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IMPAN lectures 2017/2018

Free probability and random matrices

Lecture 3. November 8, 2017
Free cumulants

Lecture 3 table of contents.

* classical cumulants, moment-cumulant formula,
  Wick formula revisited,
  lattice of (all) partitions,
  Möbius function
  independence $\approx$ vanishing cumulants.

* free cumulants $\to$ 2.2.
  freeness = vanishing free
  cumulants

classical cumulants

→ [MS] Section 1.1

- single random variable / probability measure on $\mathbb{R}$

| $k_1 = \mathbb{E}X$ | ☺ |
| $k_2 = \mathrm{Var}\, X$ | NICE! |

log of Laplace transform

$$\log \mathbb{E}\, e^{tX} = \sum_{n \geq 0} k_n \frac{t^n}{n!}$$

n-th classical cumulant.

LINK: $k_n = k(\underbrace{X, \dots, X}_{n \text{ times}})$

BETTER IDEA!

- multiple random variables

$$k(X_1, \dots, X_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \log \mathbb{E}\, e^{t_1 X_1 + \dots + t_n X_n} \Big|_{t_1 = \dots = t_n = 0} =$$

$$= [t_1 \cdots t_n] \log \mathbb{E}\, e^{t_1 X_1 + \dots + t_n X_n}$$

each cumulant k is linear with respect to each of its arguments.

$K(X, Y) = \mathrm{Cov}(X, Y)$

NICE!

INDEPENDENCE $\Longrightarrow$ VANISHING OF CUMULANTS.

if $\{A_1, A_2, \dots\}$ and $\{B_1, B_2, \dots\}$ are INDEPENDENT

then

$$K(A_{i_1}, \dots, A_{i_r}, B_{j_1}, \dots, B_{j_s}) = 0 \qquad \text{IF } r \geq 1,\ s \geq 1$$

HEURISTICS:

- "cumulants quantify violation of the naive guess

$$\mathbb{E}\, X_1 \cdots X_n = \mathbb{E}\, X_1 \cdots \mathbb{E}\, X_n$$
"

- know the mean value? naive guess

$$\mathbb{E}\, X_1 X_2 \approx \mathbb{E} X_1 \cdot \mathbb{E} X_2$$

Covariance measures the violation of this guess.

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}\, X_1 X_2 - \mathbb{E} X_1 \cdot \mathbb{E} X_2$$

- know the mean value <u>and</u> covariance?

naive guess:

$$\begin{aligned} \mathbb{E}\, X_1 X_2 X_3 = {} & \mathbb{E} X_1 \cdot \mathbb{E} X_2 \cdot \mathbb{E} X_3 + \\ & + \mathbb{E} X_1 \; \mathrm{Cov}(X_2, X_3) + \\ & + \mathbb{E} X_2 \; \mathrm{Cov}(X_1, X_3) + \\ & + \mathbb{E} X_3 \; \mathrm{Cov}(X_1, X_2) \end{aligned}$$

$$\kappa(X_1, X_2, X_3) = \ldots$$

→ [MS]
Section 1.2

Exercise:

- find cumulants of the normal distribution $N(0,1)$

Hint: $$\mathbb{E}\, e^{tX} = \int_{-\infty}^{\infty} e^{ts} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds =$$

linear change of variables

$$=$$

$$= e^{\frac{1}{2}t^2}$$

$$\log \mathbb{E}\, e^{tX} = \frac{1}{2} t^2$$

- find cumulants of multidimensional Gaussian distribution.

Gaussian ⇒ cumulants ZERO (except for $\mathbb{E}$ and Cov).

"cumulants measure deviation from Gaussianity"

CLASSICAL MOMENT–CUMULANT FORMULA

→ example NEXT PAGE

$$\mathbb{E}\, X_1 \cdots X_n = [t_1 \cdots t_n]\ \mathbb{E}\, e^{t_1 X_1 + \dots + t_n X_n} =$$

$$= [t_1 \cdots t_n]\ \exp \boxed{\log \mathbb{E}\, e^{t_1 X_1 + \dots + t_n X_n}} =$$

t-free term = 0 NICE.

"set-partitions to k ordered non-empty parts"

$$= [t_1 \cdots t_n] \sum_{k \geq 0} \frac{1}{k!} \underbrace{\boxed{} \cdots \boxed{}}_{k \text{ factors.}} =$$

Hint: if $i_1, i_2, \dots$ are all different

$$[t_{i_1} t_{i_2} \cdots t_{i_\ell}] \boxed{} = \kappa(X_{i_1}, \dots, X_{i_\ell})$$

$$= \sum_{\pi \in P(n)} \underbrace{\prod_{b \in \pi} \kappa(X_i : i \in b)}_{\text{we may denote it by } \kappa_\pi(X_1, \dots, X_n)}$$

Exercise: revisit Wick formula for moments of multidimensional Gaussian distribution.

Example

$$\mathbb{E}\, X_1 = \kappa(X_1)$$

$$\mathbb{E}\, X_1 X_2 = \kappa(X_1, X_2) + \kappa(X_1)\,\kappa(X_2)$$

$$\begin{aligned}\mathbb{E}\, X_1 X_2 X_3 = {} & \kappa(X_1, X_2, X_3) + \\ & + \kappa(X_1)\,\kappa(X_2, X_3) + \\ & + \kappa(X_2)\,\kappa(X_1, X_3) + \\ & + \kappa(X_3)\,\kappa(X_1, X_2) + \\ & \quad + \kappa(X_1)\,\kappa(X_2)\,\kappa(X_3)\end{aligned}$$

Resources which might be useful for the lecturer only = IGNORE THIS PAGE

- Sandrine Dudoit

https://www.stat.berkeley.edu/~sandrine/Docs/TerrySelectedWorksSpringer/Version1/McCullagh/McCullagh.pdf

interesting personal insight into the combinatorics behind classical cumulants

- Terence Paul Speed
„Cumulants and partition lattices"
Australian J. Statist 25 (1983), no. 2 378-388

legendary paper which shows the link between cumulants and the lattices, Möbius function, etc.

- $\to$ [NS] p. 144 Corollary 9.13

$$\#\{\pi \in NC(n) \mid \pi \text{ has } k \text{ blocks}\} =$$

$$= \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

NARAYANA NUMBERS.

IGNORE THIS PAGE

NON-CROSSING PARTITIONS, revisited

→ MS, Section 2.2

→ NS, Lecture 9

non-crossing partitions appeared already in Lecture 1 (at the very end)

- non-crossing partition of an ordered set
- blocks
- partial order on NC "reverse refinement order"
- meet $\wedge$ and join $\vee$

$\pi \wedge \sigma$ = maximum of elements which are smaller than both π and σ.

Hint: take intersections of all blocks of π and σ.

Hint: $\pi \vee \sigma = ?$ take all partitions bigger than π AND σ, then calculate their MEET $\wedge$

JOIN in $\mathcal{P}$ and NC are the same.

MEET in NC and $\mathcal{P}$ are NOT the same.

- maximal / minimal element 1 and 0
- lattice = unique supremum, unique infimum

? why [NS] Proposition 9.17 does not show existence of meet $\vee$ for non-crossing partitions ?

MEET in NC and P are NOT the same.

BUT! if τ - INTERVAL partition

π - NC-partition

THEN

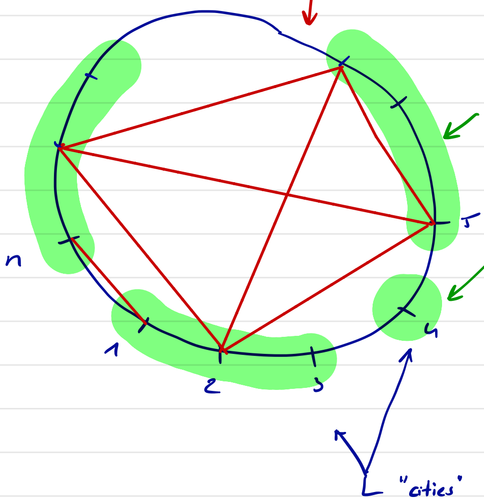
$$\tau \vee_{NC} \pi = \tau \vee_{P} \pi$$

Hint: calculating $\tau \vee_P \pi$ in a few simple steps...

1

STARTING POINT:

partition π defines an equivalence relation

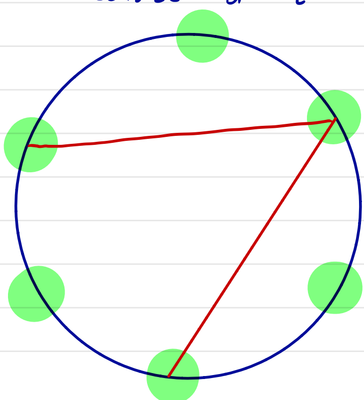


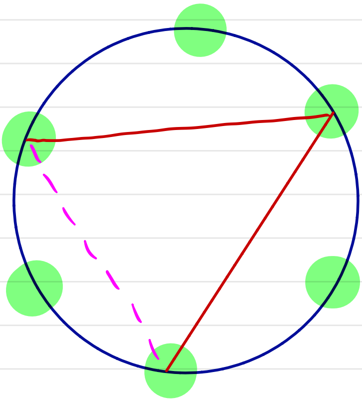
blocks of partition τ "countries"

2

two equivalence classes of τ are connected by new relation $\tilde{\pi}$ if some pair of their elements is connected.

PASS TO EQUIVALENCE CLASSES OF τ





3 not transitive?

MAKE IT TRANSITIVE by iterative closing the TRIANGLES.

at each step property "NC" is preserved.

$\tau \vee_{\beta} \delta$ is NC ✓.

FREE CUMULANTS

→ MS, Section 2.2

→ NS, lecture 11

inspired by the classical moment-cumulant formula ...

$\boxed{\text{WE DECLARE}}$

$$\varphi(a_1 a_2 \cdots a_n) = \sum_{\pi \in NC(n)} \underbrace{K_\pi(a_1, \dots, a_n)}$$

"multiplicative extension" $$:= \prod_{b \in \pi} K(a_i : i \in b)$$

we like multiplicative extensions so much that we will define multiplicative extension of φ as well → NEXT PAGE.

INSIGHT: this is an upper-triangular system of equations which CAN be inductively solved. Gives a DEFINITION of free cumulants.

Example:

$$\varphi(a_1) = K(a_1)$$

Really interesting things start to happen for 4 factors.

$$\varphi(a_1 a_2) = K(a_1, a_2) + K(a_1) K(a_2)$$

$$\varphi(a_1 a_2 a_3) = K(a_1, a_2, a_3) + \dots$$

⚠ each free cumulant is LINEAR with respect to each of its arguments

$\boxed{\text{MORE GENERAL SETUP}}$ Fix $a_1, \dots, a_n \in \mathcal{A}$

φ and κ are now functions on $NC(n)$

$$\underbrace{\varphi_\sigma(a_1 a_2 \cdots a_n)}_{:= \prod_{b \in \sigma} \varphi\left(\prod_{i \in b} a_i\right)} = \sum_{\substack{\pi \in NC(n) \\ \pi \leq \sigma}} \kappa_\pi(a_1, \dots, a_n) \qquad (\star)$$

$\boxed{\text{this is \underline{NOT} a definition; this is a COROLLARY.}}$

IMPORTANT

However ⚠ TWISTED LOGIC AHEAD

if some function $\tilde{\kappa}_\pi$ fulfills the system of equations $(\star)$ THEN $\tilde{\kappa}_\pi$ is equal to the true free cumulant κ_π

Hint: an upper-triangular system of equations has a unique solution.

convenient trick for proving Leonov–Shiryaev

IMPORTANT

Möbius inversion formula.

→ [NS] Lecture 10.

ABSTRACT NONSENSE AHEAD. FEEL FREE TO SKIP THIS PART

if P is a finite poset...

$$P^{(2)} := \{ (\pi, \sigma) : \pi \leq \sigma, \ \pi, \sigma \in P \}$$

think: $P = NC(n)$

or $P =$ all partitions of $[n]$

we are interested in the class of functions from $P^{(2)}$ to $\mathbb{C}$

CONVOLUTIONS:

- for $F, G : P^{(2)} \to \mathbb{C}$

$$(F * G)(\pi, \sigma) := \sum_{\pi \leq \rho \leq \sigma} F(\pi, \rho)\, G(\rho, \sigma)$$

this convolution can be interpreted as a matrix multiplication.

$\Rightarrow$ associativity

Homework: is it true that

$F * G = G * F$?

- for $f : P \to \mathbb{C}$

$G : P^{(2)} \to \mathbb{C}$

$$(f * G)(\sigma) := \sum_{\rho \leq \sigma} f(\rho)\, G(\rho, \sigma)$$

- $\delta : P^{(2)} \longrightarrow \mathbb{C}$

$$\delta(\pi, \sigma) = [\pi = \sigma]$$

is the unit of this convolution:

$F * \delta = \delta * F = F$

$f * \delta = f$

- $\zeta : P^{(2)} \longrightarrow \mathbb{C}$ zeta function

$$\zeta(\pi, \sigma) = 1 \qquad (\text{for } \pi \leq \sigma)$$

- $\mu : P^{(2)} \longrightarrow \mathbb{C}$ Möbius function is the inverse of ζ

$$\mu * \zeta = \zeta * \mu = \delta$$

$\Updownarrow$

left- and right- inverse (if they exist) must be equal.

$\forall \pi \leq \sigma$

$$\sum_{\pi \leq \tau \leq \sigma} \mu(\pi, \tau) = [\pi = \sigma]$$

fix π.
use induction over σ to show existence and uniqueness of $\mu(\pi, \sigma)$

END OF ABSTRACT NONSENSE.

Magic fact: Möbius function for NC is given by

$$\boxed{\mu(\varrho, \sigma) = \prod_{b \in \sigma} (-1)^{\#\varrho|_b - 1} C_{\#\varrho|_b - 1}}$$

Catalan numbers

blocks of ϱ which are sitting inside b.

any nice, conceptual proof ?

Back to free cumulants

functions on NC(n)

$$\varphi = \kappa * \zeta$$

moments — free cumulants.

$$\kappa = \varphi * \mu$$

→ [MN], Section 2.2

$$\boxed{\kappa(\sigma) = \sum_{\varrho \leq \sigma} \varphi(\varrho)\, \mu(\varrho, \sigma)}$$

THIS is THE ONLY OUTCOME OF THE ABSTRACT NONSENSE THAT WE CARE.

more concrete version: take $\sigma = 1_n$

$$\kappa(a_1, \dots, a_n) = \sum_{\varrho \in NC(n)} \varphi_\varrho(a_1, \dots, a_n)\, \mu(\varrho, 1_n)$$

Example

$$\kappa(a_1) = \varphi(a_1)$$

$$\kappa(a_1, a_2) = \varphi(a_1, a_2) - \varphi(a_1)\,\varphi(a_2)$$

$$\begin{aligned}\kappa(a_1, a_2, a_3) = \varphi(a_1, a_2, a_3) & - \\ & - \varphi(a_1)\,\varphi(a_2\, a_3) \\ & - \varphi(a_2)\,\varphi(a_1\, a_3) \\ & - \varphi(a_3)\,\varphi(a_1\, a_2) \\ & + 2\,\varphi(a_1)\,\varphi(a_2)\,\varphi(a_3)\end{aligned}$$

Free cumulants and freeness

"what free cumulants are good for?"

important and FANTASTIC Theorem

→ [MS] Section 2.2, Thm 16
→ [NS] Lecture 11 Theorem 11.16

Sets $A_1, A_2, \ldots \subseteq \mathcal{A}$ are free
(= unital algebras which they generate are free)
if and only if 'all mixed cumulants vanish',
i.e.

$$\kappa(x_1, \ldots, x_n) = 0 \quad \text{whenever} \quad x_k \in A_{i_k}$$

and $i_1, \ldots, i_n$ are not all equal.

FANTASTIC!

NO assumption that neighbors different

NO assumption on centeredness.

enough to take generators.

this characterization of freeness is more convenient than "vanishing of state on alternating product of centered elements"

PART ⇐

'want to check freeness? enough to verify on generators'.

PROOF → LATER!

PART ⇒

for set A_i you can take the whole unital algebra generated by A_i

$\rightarrow$ [MS] Section 2.2. Proposition 15.

Theorem. If $r \geq 2$! and $a_i \in \mathbb{C}$ for some i

THEN
$$\kappa(a_1, a_2, \dots, a_r) = 0.$$

Proof. use trick

remove singleton i

$$\text{Define } \widetilde{\kappa}_\pi := \begin{cases} \underbrace{\kappa_\pi(a_1, \dots, \cancel{a_i}, \dots)}_{\text{the usual free cumulant.}} \cdot a_i & \text{if } i \text{ is a singleton in } \pi \\ 0 & \text{otherwise.} \end{cases}$$

?
$$\varphi_\sigma \overset{?}{=} \sum_{\pi \leq \sigma} \widetilde{\kappa}_\pi \qquad \checkmark \quad \text{Yes!}$$

So $\widetilde{\kappa}_\pi = \kappa_\pi$ gives free cumulants.

$\square$

TOWARDS THE PROOF

→ [MS] Section 2.2 Thm 13.

LEONOV - SHIRYAEV - KRAWCZYK - SPEICHER

"how to calculate cumulants of products"?

Theorem $a_1, \dots, a_n \in \mathcal{A}$, τ is an interval partition

$$\kappa\Big(\prod_{i \in b} a_i \ : b \in \tau \Big) = \sum_{\substack{\sigma : NC(n) \\ \sigma \vee \tau = 1_n}} \kappa_\sigma (a_1, \dots, a_n)$$

cumulant of PRODUCTS

USUAL free cumulants.

Example

$\tau =$ 

$$\kappa(ab, c) = \kappa(a, b, c)$$

$$+ \kappa(a)\ \kappa(b, c)$$

$$+ \kappa(b)\ \kappa(a, c)$$

Proof. use ()

$\boxed{\text{DEFINITION}}$

$$\tilde{K}_\pi := \underbrace{\sum_{\substack{\sigma : \\ \sigma \vee \tau = \hat{\pi}}}}\ K_\sigma(a_1, \dots, a_n)$$

USUAL free cumulants.

$\sigma, \tau, \hat{\pi}$ are NC partitions of $\{1, \dots, n\}$

NC partition of $\{1, \dots, \#\tau\} =$ = blocks of τ

$$\varphi_\mu \overset{?}{=} \sum_{\pi \le \mu} \tilde{K}_\pi$$

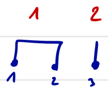
$\longrightarrow$ PROOF next page

passage from π to $\hat{\pi}$:

replace each block of τ by its entries.

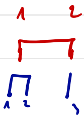
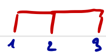
"PARTITION OF COUNTRIES $\mapsto$ PARTITION OF CITIES"

Example

for $\tau =$ 

two choices:

a) $\pi =$  $\quad \hat{\pi} =$ 

b) $\pi =$  $\quad \hat{\pi} =$ 

μ, π - partitions of $\{1, \dots, \#\tau\}$

$\hat{\pi}, \delta$ - partition of $\{1, \dots, n\}$

$$\varphi_\mu = \sum_{\pi \leq \mu} \tilde{\kappa}_\pi$$

$$L = \sum_{\delta \leq \hat{\mu}} \kappa_\delta$$

$$R = \sum_{\pi \leq \mu} \sum_{\delta : \delta \vee \tau = \hat{\pi}} \kappa_\delta =$$

$$= \sum_{\delta} \sum_{\substack{\pi \leq \hat{\mu}, \\ \delta \vee \tau = \hat{\pi}}} \kappa_\delta$$

if $\delta \not\leq \hat{\mu}$ then such π does not exist

if $\delta \leq \hat{\mu}$ then such π is unique

$[\delta \leq \hat{\mu}]$

$\pi := \delta \vee \tau$ | glue elements of τ.

"TURN PARTITION OF CITIES TO A PARTITION OF COUNTRIES"

Free cumulants and freeness

"what free cumulants are good for?"

→ [MS] Section 2.2, Thm 16

→ [NS] Lecture 11
Theorem 11.16
RECOMMENDED

important and FANTASTIC Theorem

Sets $A_1, A_2, \ldots \subseteq \mathcal{A}$ are free
(= unital algebras which they generate are free)
if and only if 'all mixed cumulants vanish',

i.e.

$$\kappa(x_1, \ldots, x_n) = 0 \quad \text{whenever} \quad x_k \in A_{i_k}$$

and $i_1, \ldots, i_n$ are not all equal.

→ PROOF ...

proof

PART $\Leftarrow$

"if mixed cumulants involving generators vanish,
more complex cumulants vanish as well"

(1) show that

$$\kappa(x_1, \dots, x_n) = 0 \quad \text{whenever} \quad x_k \in A_{i_k}$$

and $i_1, \dots, i_n$ are <u>not</u> all equal.

$$\Downarrow$$

$\mathcal{A}_{i_k} :=$

$$\kappa(x_1, \dots, x_n) = 0 \quad \text{whenever} \quad x_k \in \mathrm{Alg}(1, A_{i_k})$$

and $i_1, \dots, i_n$ are <u>not</u> all equal.

HINT:

use $\longrightarrow$ (a) free cumulants involving $\mathbb{C}$ are ZERO.
(b) Leonov – Shiryaev – Krawczyk – Speicher

(2) alternating product of centered random variables $x_k \in \mathcal{A}_{i_k}$

$$\varphi(x_1 x_2 \cdots x_n) = \sum_{\pi \in NC(n)} \kappa_\pi(x_1, \dots, x_n) = 0$$

BECAUSE...

"each NC partition contains a block which is an interval"

- centered $\Rightarrow$ singletons forbidden
- mixed cumulants vanish $\Rightarrow$ π respects i
- alternating $\Rightarrow$ no connections to neighbors

proof

[NS] Thm 11.16

PART $\Rightarrow$

$$\kappa(a_1, a_2, \dots, a_n) \overset{?}{=} 0$$

- if $a_1, \dots, a_n$ centered and alternating

$$\kappa(a_1, \dots, a_n) = \sum_{\pi \in NC} \mu(\pi, 1_n) \underbrace{\varphi_\pi(a_1, \dots, a_n)}_{=0}$$ ✓

Hint: the minimal block

- if $a_1, \dots, a_n$ ~~centered and~~ alternating ✓

Hint: $\kappa(\dots, 1, \dots) = 0$

- if $a_1, \dots, a_n$ ~~alternating~~ ✓

$i_1, i_2, \dots, i_n$ not all equal

Hint: $\tau :=$ interval partition

$a < b$ are connected by τ if

$i_a = i_{a+1} = \dots = i_b$

$\boxed{\text{INDUCTION OVER } n.}$

Leonov & Shiryaev

$$0 = \kappa\left(\prod_{i \in b} a_i : b \in \tau\right) = \kappa(a_1, \dots, a_n) +$$

alternating product

$$+ \underbrace{(\text{remaining terms})}_{=0 \text{ by inductive hypothesis}}$$

products of free random variables & Kreweras complement

→ [MS] Section 2.3.

if $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ are free...

$$\varphi(a_1 b_1 a_2 b_2 \cdots a_r b_r) = \sum_{\pi \in NC(2r)} \kappa_\pi (\overbrace{a_1, b_1, \dots, a_r, b_r}^{\pi_A}) =$$

π_B

$$= \sum_{\pi_A \in NC(r)} \kappa_{\pi_A}(a_1, \dots, a_r) \sum_{\pi_B \in NC(r)} \kappa_{\pi_B}(b_1, \dots, b_r)$$

⚠ $\pi_A \cup \pi_B$ is non-crossing

$$= \varphi_{K(\pi_A)}(b_1, \dots, b_r)$$

π_A - partition on $1, \quad 2, \quad 3$

π_B - partition on $\quad \bar{1}, \quad \bar{2}, \quad \bar{3} \; . \; .$

Wow: there exists a MAXIMAL non-crossing partition π_B such that $\pi_A \cup \pi_B$ is non-crossing.

We call it Kreweras complement of π_A $\qquad K(\pi_A)$

→ [MN], Sect. 2.4.

Functional relation

fix $a \in \mathcal{A}$

$$M(z) = 1 + \sum_{n\geq 1} \varphi(a^n)\, z^n$$

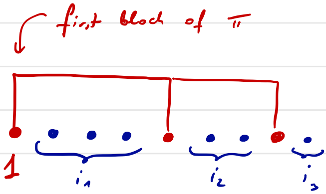
formal power series

$$C(z) = 1 + \sum_{n\geq 1} \kappa_n(\underbrace{a,\dots,a}_{n})\, z^n$$

Thm.:
$$M(z) = C(zM(z))$$

Proof coefficient at z^n:

$$[z^n]\, M(z) = \varphi(a^n) = \sum_{\pi \in NC(n)} \kappa_\pi =$$



$$= \sum_{s\geq 1} \quad \sum_{\substack{i_1,\dots,i_s \geq 0 \\ s+i_1+\dots+i_s = n}}$$

↑ number of elements in the first block

$$\kappa_s \cdot \underbrace{\sum_{\pi_1 \in NC(i_1)} \kappa_{\pi_1}}_{\varphi(a^{i_1})} \cdots\cdots \cdot$$

$$= [z^n] \sum_{s \geq 1} \sum_{\substack{i_1, \dots, i_s \geq 0 \\ \cancel{s} + \cancel{i_1 + \cdots + i_s} = n}}$$

number of elements in the first block

$$K_s \underbrace{\left(z\, z^{i_1} \underbrace{\sum_{\pi_1 \in NC(i_1)} K_{\pi_1}}_{z \cdot z^{i_1} \varphi(a^{i_1})} \right) \left(z\, z^{i_2} \cdots \right) \cdots}_{s \text{ factors}}$$

$$= [z^n] \; C\big(z\, M(z)\big)$$