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IMPAN lectures 2017/2018

Free probability and random matrices

Lecture 5. December 20, 2017.
Weingarten calculus.

Quick reminder from November 22

$$\underbrace{\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{k \text{ copies}} = (\mathbb{C}^N)^{\otimes k}$$

S_n acts by permuting the factors

$$\mathbb{C}[S_n] \hookrightarrow \mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$$

the group algebra of the symmetric group can be viewed as a subalgebra *

Your Mileage May Vary.
for best results $N \geq k$

* for a PRECISE statement
→ PREVIOUS LECTURE.

Quick reminder

fantastic linear map

$$\Pi : \mathrm{End}\left[(\mathbb{C}^N)^{\otimes k}\right] \longrightarrow \mathrm{End}\left[(\mathbb{C}^N)^{\otimes k}\right]$$

$$\Pi : X \longmapsto \int_{U(N)} (U \otimes \cdots \otimes U)\, X\, (U^{-1} \otimes \cdots \otimes U^{-1})\, dU$$

on elementary tensors:

$$\Pi : A_1 \otimes \cdots \otimes A_k \longmapsto \int_{U(N)} (U A_1 U^{-1}) \otimes \cdots \otimes (U A_k U^{-1})\, dU$$

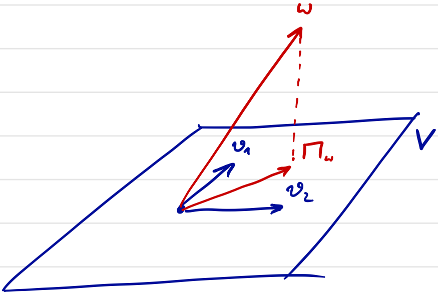
Fantastic simple fact

$$\Pi : \mathrm{End}\,(\mathbb{C}^N)^{\otimes k} \longrightarrow \mathbb{C}[S_k]$$

is an orthogonal projection

Hint: you need to know how to view $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ as a space with a scalar product.

Flashback from linear algebra



V - linear space with basis $v_1, v_2, \dots, v_d$

this formula might require some correction if $\langle v_i, v_j \rangle \notin \mathbb{R}$

orthogonal projection

$$\Pi_V w = \sum_{ij} C_{ij} \langle v_i, w \rangle v_j$$

(2) $V^\perp$ is mapped to 0

(3) restriction to $V \overset{?}{=} id$?

$$C^{-1} = \left[\langle v_i, v_j \rangle \right]_{ij}$$

GRAMM MATRIX.

Ad (3):

$$v_k \longmapsto \sum_{ij} \underbrace{\overbrace{C_{ij} \langle v_i, v_k \rangle}^{\text{MATRIX PRODUCT = ID MATRIX}}}_{?\ [j=k]\ \checkmark} v_j \quad \checkmark$$

ok!

if $v_1, \dots, v_d$ are not linearly independent, bad things happen.

Quick reminder

Scalar product on

$$\mathrm{End}\, \mathbb{C}^N = \{ f : \mathbb{C}^N \longrightarrow \mathbb{C}^N \}$$

LINEAR MAPS

Hermitian conjugation

$$\langle f, g \rangle_{\mathrm{End}} := \mathrm{Tr}\, f^* g =$$

$$= \sum_j \langle f e_i, g e_i \rangle_{\mathbb{C}^N}$$

How the basis of $\mathrm{End}\,(\mathbb{C}^N)^{\otimes n}$ is related to the ~~basis~~ ? of $\mathbb{C}[S_n]$?

$$\langle \underbrace{e_{i i'}}_{\text{"matrix unit"}} , \sigma \rangle_{\mathrm{End}} = \sum_j \langle e_{ii'} e_j , \sigma e_j \rangle_{(\mathbb{C}^N)^{\otimes n}}$$

$$= \langle e_i , \sigma e_{i'} \rangle_{(\mathbb{C}^N)^{\otimes n}} = [\, i = \sigma i' \,] =$$

S_n acts on $\{ (i_1, \dots, i_n) : i_1, \dots, i_n \in [N] \}$ by permuting factors.

$$= [\, i_m = i'_{\sigma^{-1}(m)} \,]$$

$$= [\, i'_m = i_{\sigma(m)} \,]$$

$$\langle \pi , \sigma \rangle_{\mathrm{End}} = \sum_j \langle \pi e_j , \sigma e_j \rangle_{(\mathbb{C}^N)^{\otimes n}} =$$

$$= \sum_j [\, \pi j = \sigma j \,] = \sum_j [\, \sigma^{-1} \pi j = j \,] =$$

$$N^{\# \sigma^{-1} \pi}$$

Weingarten formula.

See → NEXT PAGE

$$\int_{U(N)} u_{i_1 j_1} \cdots u_{i_k j_k} \overline{u_{i_1' j_1'}} \cdots \overline{u_{i_k' j_k'}} =$$

flashback from linear algebra

$$= \langle e_{i i'}, \underbrace{\Pi e_{j j'}} \rangle =$$

$$= \langle e_{i i'}, \underbrace{\sum_{\pi, \sigma \in S_k} Wg_{\pi,\sigma} \langle \pi, e_{j j'} \rangle \sigma}_{\text{the projection formula}} \rangle =$$

$$= \sum_{\pi, \sigma \in S_k} [i = \sigma i'] [j = \pi j'] Wg_{\pi,\sigma}$$

$= Wg_{\pi\sigma^{-1}}$

→ NEXT PAGE.

Weingarten function = "inverse of Gramm matrix"

$$Wg^{-1} = \left[\langle \pi, \sigma \rangle \right]_{\pi, \sigma \in S_k} = \left[N^{\#\pi\sigma^{-1}} \right]_{\pi, \sigma \in S_k}$$

Quick reminder

on elementary tensors:

$$\Pi : A_1 \otimes \dots \otimes A_k \longmapsto \int_{U(N)} (U A_1 U^{-1}) \otimes \dots \otimes (U A_k U^{-1}) \, dU$$

$$\langle e_{i\, i'}, \Pi\, e_{j\, j'} \rangle =$$

$$= \int_{U(N)} \langle e_{i_1 i_1'}, \quad U\, e_{j_1 j_1'}\, U^* \otimes \dots \rangle =$$

$$= \int_{U(N)} U_{i_1 j_1} \dots (U^*)_{j_1' i_1'} \dots =$$

$$= \int_{U(N)} U_{i_1 j_1} \dots \overline{U_{i_1' j_1'}} \dots$$

Weingarten function and $\mathbb{C}[S_n]$

"left-regular representation"

operator on $\mathbb{C}[S_n]$ of multiplication from the left by

$$\underbrace{\sum_{\pi \in S_n} N^{\#\pi} \, \pi}_{\in \mathbb{C}[S_n]}$$

Q: what is its MATRIX in the basis $(\sigma : \sigma \in S_n)$?

A:

$$\left[N^{\#\pi\sigma^{-1}} \right]_{\pi, \sigma \in S_n}$$

"in order to map σ to the base vector π we must multiply by $\pi\sigma^{-1}$"

Conclusion: $\left(Wg_{\pi,\sigma} \right)$ is the MATRIX of the OPERATOR

function on $S_n \times S_n$

function on S_n

$$\left(\sum_{\pi \in S_n} N^{\#\pi} \, \pi \right)^{-1} = \sum_{\pi \in S_n} Wg_\pi \, \pi \quad \in \mathbb{C}[S_n]$$

inverse in the symmetric group algebra

$$\boxed{Wg_{\pi,\sigma} = Wg_{\pi\sigma^{-1}}}$$

ONLY ONE ARGUMENT is NECESSARY

PROBLEM: information about Wg ?

Fix integer $k \geq 1$. For each $\pi \in S_k$

$$\boxed{Wg_\pi \text{ is a RATIONAL FUNCTION in } N}$$

Toy example. $\boxed{k=2}$

This calculation is legal only for $N \geq 2$!

$$\left(Wg_{\pi,\sigma}\right) = \begin{bmatrix} N^2 & N \\ N & N^2 \end{bmatrix}^{-1} = \frac{1}{N^4 - N^2} \begin{bmatrix} N^2 & -N \\ -N & N^2 \end{bmatrix}$$

$$Wg_{(1)(2)} = \frac{N^2}{N^2(N-1)(N+1)} = \frac{1}{(N-1)(N+1)}$$

interesting denominators!

$$Wg_{(1,2)} = -\frac{N}{N^2(N-1)(N+1)} = -\frac{1}{(N-1)N(N+1)}$$

$$\int_{U(N)} |u_{11}|^4 \, dU = \int u_{11} u_{11} \overline{u_{11}} \, \overline{u_{11}} = \sum_{S_2 \times S_2} \dots \quad (4 \text{ summands})$$

$$= 2\, Wg_{(1)(2)} + 2\, Wg_{(1,2)} =$$

magic cancellation for $N=1$.

$$= \frac{2N - 2}{(N-1)\,N\,(N+1)} = \frac{2}{N(N+1)}$$

Mysterious!

this formula gives a correct value for $N \geq 1$

BUT the proof DOES NOT WORK for $N=1$!

Asymptotics of Weingarten function.

algebra for today:

$$\mathcal{A} = \Big\{ \sum_{\pi \in S_k} f_\pi \frac{1}{N^{|\pi|}} \pi \; : \; f_\pi \in \mathbb{C}[[\tfrac{1}{N^2}]] \Big\}$$

formal power series in $\frac{1}{N^2}$

Yes, it is an algebra!

$$\Big(f_\pi \frac{1}{N^{|\pi|}} \pi\Big)\Big(g_\sigma \frac{1}{N^{|\sigma|}} \sigma\Big) =$$

$$= f_\pi g_\sigma \frac{1}{N^{\underbrace{|\pi|+|\sigma|-|\pi\sigma|}_{\geq 0 \text{ and EVEN}}}} \cdot \frac{1}{N^{|\pi\sigma|}} \pi\sigma$$

TRIANGLE INEQUALITY.

$$\sum_{\pi \in S_k} N^{\#\pi} \pi = N^k \cdot \underbrace{\sum_{\pi \in S_k} \frac{1}{N^{|\pi|}} \pi}_{\in \mathcal{A}}$$

$$\sum_{\pi} \mathrm{Wg}_\pi \cdot \pi = N^{-k} \underbrace{\Big[\sum_{\pi \in S_k} \frac{1}{N^{|\pi|}} \pi\Big]^{-1}}_{\in \mathcal{A}}$$

CONCLUSION.

$$\boxed{\mathrm{Wg}_\pi = O\Big(\frac{1}{N^{k+|\pi|}}\Big)}$$

quick check: why inverse in $\mathcal{A}$ exists?

→ NEXT PAGE.

Ideal $I = \left\{ \sum_{\pi \in S_k} f_\pi \cdot \frac{1}{N^{|\pi|+2}} \, \pi \ : \ f_\pi \in \mathbb{C}[\tfrac{1}{N}] \right\}$

Yes, it's an ideal.
SAME PROOF AS ☆

"first order asymptotics"

$$\mathcal{A}/I \cong \left\{ \sum_{\pi \in S_k} f_\pi \, \pi \ : \ f_\pi \in \mathbb{C} \right\} \ \triangle!$$

= symmetric group algebra $\mathbb{C}[S_k]$ WITH A NEW PRODUCT

$$\pi \cdot \sigma = \begin{cases} \pi\sigma & \text{if } |\pi\sigma| = |\pi| + |\sigma| \\ 0 & \text{if } |\pi\sigma| < |\pi| + |\sigma| \end{cases}$$

$$\left(\sum_{\pi \in S^k} 1 \, \frac{1}{N^{|\pi|}} \, \pi \right) \left(\sum_{\pi} \left(Wg_\pi \, N^k \right) \pi \right) = 1$$

equality in $\mathcal{A}/I$

$\Updownarrow$

$$\sum_{\sigma \in S_k} \sigma \ \boxed{\sum_{\substack{\pi_1 \cdot \pi_2 = \sigma \\ |\pi_1| + |\pi_2| = |\sigma|}} 1 \cdot \left(\lim_{N \to \infty} Wg_{\pi_2} \, N^{k+|\pi_2|} \right)} = 1$$

$= [\sigma = id]$

equality in $\mathbb{C}[S_k]$ $\triangle!$

what does it tell if $\sigma \in NC$ corresponds to a NONCROSSING PARTITION?

$\boxed{\text{Möbius inversion formula.}}$

→ [NS] Lecture 10.

if P is a finite poset... | think: $P = NC(n)$

$$P^{(2)} := \{ (\pi, \sigma) : \underset{\pi, \sigma \in P}{\pi \leq \sigma ,} \}$$

we are interested in the class of functions from $P^{(2)}$ to $\mathbb{C}$

CONVOLUTIONS:

- for $F, G : P^{(2)} \longrightarrow \mathbb{C}$

$$(F * G)(\pi, \sigma) := \sum_{\pi \leq \rho \leq \sigma} F(\pi, \rho)\, G(\rho, \sigma)$$

- for $f : P \longrightarrow \mathbb{C}$
  $G : P^{(2)} \longrightarrow \mathbb{C}$

$$(f * G)(\sigma) := \sum_{\rho \leq \sigma} f(\rho)\, G(\rho, \sigma)$$

→ LECTURE 3 RECALLED

- $\delta : P^{(2)} \longrightarrow \mathbb{C}$

  $$\delta(\pi, \sigma) = [\pi = \sigma]$$

  is the unit of this convolution:

  $F * \delta = \delta * F = F$

  $f * \delta = f$

- $\zeta : P^{(2)} \longrightarrow \mathbb{C}$ zeta function

  $$\zeta(\pi, \sigma) = 1 \qquad (\text{for } \pi \leq \sigma)$$

- $\mu : P^{(2)} \longrightarrow \mathbb{C}$ Möbius function is the inverse of ζ

  $$\mu * \zeta = \zeta * \mu = \delta$$

  left- and right- inverse (if they exist) must be equal.

  $$\Updownarrow$$

  $\forall \pi \leq \sigma$ $$\sum_{\pi \leq \tau \leq \sigma} \mu(\pi, \tau) = [\pi = \sigma]$$

  fix π.
  use induction over σ to show existence and uniqueness of $\mu(\pi, \sigma)$

σ - non-crossing partition

→ Lecture 3
"Möbius inversion formula"

$0_k = e =$
= minimal partition.

$$\sum_{\substack{\pi_1 \cdot \pi_2 = \sigma \\ |\pi_1| + |\pi_2| = |\sigma|}} \underbrace{\left(\lim_{N\to\infty} Wg_{\pi_1} N^{k+|\pi_1|} \right)}_{F(\pi_2):=} \underbrace{1}_{\zeta(\pi_2, \sigma)} = [\sigma = id]$$

sum over non-crossing partitions π_2

$$F * \zeta = \delta(0_n, \bullet)$$

$$F = F * \underbrace{\zeta * Moeb}_{\delta} = Moeb(0_n, \bullet)$$

$$\lim_{N\to\infty} \left(Wg_\sigma \cdot N^{k+|\sigma|} \right) = Moeb(0_n, \sigma) =$$

$$= \prod_{\substack{c \in \sigma \\ \text{cycles of } \sigma}} (-1)^{|c|-1} C_{|c|-1}$$

(Catalan number)

$$\boxed{Wg_\sigma = \frac{1}{N^{k+|\sigma|}} \left[Moeb_{(0_n, \sigma)} + O\left(\frac{1}{N^2}\right) \right]}$$

$|c|$ = length of cycle =
= how many elements are permuted.

Weingarten function is a generating function of ... ?

Exercise

$$\sum_{\pi \in S_k} x^{|\pi|} \pi = (1 + x J_1)(1 + x J_2) \cdots (1 + x J_k)$$

Jucys-Murphy elements $\in \mathbb{C}[S_k]$

$J_1 = 0$

$J_2 = (1,2)$

$J_3 = (1,2) + (1,3)$

$\vdots$

$J_i = (1,i) + (2,i) + \cdots + (i-1,i)$

$$\sum_{\pi \in S_k} Wg_\pi \, \pi = \left(N^k \sum_{\pi \in S_k} \frac{1}{N^{|\pi|}} \, \pi \right)^{-1} =$$

$$= N^{-k} \left(1 + \frac{1}{N} J_1\right)^{-1} \left(1 + \frac{1}{N} J_2\right)^{-1} \cdots \left(1 + \frac{1}{N} J_k\right)^{-1} =$$

$$= N^{-k} \left(1 - \frac{1}{N} J_1 + \frac{1}{N^2} J_1^2 - \cdots \right) \cdots$$
$$\left(1 - \frac{1}{N} J_k + \frac{1}{N^2} J_k^2 - \cdots \right)$$

each series is absolutely convergent for $|N| > k-1$

information about the poles of $Wg(N)$

$$N^k Wg_\pi = \sum_{i \geq 0} \underbrace{\left(-\frac{1}{N}\right)^{i}}_{(-1)^{\pi} \cdot \frac{1}{N^i}} \# \left\{ (a_1, b_1) \cdots (a_i, b_i) = \pi \right\}$$

$a_1 < b_1 \; \ldots \; , a_i < b_i$

$b_1 \leq b_2 \leq \cdots \leq b_i$

A BIT OF THEORY

$G = S_k$

Fourier transform in $\mathbb{C}[G]$

"if you want to invert a matrix...
... better it be a 1x1 matrix"

by abstract nonsense. Concretely...

our beloved commutative algebra: $Z\mathbb{C}[G] = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{\text{indexed by } \hat{G} \,=\, \text{irreducible representations of } G}$

ALGEBRA ISOMORPHISM

$$Z\mathbb{C}[G] \ni f \longmapsto \hat{f} \in F[\hat{G}]$$

complex functions on $\hat{G}$

how this works: from left to right:

$$\hat{f}(\lambda) = \frac{\mathrm{Tr}\, \rho^\lambda(f)}{\mathrm{Tr}\, \rho^\lambda(id)} = \sum_{\pi \in G} \frac{\mathrm{Tr}\, \rho^\lambda(\pi)}{\mathrm{Tr}\, \rho^\lambda(id)} f(\pi)$$

"convert multiple of id matrix to a complex number" = NORMALIZED TRACE

from right to left

$$ZC[G] \ni \; ? \longmapsto (0, \dots, 0, 1, 0, \dots 0) =$$

$$= \delta_\mu$$

"first try..."

$$\sum_{\pi \in G} \operatorname{Tr} \varrho^\mu(\pi) \; \pi \longmapsto \quad (\star)$$

$$\left(\lambda \longmapsto \sum_{\pi \in G} \operatorname{Tr} \varrho^\mu(\pi) \frac{\operatorname{Tr} \varrho^\lambda(\pi)}{\operatorname{Tr} \varrho^\lambda(id)} = \right.$$

$$\left. = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \dfrac{|G|}{\operatorname{Tr} \varrho^\lambda(id)} & \text{for } \lambda = \mu \end{cases} \right)$$

„almost right"

so...

$$= \sum_{\pi \in G} \frac{\operatorname{Tr} \varrho^\mu(id) \cdot \operatorname{Tr} \varrho^\mu(\pi)}{|G|} \; \pi$$

p_μ — "minimal central projection in $C[G]$"

$$\longmapsto (0, \dots, 0, 1, 0, \dots 0) =$$

$$= \delta_\mu$$

A BIT OF THEORY

Schur-Weyl duality:

$$(\mathbb{C}^N)^{\otimes k} = \bigoplus_{\substack{\mu \vdash k \\ \ell(\mu)\leq N}} V^{\mu}_{S_k} \otimes \underbrace{V^{\mu}_{GL_N}}$$

DIMENSION = . . .

$= s_\mu(1^N) =$ Schur polynomial

$= s_\mu(\underbrace{1,\dots,1}_{N \text{ times}})$

$= \# SSYT(\mu, N)$

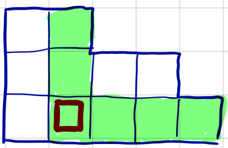
"semistandard Young tableaux with shape μ and entries in $\{1,2,\dots,N\}$"

each column strictly increasing

4	7			
2	3	3	6	
1	1	2	3	5

each row weakly increasing

"hook length"



$h_\square = 6$

= "hook-content formula"

$x-y$ = "content of $\square$"

$$\prod_{\square=(x,y)\in\mu} \frac{N + x - y}{h_\square}$$

Schur-Weyl duality:

$$(\mathbb{C}^N)^{\otimes k} = \bigoplus_{\substack{\mu \vdash k \\ \ell(\mu) \leq N}} V^\mu_{S_k} \otimes \underbrace{V^\mu_{GL_N}}_{\text{dimension} = s_\mu(1^N)}$$

representations of S_k. Compare characters:

$$\sum_\pi N^{\#\pi}\,\pi = \sum_\mu s_\mu(1^N)\,\underbrace{\operatorname{Tr} \rho^\mu(\pi)\cdot \pi}$$

$$\xrightarrow[(\bigstar)]{\text{Fourier transform}} \left(\lambda \longmapsto s_\lambda(1^N)\,\frac{k!}{\underbrace{\operatorname{Tr} \rho^\lambda(id)}}\right)$$

$= 0 \iff \ell(\lambda) > N$

$(*)$

the inverse is easy...

$$\sum_\pi \mathrm{Wg}_\pi\,\pi \xrightarrow{\text{Fourier trf}} \left(\lambda \longmapsto \frac{\operatorname{Tr} \rho^\lambda(id)}{s_\lambda(1^N)\quad k!}\right)$$

the inverse is easy...

$$\sum_{\pi} Wg_{\pi} \, \pi \xrightarrow{\text{Fourier transf}} \left(\lambda \longmapsto \frac{\operatorname{Tr} \varrho^{\lambda}(id)}{S_{\lambda}(1^{N}) \quad k!} \right)$$

$$\sum_{\pi \in S_n} \frac{\operatorname{Tr} \varrho^{\mu}(id) \cdot \operatorname{Tr} \varrho^{\mu}(\pi)}{k!} \, \pi$$

p_{μ} — "minimal central projection in $\mathbb{C}[G]$"

$$\longmapsto (0, \dots, 0, 1, 0, \dots 0) = \\ = \delta_{\mu}$$

so...

$$Wg_{\pi} = \sum_{\lambda} \frac{\operatorname{Tr} \varrho^{\lambda}(id)}{S_{\lambda}(1^{N}) \quad k!} \; \frac{\operatorname{Tr} \varrho^{\lambda}(id) \cdot \operatorname{Tr} \varrho^{\lambda}(\pi)}{k!}$$

$$= \sum_{\lambda} \frac{\left[\operatorname{Tr} \varrho^{\lambda}(id) \right]^{2}}{(k!)^{2} \quad S_{\lambda}(1^{N})} \operatorname{Tr} \varrho^{\lambda}(\pi)$$

so...

$$\mathrm{Wg}_\pi = \frac{1}{(k!)^2} \sum_\lambda \frac{\left[\operatorname{Tr} \varrho^\lambda(\mathrm{id})\right]^2}{s_\lambda(1^N)} \operatorname{Tr} \varrho^\lambda(\pi)$$

$\rightarrow$ Collins, IMRN 2003.

not very convenient for asymptotics $N \to \infty$

<u>Example.</u> $k=2$.

$$\mathrm{Wg}_{(1)(2)} = \frac{1}{(2!)^2} \Bigg[\underbrace{\frac{1^2}{\frac{N \cdot (N+1)}{2 \cdot 1}} \cdot 1}_{\square\!\square} + \underbrace{\frac{1^2}{\frac{N \cdot (N-1)}{2 \cdot 1}} \cdot 1}_{\substack{\square\\ \square}} \Bigg] =$$

$$= \frac{1}{2} \frac{(N-1)+(N+1)}{(N-1)\,N\,(N+1)} = \frac{1}{(N-1)(N+1)} \quad \checkmark$$

$$\mathrm{Wg}_{(1,2)} = \frac{1}{(2!)^2} \Bigg[\underbrace{\frac{1^2}{\frac{N \cdot (N+1)}{2 \cdot 1}} \cdot 1}_{\square\!\square} + \underbrace{\frac{1^2}{\frac{N \cdot (N-1)}{2 \cdot 1}} \, (-1)}_{\substack{\square\\ \square}} \Bigg] =$$

$$= \frac{1}{2} \frac{(N-1)-(N+1)}{(N-1)\,N\,(N+1)} = -\frac{1}{(N-1)\,N\,(N+1)} \quad \checkmark$$

magic cancellation

Small dimension N

$$\mathbb{C}[S_k] = \bigoplus_{\lambda \vdash k} \operatorname{End} V^{\lambda}_{S_n}$$

ideal in $\mathbb{C}[S_n]$

this is the right algebra to calculate Wg

$$\mathbb{C}[S_k]_N := \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq N}} \operatorname{End} V^{\lambda}_{S_n}$$

NEW!

Idea: Weingarten calculus works (usually) fine if instead of ~~$\mathbb{C}[S_n]$~~ we work with $\mathbb{C}[S_n]_N$.

things not invertible in $\mathbb{C}[S_n]$ might be invertible in $\mathbb{C}[S_n]_N$

Fourier transform

$$Z\mathbb{C}[S_n] = \mathbb{C} \oplus \cdots \oplus \mathbb{C} \oplus \cdots\cdots \oplus \mathbb{C}$$

$$Z\mathbb{C}[S_n]_N = \mathbb{C} \oplus \cdots \oplus \mathbb{C} \oplus \cancel{0} \oplus \cdots \cancel{\oplus} \cancel{0}$$

Schur-Weyl duality, for small N

$$(\mathbb{C}^N)^{\otimes k} = \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq N}} V^{\lambda}_{S_k} \otimes V^{\lambda}_{GL(N)}$$

$\mathbb{C}[S_k]_N$ acts here in a faithful way.

$$\mathbb{C}[S_k]_N \hookrightarrow \mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$$

is an embedding (NO CHEATING HERE)

¿ How to compute Π - orthogonal projection from $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ to $\mathbb{C}[S_k]_N$?

⚠ the linear algebra for projection doesn't work: permutations viewed as elements of $\mathrm{End}\,(\mathbb{C}^N)^{\otimes k}$ ARE LINEARLY DEPENDENT

ZONK!

¿ How to compute Π - orthogonal projection from $\mathrm{End}(\mathbb{C}^N)^{\otimes k}$ to $\mathbb{C}[S_k]_N$?

(0) STARTING POINT: $\vartheta \in \mathrm{End}(\mathbb{C}^N)^{\otimes k}$

(1) $\pi \in S_k$ do not form a basis of $\mathbb{C}[S_k]_N$ but we don't care...

information about scalar products $\langle \pi, \vartheta \rangle$ encoded as...

$$\sum_{\pi \in S_k} \langle \pi, \vartheta \rangle \, \pi \in \cancel{\mathbb{C}[S_k]} \quad \mathbb{C}[S_k]_N$$

this information is sufficient for calculating $\Pi \vartheta$

Hint: if $\vartheta = (id) \in S_k$

$$\sum_{\pi \in S_k} \langle \pi, id \rangle \pi = \sum_{\pi \in S_k} N^{\#\pi} \, \pi$$

Fourier transform $\longrightarrow$ $(*)$

simple exercise: general $\vartheta \in S_k$ is also OK.

in the restricted setup...

$$\mathbb{C}[S_k] \ni \vartheta \longmapsto \sum_{\pi \in S_k} \langle \pi, \vartheta \rangle \, \pi$$

viewed as element in $\mathrm{End}(\mathbb{C}^N)^{\otimes k}$

viewed as element of $\mathbb{C}[S_k]_N$

...and sometimes viewed as element of $\mathbb{C}[S_k]$

$$\delta \longmapsto \sum_{\pi \in S_k} N^{\#\pi\delta^{-1}} \, \pi =$$

$$= \underbrace{\left(\sum_{\pi \in S_k} N^{\#\pi\delta^{-1}} \, \pi\delta^{-1} \right) \delta}_{\text{Multiplication from the left by } \sum_{\pi \in S_k} N^{\#\pi} \pi}$$

the Inverse exists in $\cancel{\mathbb{C}[S_k]}$ $\mathbb{C}[S_k]_N$ and is equal to...

NEW!

$$\mathrm{Wg}_\pi = \frac{1}{(k!)^2} \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq N}} \frac{\left[\mathrm{Tr}\, \varrho^\lambda(\mathrm{id}) \right]^2}{s_\lambda(1^N)} \, \mathrm{Tr}\, \varrho^\lambda(\pi)$$

inverse in $\mathbb{C}[S_k]_N$!

$$\int_{U(N)} u_{i_1 j_1} \cdots u_{i_k j_k} \, \overline{u_{i_1' j_1'}} \cdots \overline{u_{i_k' j_k'}} =$$

$$= \sum_{\pi, \delta \in S_k} [i = \delta i'] \, [j = \pi j'] \; \mathrm{Wg}_{\pi\delta^{-1}}$$

NEW!

Example $\boxed{N=1}$ $k=2$

$$\mathrm{Wg}^{\mathrm{NEW}}_{(1)(2)} = \frac{1}{(2!)^2}\underbrace{\left[\frac{1^2}{\frac{1\cdot(1+1)}{2\cdot 1}}\cdot 1\right]}_{\square\square} = \frac{1}{4}$$

$$\mathrm{Wg}^{\mathrm{NEW}}_{(1,2)} = \frac{1}{(2!)^2}\underbrace{\left[\frac{1^2}{\frac{1\cdot(1+1)}{2\cdot 1}}\cdot 1\right]}_{\square\square} = \frac{1}{4}$$

$$\int |u_{11}|^2 = \int u_{11}\, u_{11}\, \bar{u}_{11}\, \bar{u}_{11} = 2\, \mathrm{Wg}^{\mathrm{NEW}}_{(1)(2)} + 2\, \mathrm{Wg}^{\mathrm{NEW}}_{(12)} = 1$$

✓ ok.

Amazingly strange

$$\int_{U(N)} u_{i_1 j_1} \cdots u_{i_k j_k} \overline{u_{i'_1 j'_1}} \cdots \overline{u_{i'_k j'_k}} =$$

$$= \lim_{N' \to N} \underbrace{\sum_{\pi, \sigma \in S_k} [i = \sigma i'] \, [j = \pi j'] \, Wg^{OLD}_{\pi \sigma^{-1}}}$$

rational function in N'

does not matter if OLD or NEW

Hint: Fourier transform OR $\times$ $\mathbb{C}[S_k]_N$ is an IDEAL

$$[id] \underbrace{\left[\sum_{\pi} [j = \pi j'] \, \pi^{-1} \right]}_{\in \mathbb{C}[S_k]_N} \left[\sum_{\nu} Wg_{\nu} \, \nu \right] \underbrace{\left[\sum_{\sigma} [i = \sigma i'] \, \sigma \right]}_{\mathbb{C}[S_k]_N}$$

Integration over other classical groups?

* the symplectic group
* the orthogonal group

Hint: use Schur-Weyl duality

$$\mathbb{R}^N \otimes \cdots \otimes \mathbb{R}^N$$

O(N) acts on each factor separately

~~Symmetric group algebra~~
Brauer algebra