Piotr Śniady

IMPAN lectures 2017/2018

Free probability and random matrices

Lecture 4b. November 22, 2017.
Kreweras complement.
Freeness of GUE and deterministic.
Weingarten calculus.

$\boxed{\text{Laziness notice}}$

I skipped
[MS] Chapter 3 "Free harmonic analysis".

Not in the mood for complex analysis.
Maybe it is a lifetime mistake?

Want to make true concrete calculations?
→ need the computation machinery.
Too bad.

→ [MN] Section 4.1

Asymptotic freeness:

averaged vs almost sure

TODO

products of free random variables & Kreweras complement

→ [MS] Section 2.3.

if $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ are free...

$$\varphi(a_1 b_1 a_2 b_2 \cdots a_r b_r) = \sum_{\pi \in NC(2r)} K_\pi(a_1, b_1, \dots, a_r, b_r) =$$

π_A (upper blocks), π_B (lower blocks)

$$= \sum_{\pi_A \in NC(r)} K_{\pi_A}(a_1, \dots, a_r) \boxed{\sum_{\pi_B \in NC(r)} K_{\pi_B}(b_1, \dots, b_r)}$$

⚠ $\pi_A \cup \pi_B$ is non-crossing

$= \varphi_{K(\pi_A)}(b_1, \dots, b_r)$

π_A - partition on $1, \quad 2, \quad 3$

π_B - partition on $\bar{1}, \quad \bar{2}, \quad \bar{3} \quad . \; .$

Wow: there exists a MAXIMAL non-crossing partition π_B such that $\pi_A \cup \pi_B$ is non-crossing.

We call it Kreweras complement of π_A $\qquad K(\pi_A)$

$$\boxed{\varphi(a_1 b_1 \cdots a_r b_r) = \sum_{\pi \in NC} K_\pi(a_1, \dots, a_r)\, \varphi_{K(\pi)}(b_1, \dots, b_r)}$$

$$\boxed{\varphi(a_1 b_1 \cdots a_r b_r) = \sum_{\pi \in NC} \kappa_\pi(a_1, \dots, a_r)\, \varphi_{K(\pi)}(b_1, \dots, b_r)}$$

Same proof, another formula:

$$\boxed{\varphi(a_1 b_1 \cdots a_r b_r) = \sum_{\pi \in NC} \varphi_{K^{-1}(\pi)}(a_1, \dots, a_r)\, \kappa_\pi(b_1, \dots, b_r)}$$

Partitions and permutations.

Fantastic idea!

$$\gamma = (1,2,\dots,n)$$
"THE LONG CYCLE"

partition of $[n]$
$\longmapsto$ permutation in S_n
⚠ CHOOSE THE CORRECT DIRECTION OF ROTATION

$\longrightarrow$ Lecture 1, page 21
length on the symmetric group

$$(*) \qquad |\pi| + |\gamma\pi^{-1}| = |\pi| + |\pi^{-1}\gamma| \geq |\gamma|$$

MAGIC FACT: equality iff π is NC

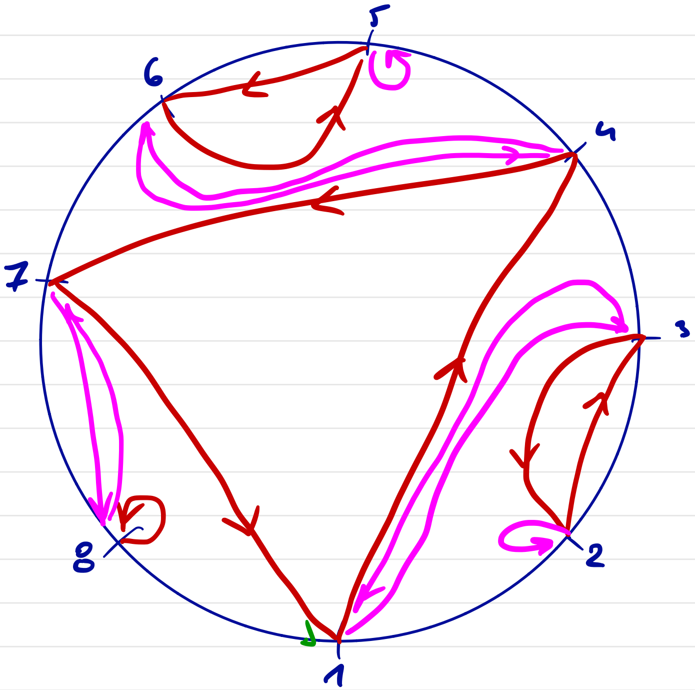
Wow: if π fulfills $(*)$ then

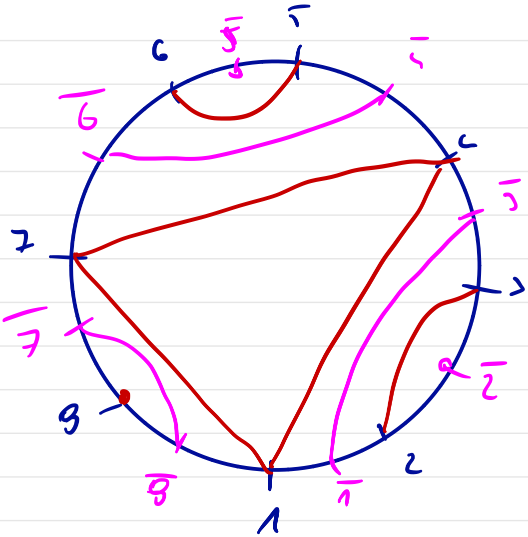
$\pi' := \pi^{-1}\gamma$
and
$\pi'' := \gamma\pi^{-1}$

also fulfill $(*)$

MYSTERIOUS NONCROSSING PARTITIONS.

π



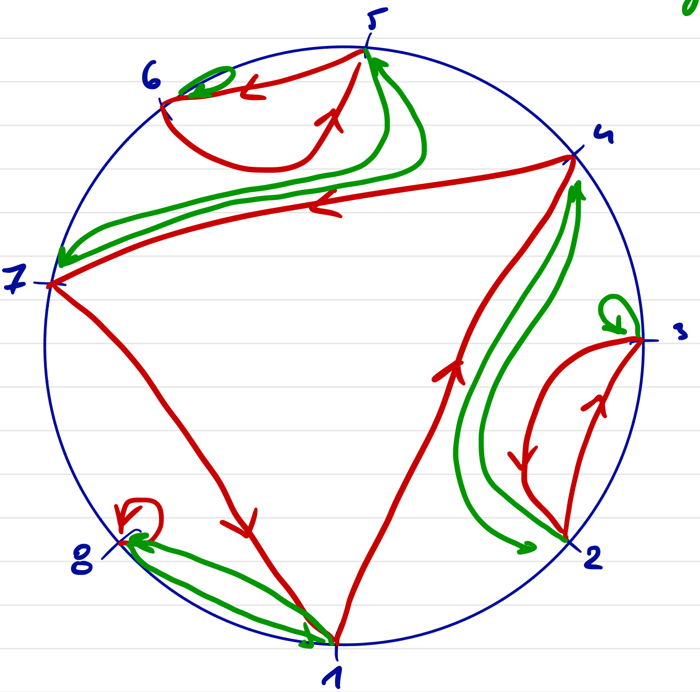


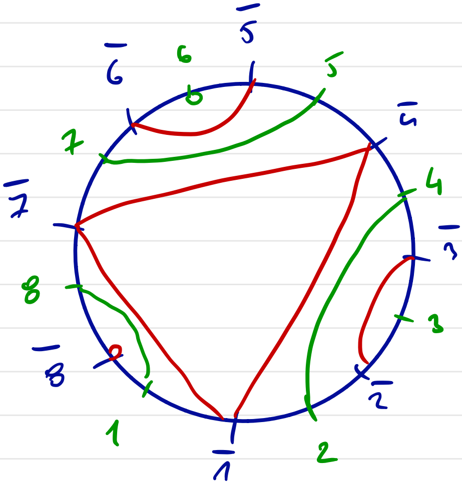
$$\boxed{\pi^{-1}\gamma = K(\pi)}$$

"proof by experiment"

$\gamma\pi^{-1}$

π





$$\boxed{\gamma\pi^{-1} = K^{-1}(\pi)}$$

Asymptotic freeness of GUE and deterministic matrices.

(A_N) - sequence of GUE random matrices

"$A_N \xrightarrow{\text{dist}} s$" convergence to the SEMICIRCULAR element

assume (D_N) - sequence of DETERMINISTIC matrices, s.t.

$$\lim_{N\to\infty} tr\, D_N^m \quad \text{exists for} \quad m = 1, 2, \dots$$

"$D_N \xrightarrow{\text{dist}} d$" the limit of the distributions exists in the world of non-commutative probability spaces

¿ joint distribution of A_N and D_N for $N \to \infty$?

$$\lim_{N\to\infty} \mathbb{E}\, tr\, D^{q(1)} A_N D^{q(2)} \cdots D^{q(m)} A_N = ?$$

BEFORE THE LIMIT

$$\mathbb{E}\, tr\, D^{q(1)} A_N D^{q(2)} \cdots D^{q(m)} A_N =$$

pass to ENTRIES of the matrices

$$\left[d_{ij}^{(k)}\right]_{ij} = D_N^{q(k)}$$

$$= \frac{1}{N} \sum_{i,j} d^{(1)}_{j_1 i_1} d^{(2)}_{j_2 i_2} \cdots d^{(m)}_{j_m i_m} \; \mathbb{E}\left[a_{i_1 j_2} a_{i_2 j_3} \cdots a_{i_m j_1}\right] =$$

Wick formula: $\underbrace{a_{i_r j_{r+1}} \quad a_{i_s j_{s+1}}}_{\pi}$

$$= \frac{1}{N^{1+m/2}} \sum_{\pi \in \mathcal{P}_2(m)} \sum_{i,j} d^{(1)}_{j_1 i_1} \cdots d^{(m)}_{j_m i_m} \prod \left[i_r = j_{\gamma\pi(r)} \right] =$$

$\gamma = (1, 2, \ldots, m)$ THE LONG CYCLE

PAIR PARTITIONS = PERMUTATIONS s.t. EACH CYCLE HAS LENGTH 2.

$$= \frac{1}{N^{1+m/2}} \sum_{\pi \in \mathcal{P}_2(m)} \underbrace{\sum_{j} d^{(1)}_{j_1 j_{\gamma\pi(1)}} \cdots d^{(m)}_{j_m j_{\gamma\pi(m)}}}_{?}$$

NOTATION.

$$\underbrace{tr_{(1,6,3)(4)(2,5)}}_{\in S_6}[D_1, \dots, D_6] = \text{"multiplicative extension of the normalized trace"}$$

$$= tr\, D_1 D_6 D_3 \cdot tr\, D_4 \cdot tr\, D_2 D_5$$

EXERCISE:

$$tr_\gamma(A_1, \dots, A_n) = \frac{1}{N^{\#\gamma}} \sum \underbrace{a^{(1)}_{i_1 i_{\gamma(1)}}}_{\text{entries of } A^{(1)}} \cdots a^{(n)}_{i_n i_{\gamma(n)}}$$

$$\mathbb{E}\, tr\, D^{q(1)} A_N D^{q(2)} \cdots D^{q(m)} A_N = \dots$$

$$= \frac{1}{N^{1+m/2}} \sum_{\pi \in P_2(m)} \underbrace{\sum_j d^{(1)}_{j_1 j_{\gamma\pi(1)}} \cdots d^{(m)}_{j_m j_{\gamma\pi(m)}}}_{N^{\#\gamma\pi} \cdot tr_{\gamma\pi}\left[D_N^{q(1)}, \dots, D_N^{q(m)}\right]} =$$

$$= \sum_{\pi \in P_2(m)} N^{\#\gamma\pi - 1 - m/2} \; tr_{\gamma\pi}\left[D_N^{q(1)}, \dots, D_N^{q(m)}\right]$$

→ Lecture 1, page ≈ 21

$=0$ if γ is non-crossing

<0 if γ is crossing.

$$\lim_{N\to\infty} \mathbb{E}\,\mathrm{tr}\, D^{q(1)} A_N D^{q(2)} \cdots D^{q(m)} A_N = \cdots$$

$$= \sum_{\pi \in NC_2(m)} \varphi_{\gamma_\pi}\left(d^{q(1)}, \ldots, d^{q(m)}\right) \quad 1$$

$$\boxed{K_\pi(\underbrace{s, \cdots, s}_{m \text{ times}})}$$

$= 1$ if $\pi \in NC_2(m)$

$$= \sum_{\pi \in NC(m)} \varphi_{\underbrace{\gamma_\pi}_{=\gamma\pi^{-1} = K^{-1}(\pi)}}\left(d^{q(1)}, \ldots, d^{q(m)}\right) K_\pi(\underbrace{s, \cdots, s}_{m \text{ times}})$$

no restriction to 2-partitions !

REMEMBER? if $\{a_1, \ldots, a_l\}$ and $\{b_1, \ldots, b_l\}$ are free then...

$$\boxed{\varphi(a_1 b_1 \cdots a_l b_l) = \sum_{\pi \in NC} \varphi_{K^{-1}(\pi)}(a_1, \ldots, a_l)\, K_\pi(b_1, \ldots, b_l)}$$

put this proof on STEROIDS. Then...

→ [MS] Section 4.2 Thm 4

CONCLUSION:

if $A_N^{(1)}, \dots, A_N^{(p)}$ are independent GUE matrices

$D_N^{(1)}, \dots, D_N^{(q)}$ are deterministic $N \times N$ matrices s.t.

$$D_N^{(1)}, \dots, D_N^{(q)} \xrightarrow{\text{distr}} d_1, \dots, d_q$$

THEN

$$A_N^{(1)}, \dots, A_N^{(p)}, D_N^{(1)}, \dots, D_N^{(q)} \xrightarrow{\text{distr}} s_1, \dots, s_p, d_1, \dots, d_q$$

WHERE

$s_1, \dots, s_p$ – semicircular and

$s_1, \dots, s_p, \{d_1, \dots, d_q\}$ are FREE.

Convergence:

- in the mean value ✓
- almost surely ✓

Hint: prove that variance is $O(1/N^2)$.

MORE STEROIDS

→ [MS] Section 4.2
Thm 5

if $A_N^{(1)}, \dots, A_N^{(p)}$ are independent GUE matrices

$D_N^{(1)}, \dots, D_N^{(q)}$ are ~~deterministic~~ RANDOM $N \times N$ matrices s.t.

$$D_N^{(1)}, \dots, D_N^{(q)} \xrightarrow{\text{distr}} d_1, \dots, d_q \qquad \text{ALMOST SURELY}$$

THEN

$$A_N^{(1)}, \dots, A_N^{(p)}, D_N^{(1)}, \dots, D_N^{(q)} \xrightarrow{\text{distr}}$$

$$s_1, \dots, s_p, d_1, \dots, d_q$$

convergence:
- in the ~~mean value~~
- almost surely ✓

WHERE

$s_1, \dots, s_p$ — semicircular and

$s_1, \dots, s_p, \{d_1, \dots, d_q\}$ are FREE **ALMOST SURELY**

Hint: use CONDITIONAL EXPECTED VALUE
CONDITIONAL VARIANCE

Haar measure on $\mathcal{U}(N)$

Our favorite functions on $\mathcal{U}(N)$:

$$\mathcal{U}(N) \ni U = [u_{ij}]$$

$$\left.\begin{matrix} u_{ij} \\ \overline{u_{ij}} \end{matrix}\right\} 2N^2 \text{ nice functions on } \mathcal{U}(N)$$

Problem: how to integrate polynomial functions on $\mathcal{U}(N)$?

$$\int_{\mathcal{U}(N)} u_{i_1 j_1} \cdots u_{i_n j_n} \overline{u_{i_1' j_1'}} \cdots \overline{u_{i_m j_m}} \, dU = ?$$

$\boxed{\text{EASY CASE}}$

$$= 0 \quad \text{if } n \neq m$$

$$= ? \quad \text{if } n = m$$

MOTTO:

INTEGRATION OVER $\mathcal{U}(N)$ =
LINEAR ALGEBRA (and no steroids)

CRASH COURSE ON TENSOR PRODUCTS

TODO

private note for lecturer's self-tormenting: can we speak about Weingarten calculus WITHOUT tensor products?

$\mathbb{C}^N$ — vector space with a scalar product

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \right\rangle = \sum \overline{x_i}\, y_i$$

orthonormal basis $e_1, \dots, e_N$ $\qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$(\mathbb{C}^N)^{\otimes k}$ — vector space with a scalar product

orthonormal basis $(e_{\mathbf{i}})$ $\qquad \mathbf{i} = (i_1, \dots, i_k)$

$$e_{\mathbf{i}} = e_{i_1} \otimes \cdots \otimes e_{i_k}$$

if you don't feel easy about the tensor products you may take this as a DEFINITION of $(\mathbb{C}^N)^{\otimes k}$

★ Fantastic vector space with a scalar product

$$\mathrm{End}\, \mathbb{C}^N = \{ f : \mathbb{C}^N \longrightarrow \mathbb{C}^N \}$$

LINEAR MAPS "MATRICES"

Hermitian conjugation

$$\langle f, g \rangle := \mathrm{Tr}\, f^* g =$$

$$= \sum_{\mathbf{j}} \langle f e_{\mathbf{j}} , g e_{\mathbf{j}} \rangle_{\mathbb{C}^N}$$

orthonormal basis

$$e_{\mathbf{ij}} = \begin{array}{cc} & \mathbf{j}\text{-th column} \\ \mathbf{i}\text{-th row} & \begin{bmatrix} 0 & 0 & \vdots & 0 \\ -0 & - & 1 & 0 \\ 0 & 0 & & 0 \end{bmatrix} \end{array}$$

$$\mathbf{i} = (i_1, \dots, i_n)$$
$$\mathbf{j} = (j_1, \dots, j_n)$$

HUGE MATRIX OF SIZE N^n

~~Schur-Weyl duality~~

AND ITS CONSEQUENCES

$$\underbrace{\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{k \text{ copies}} = (\mathbb{C}^N)^{\otimes k}$$

S_k acts by permuting the factors

THE SAME ACTION

S_k acts on the set of sequences $[N]^k$
$= \{\mathbf{i} = (i_1, \dots, i_k)\}$
$\pi \mathbf{i} = (i_{\pi^{-1}(1)}, \dots)$

action on the basis
$\pi e_{\mathbf{i}} := e_{\pi \mathbf{i}}$

$$\mathbb{C}[S_k] \hookrightarrow \mathrm{End}\left((\mathbb{C}^N)^{\otimes k}\right)$$

the group algebra of the symmetric group can be viewed as a subalgebra

more precisely: there is an algebra homomorphism
$\mathbb{C}[S_k] \longrightarrow \mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$
which might not be an embedding

? How to identify the image of $\mathbb{C}[S_k]$ here ?

"MYSTERIOUS LINEAR SUBSPACE"

Yes, embedding for $N \geq k$ ✓

Not discussed on 22 XI

How the basis of $\mathrm{End}\,(\mathbb{C}^N)^{\otimes n}$ is related to ~~basis~~ ? of $\mathbb{C}[S_n]$?

$$\langle e_{i i'}, \sigma \rangle = \sum_j \langle e_{i i'} e_j, \sigma e_j \rangle_{(\mathbb{C}^N)^{\otimes n}}$$

$$= \langle e_i, \sigma e_{i'} \rangle = \underline{[\, i = \sigma i' \,]} =$$

S_n acts on $\{(i_1, \dots, i_n) : i_1, \dots, i_n \in [N]\}$ by permuting factors.

$$= [\, i_m = i'_{\sigma^{-1}(m)} \,]$$

$$= [\, i'_m = i_{\sigma(m)} \,]$$

$$\langle \pi, \sigma \rangle = \sum_j \langle \pi e_j, \sigma e_j \rangle_{(\mathbb{C}^N)^{\otimes n}} =$$

$$= \sum_j [\, \pi j = \sigma j \,] = \sum_j [\, \sigma^{-1} \pi j = j \,] =$$

$$N^{\# \sigma^{-1} \pi}$$

Schur-Weyl duality

AND ITS CONSEQUENCES

$U(N)$ acts diagonally, on each coordinate

$$\underbrace{\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{k \text{ copies}} = \left(\mathbb{C}^N\right)^{\otimes k}$$

S_n acts by permuting the factors

EASY: action of $U(N)$ commutes with the action of S_k

$$\mathbb{C}[S_n] \hookrightarrow \mathrm{End}\left(\mathbb{C}^N\right)^{\otimes k}$$

THE IMAGE OF $\mathbb{C}[S_n]$ is EQUAL to the commutant of $U(N)$

$\boxed{\text{MORE CONCRETELY:}}$

for $X \in \mathrm{End}\left(\mathbb{C}^N\right)^{\otimes k}$

$X \in \mathbb{C}[S_n] \iff \forall U \in U(N)$ actions of X and U on $\left(\mathbb{C}^N\right)^{\otimes k}$ commute

Schur-Weyl duality

Works fine also for $GL(N)$

REPRESENTATION OF $S_k \times U(N)$

$$\underbrace{\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{k \text{ copies}} = \left(\mathbb{C}^N\right)^{\otimes k} =$$

$$= \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq N}} V_{S_k}^{\lambda} \otimes V_{U(N)}^{\lambda}$$

Irreducible representations of S_k are indexed by YOUNG DIAGRAMS WITH k BOXES.

POLYNOMIAL irreducible representations of $U(N)$ are indexed by Young diagrams with at most N rows.

We don't care about other representations

MULTIPLICITY-FREE! Howe explains why this is fantastic.

DETAILS

UNNECESSARY

fantastic linear map

$$\Pi : \mathrm{End}\left[(\mathbb{C}^N)^{\otimes k}\right] \longrightarrow \mathrm{End}\left[(\mathbb{C}^N)^{\otimes k}\right]$$

for fans of abstract nonsense:

$$\Pi \in \mathrm{End}\left(\mathrm{End}\left[(\mathbb{C}^N)^{\otimes k}\right]\right)$$

COOL

$$\Pi : X \longmapsto \int_{U(N)} (U \otimes \cdots \otimes U)\, X\, (U^{-1} \otimes \cdots \otimes U^{-1})\, dU$$

on elementary tensors:

$$\Pi : A_1 \otimes \cdots \otimes A_k \longmapsto \int_{U(N)} (U A_1 U^{-1}) \otimes \cdots \otimes (U A_k U^{-1})\, dU$$

on elementary tensors:

$$\Pi : A_1 \otimes \dots \otimes A_k \longmapsto \int_{U(N)} (U A_1 U^{-1}) \otimes \dots \otimes (U A_k U^{-1}) \, dU$$

Why useful?

$$\langle e_{\mathbf{i}\mathbf{i}'}, \Pi e_{\mathbf{j}\mathbf{j}'} \rangle =$$

$$= \int_{U(N)} \langle e_{\mathbf{i}\mathbf{i}'}, \; U e_{j_1 j_1'} U^* \otimes \dots \rangle =$$

$$= \int_{U(N)} U_{i_1 j_1} \cdots (U^*)_{j_1' i_1'} \cdots =$$

$$= \int_{U(N)} U_{i_1 j_1} \cdots \overline{U_{i_1' j_1'}} \cdots = ♡$$

fantastic linear map

$$\Pi : \mathrm{End}\left[(\mathbb{C}^N)^{\otimes k}\right] \longrightarrow \cancel{\mathrm{End}\left[(\mathbb{C}^N)^{\otimes k}\right]}$$

image $\subseteq$ $\mathbb{C}[S_k]$

on elementary tensors:

$$\Pi : A_1 \otimes \dots \otimes A_k \longmapsto \int_{U(N)} (U A_1 U^{-1}) \otimes \dots \otimes (U A_k U^{-1}) \, dU$$

$\to$ Schur-Weyl duality

¿ does it belong to the commutant ?

PROOF. We check on elementary tensors (general tensors via linearity)

$$(V \otimes \dots \otimes V) \times \int_{U(N)} (U A_1 U^{-1}) \otimes \dots \otimes (U A_k U^{-1}) \, dU \overset{?}{=}$$

$$\int_{U(N)} (U A_1 U^{-1}) \otimes \dots \otimes (U A_k U^{-1}) \, dU \quad (V \otimes \dots \otimes V)$$

$\Updownarrow$

$$\int \underbrace{(VU)}_{W := VU} A_1 (VU)^{-1} \otimes \dots \otimes (VU) A_k (VU)^{-1} \, dU = \int_{U(N)} (U A_1 U^{-1}) \otimes \dots \otimes (U A_k U^{-1}) \, dU \quad ✓$$

change of variables.

Haar measure is left and right invariant.

Fantastic simple fact

$$\Pi : \mathrm{End}\,(\mathbb{C}^N)^{\otimes k} \longrightarrow \mathbb{C}[S_k]$$

is an <u>orthogonal</u> projection

Why?

- $\mathrm{Im}\,\Pi \subseteq \mathbb{C}[S_k]$ ✓ already done

- $$\langle \sigma, \Pi\, A_1 \otimes \cdots \otimes A_k \rangle \overset{?}{=} \langle \sigma, A_1 \otimes \cdots \otimes A_k \rangle$$

$\sigma \in S_k$, $A_1 \otimes \cdots \otimes A_k \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$

$$\sum_{i} \langle \sigma (e_{i_1} \otimes \cdots), (A_1 \otimes \cdots) e_{i_1} \otimes \cdots \rangle = \sum \langle e_{i_{\sigma^{-1}(1)}} \otimes \cdots, A_1 e_{i_1} \otimes \cdots \rangle =$$

$$= \sum (A_1)_{i_{\sigma^{-1}(1)}\, i_1} \cdots = \sum (A_1)_{i_4 i_1} (A_2)_{i_1 i_2} \cdots = \mathrm{Tr}\, A_1 A_2 A_3 A_4 =$$

"PROOF BY EXAMPLE" if $\sigma = (1\ 2\ 3\ 4)$ is a cycle

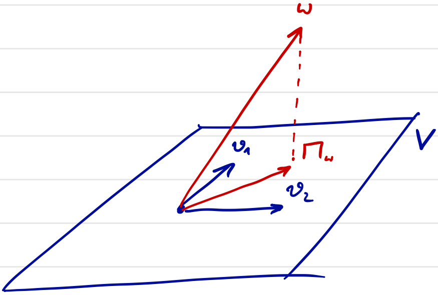
$$= \mathrm{Tr}_\sigma (A_1 \otimes \cdots \otimes A_k) \leftarrow$$

MULTIPLICATIVE EXTENSION OF THE TRACE OVER CYCLES

adding Π does not change this quantity ✓

Not discussed on 22 XI

Flashback from linear algebra



V - linear space with basis $v_1, v_2, \dots, v_d$

orthogonal projection

$$\Pi_V w = \sum_{ij} C_{ij} \langle v_i, w \rangle v_j$$

① image $\subseteq V$

② $V^\perp$ is mapped to 0

③ restriction to $V \overset{?}{=} id$?

$$C^{-1} = \left[\langle v_i, v_j \rangle \right]_{ij}$$

GRAMM MATRIX

Ad ③:

$$v_k \longmapsto \sum_{ij} \underbrace{\overbrace{C_{ij} \langle v_i, v_k \rangle}^{\text{MATRIX PRODUCT = ID MATRIX}}}_{?\ [i=k]\ \checkmark} v_j \quad \checkmark$$

ok!

Not discussed on 22 XI

Weingarten formula.

$$\int_{U(N)} u_{i_1 j_1} \cdots \overline{u_{i_1' j_1'}} \cdots =$$

$$= \langle e_{i i'} , \underbrace{\Pi e_{j j'}} \rangle =$$

$$= \langle e_{i i'} , \underbrace{\sum_{\pi, \sigma \in S_n} Wg_{\pi,\sigma} \langle \pi, e_{j j'} \rangle \sigma}_{\text{the projection formula}} \rangle =$$

$$= \sum_{\pi, \sigma \in S_n} [i = \sigma i'] \, [j = \pi j'] \, \underbrace{Wg_{\pi,\sigma}}_{= Wg_{\pi\sigma^{-1}}}$$

→ NEXT PAGE.

Weingarten function = "inverse of Gramm matrix"

$$Wg^{-1} = \left[\langle \pi, \sigma \rangle \right]_{\pi,\sigma \in S_n} = \left[N^{\# \pi \sigma^{-1}} \right]_{\pi,\sigma \in S_n}$$

Not discussed on 22 XI

Weingarten function and $\mathbb{C}[S_n]$

"left-regular representation"

operator on $\mathbb{C}[S_n]$ of multiplication from the left by

$$\underbrace{\sum_{\pi \in S_n} N^{\#\pi} \pi}_{\in \mathbb{C}[S_n]}$$

$$\left[N^{\#\pi\sigma^{-1}} \right]_{\pi,\sigma \in S_n}$$

Q: what is its MATRIX in the basis $(\sigma : \sigma \in S_n)$?

A:

$$\left[N^{\#\pi\sigma^{-1}} \right]_{\pi,\sigma \in S_n}$$

"in order to map σ to the base vector π we must multiply by $\pi\sigma^{-1}$"

Conclusion:

$(Wg_{\pi,\sigma})$ is the MATRIX of

function on $S_n \times S_n$

function on S_n

$$\left(\sum_{\pi \in S_n} N^{\#\pi} \pi \right)^{-1} = \sum_{\pi \in S_n} Wg_\pi \, \pi \in \mathbb{C}[S_n]$$

inverse in the symmetric group algebra

$$\boxed{Wg_{\pi,\sigma} = Wg_{\pi\sigma^{-1}}}$$

ONLY ONE ARGUMENT is NECESSARY