernel of $i:[n]\to[k]$ is a set-partition π of $[n]$ s.t.

r and s are in the same block of π $\Longleftrightarrow$ $i_r = i_s$

Claim: if $\varphi = \ker i = \ker j$ then

$\varphi(\pi) := \varphi(a_{i_1} \cdots a_{i_n}) = \varphi(a_{j_1} \cdots a_{j_n})$ is well-defined.

$\to$ [MS] Lemma 3.

Hint: freeness determines the mixed moments by the moments of each element.

$$\varphi(S_k^{\,n}) = \frac{1}{k^{n/2}} \sum_{\pi \in P(n)} k^{\underline{\#\pi}} \, \varphi(\pi)$$

falling factorial

① if π has a singleton $\{l\}$ then $\varphi(\pi) = 0$

argument of MS on page 37 is not complete.

HINT: associativity of freeness

$$\varphi(\pi) = \varphi\big(\underbrace{X_{i_1} \; X_{i_2} \cdots X_{i_{l-1}}}_{a_1} \; \underbrace{X_{i_l}}_{b} \; \underbrace{X_{i_{l+1}} \cdots X_n}_{a_2}\big) =$$

b and $\{a_1, a_2\}$ are free

$$= \varphi(a_1 a_2) \underbrace{\varphi(b)}_{=0}$$

② if $\#\pi \leq n/2$ then the summand does not contribute in the limit.

③ enough to consider only π which are PAIR-PARTITIONS.

Hint: ① + ②

④ π is crossing? $\qquad \varphi(\pi) = 0.$

Hint: associativity of freeness.

$$\pi = \; .. \; p \; .. \; q \; .. \; r \; .. \; s \; ..$$

$$\varphi(\pi) = \varphi\Big(\underbrace{X_{i_1} \cdots X_{i_p} \cdots X_{i_{q-1}}}_{a_1 \ \text{CENTERED}} \; \underbrace{X_{i_q}}_{b_1 \ \text{CENTERED}} \; \underbrace{X_{i_{q+1}} \cdots X_{i_r} \cdots X_{i_{s-1}}}_{a_2 \ \text{CENTERED}} \; \underbrace{X_{i_s}}_{b_2 \ \text{CENTERED}} \; \underbrace{X_{i_{s+1}} \cdots X_{i_n}}_{a_3} \Big) = 0$$

$\{a_1, a_2, a_3\}$ and $\{b_1, b_2\}$ are free

! NOT centered, but this is not an issue!

⑤ π is non-crossing? $\qquad \varphi(\pi) = 1$

Hint: look at the most-nested block, center, use ①, repeat!

Corollary:

$$\lim_{k\to\infty} \varphi(S_k^n) = \# NC_2([n]) =$$

$$= \int_{-2}^{2} x^n \underbrace{\frac{1}{2\pi}\sqrt{4-x^2}\, dx}_{\text{SEMICIRCULAR LAW.}}$$

MULTIDIMENSIONAL CASE

→ [NS], lecture 8, p. 125 – 131

"like multidimensional Gaussian distribution, but in the free setup"