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IMPAN lectures 2017/2018

Free probability and random matrices

Lecture 6. January 24, 2018
(Quantum) unitarily invariant random matrices

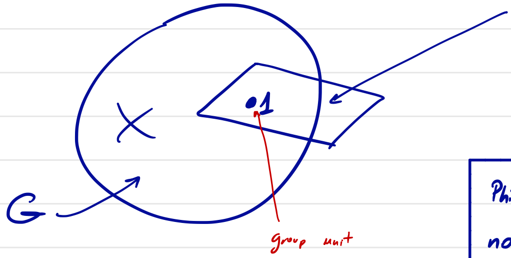
What can we learn about the representations of Lie groups from the viewpoint of random matrix theory / free probability theory?

CRASH COURSE ON LIE GROUPS AND LIE ALGEBRAS.

G - (compact) Lie group

$\mathfrak{g}$ - tangent space to G at group identity

"infinitesimal neighborhood of e"



Philosophy: group G is a difficult, nonlinear object.
Replace it with a linear object

how much of the multiplication from G can you see on the tangent space $\mathfrak{g}$?

$v, w \in \mathfrak{g}$ "$\varepsilon \in \mathbb{R}$, $\varepsilon \to 0$"

$$\underbrace{(1+\varepsilon v)}_{\substack{\text{element of } G \\ \text{in "infinitesimal neighborhood of } 1\text{"}}}(1+\varepsilon w) = 1 + \varepsilon(v+w) + \dots$$

"first order approximation: product in G becomes addition in the tangent space $\mathfrak{g}$"

NOT MUCH. :)

OR MAYBE?

in order to see interesting structures we need to look on the quadratic terms

for $v, w \in \mathfrak{g}$

this calculation is a cheating, but it can be converted to a rigorous mathematics

$$\overbrace{\underbrace{(1+\varepsilon v)}_{\substack{\text{element of } G \text{ in} \\ \text{infinitesimal neighborhood of } 1.}}(1+\varepsilon' w)(1-\varepsilon v)(1-\varepsilon' w)}^{\text{"}ghgh^{-1} \text{ for } g,h\in G,\ g,h\to 1\text{"}} =$$

$$1 + \varepsilon\varepsilon'\left(vw + (\cancel{-vw}) - wv + \cancel{vw}\right)$$

$$- \varepsilon^2\left(v^2 + w^2\right) + \dots$$

$$\frac{\partial}{\partial \varepsilon}\frac{\partial}{\partial \varepsilon'}(1+\varepsilon v)(1+\varepsilon' w)(1-\varepsilon v)(1-\varepsilon' w) = vw - wv =$$

$$= [v, w]$$

NEW Lie bracket = "second derivative of the multiplication in G"

"this is how much of the multiplication in G you see in the tangent space" the best concrete choice for today

$G = U(d)$ $\qquad \mathfrak{g}$ = REAL linear space of antihermitian matrices

HAVE THE SAME REPRESENTATION THEORY

OR

$G = GL(d)$ $\qquad \mathfrak{g}$ = linear space of all matrices

Lie bracket $\quad [X,Y] = XY - YX$

our favorite object for today:

representation of G, i.e a GROUP HOMOMORPHISM

$$\rho : G \longrightarrow \underset{\text{invertible}}{\mathrm{End}}(V^{\rho})$$

$$\rho(gh) = \rho(g)\,\rho(h)$$

by taking the derivative of this map we get a

representation of the Lie algebra

$$\rho : \mathfrak{g} \longrightarrow \underset{\text{(not necessarily invertible)}}{\mathrm{End}}(V^{\rho})$$

linear map which preserves the bracket

$$\rho([x,y]) = [\rho(x), \rho(y)]$$

bracket in $\mathfrak{g}$

bracket in $\mathrm{End}(V^{\rho})$ coming from the usual multiplication of matrices.

Surprisingly restrictive condition.

TODO: applications/motivations

TOY TOY EXAMPLE.

$G = GL(d)$ acts on $\mathbb{C}^d$

action on a linear space = representation

$\Downarrow$

action on $\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d$

not necessarily irreducible

but we don't care

by a clever choice of a subspace you may try to get irreducible representations

Moral lesson:

even if the Lie group has a matrix structure ("it is linear")

the representation may be a very nonlinear map

FOR ☆

irreducible representations indexed by signatures

$$(\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d)$$

$\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$

slightly more general than Young diagrams.

BAD NEWS: not easy to describe / construct these representations.

Your Mileage May Vary.
For best results read
User's Manual to semisimple Lie algebras.

for a general (semisimple) Lie group this corresponds to
"the highest weight"

alternative approach:
use Schur-Weyl duality

TYPICAL PROBLEM.

$$V^\lambda \otimes V^\mu = \bigoplus_\nu \underbrace{c_{\lambda\mu}^\nu}_{} V^\nu$$

irreducible representations
OF THE SAME GROUP

multiplicity,
$\in \{0, 1, 2, \dots\}$

can we understand these numbers?

if $G = U(d)$ or $G = GL(d)$

$c_{\lambda\mu}^\nu$ – are called **LITTLEWOOD–RICHARDSON** coefficients.

- can be calculated by LITTLEWOOD–RICHARDSON RULE

  BAD NEWS: complicated
  bad computational complexity.

- can we have approximate / asymptotic description which is more tractable?

! → interesting research active
→ Geometric Complexity Theory
→ Greta Panova

probabilistic viewpoint on Littlewood - Richardson

fix λ and μ

$$V^\lambda \otimes V^\mu = \bigoplus_\nu \underbrace{c_{\lambda\mu}^\nu}_{\substack{\text{multiplicity,}\\ \in\{0,1,2,\dots\}}} V^\nu$$

irreducible representations

$$\mathbb{P}(\nu) := \frac{c_{\lambda\mu}^\nu \cdot \dim V^\nu}{\dim V^\lambda \cdot \dim V^\mu}$$

probability measure on $\hat{G}$

"random irreducible representation"

¿ law of large numbers ?

¿ central limit theorem ?

" $\lambda, \mu \longrightarrow \infty$ "

tensor products of representations of Lie algebras

starting point: TWO representations of THE SAME LIE GROUP G

$$\rho_1 : G \longrightarrow \mathrm{End}\, V^{(1)}$$

$$\rho_2 : G \longrightarrow \mathrm{End}\, V^{(2)}$$

TENSOR PRODUCT

$$(\rho_1 \otimes \rho_2) : G \longrightarrow \mathrm{End}\, V^{(1)} \otimes V^{(2)}$$

$$g \longmapsto \rho_1(g) \otimes \rho_2(g)$$

IN THE NEIGHBORHOOD OF IDENTITY:

Lie group representations

$x \in \mathfrak{g}$

$$1 + \varepsilon x \longmapsto \rho_1(1 + \varepsilon x) \otimes \rho_2(1 + \varepsilon x) =$$

$$= \big(1 + \varepsilon \rho_1(x)\big) \otimes \big(1 + \varepsilon \rho_2(x)\big) =$$

Lie algebra representations

$$= 1 + \varepsilon \big(\rho_1(x) \otimes 1 + 1 \otimes \rho_2(x) \big) + \varepsilon^2 \ldots$$

Lie algebra representations

Moral lesson: TENSOR REPRESENTATION OF LIE ALGEBRA

$$x \longmapsto \rho_1(x) \otimes 1 + 1 \otimes \rho_2(x)$$

is given by "Leibnitz rule".

$\boxed{\text{TOY EXAMPLE } d=2}$

this group is not simply connected, better take its UNIVERSAL COVER $Spin(3) = \boxed{SU(2)}$

for best results use Clifford algebras

$G = SO(\cancel{3})$

have the same Lie algebra $so(3) \cong su(2)$

symmetries of the physical space $\mathbb{R}^3$ the most of us live in

$G = SU(2)$ $\qquad$ $\mathfrak{g}$ = antihermitian matrices with trace ZERO

irreducible representations of $U(2)$ are indexed by $\lambda_1 \geq \lambda_2$ $\quad \lambda_1, \lambda_2 \in \mathbb{Z}$

hint: RESTRICTION!

irreducible representations V^j of $SU(2)$ are indexed by $j \in \{0, 1, 2, \dots\}$

HINT: $j = \lambda_1 - \lambda_2$

representations of $SU(2)$ are not very complicated!

concrete version:

$$V^j = \mathrm{Sym}\,(\underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{j \text{ factors}})$$

$SU(2)$ acts on each factor "diagonally"

→ https://arxiv.org/abs/1611.01892

Appendix D

What is angular momentum in classical mechanics?

"angular momentum is the physical quantity related to the rotational symmetry of the space"

more concretely...

$x \in \mathfrak{g}$

$$\mathbb{R} \ni t \longmapsto e^{tx} \in G$$

one-parameter group of symmetries of the physical space

$\downarrow$

$\mathbb{R}$

Emma Noether:

one-parameter group of symmetries of the phase-space $\longleftrightarrow$

$\longleftrightarrow$ physical quantity which is preserved over time

for a more concrete answer you need the formalism of lagrangians

Lesson learned: angular momentum is an element of

$$[\mathfrak{so}(3)]^* \cong [\mathfrak{su}(2)]^*$$

dual space

(*)

the usual viewpoint

$$\vec{J} = (J_x, J_y, J_z)$$

related to groups of rotations along...

Oyz $\quad Ozx$ $\quad Oxy$

hint: for best results find an isomorphism of $SO(3)$-modules

$$\mathbb{R}^3 \cong [\mathfrak{so}(3)]^*$$

canonical action of $SO(3)$

coadjoint action of $SO(3)$

don't like dual spaces?

fix the isomorphism $su(2)^* \cong su(2) \subseteq M_2(\mathbb{C})$.

Hint: use a bilinear form on $M_2(\mathbb{C})$

$$\langle x, y \rangle = \mathrm{Tr}\, x^T y$$

two natural choices available!

good! matrices!

angular momentum is an element of...

$$so(3) \cong [so(3)]^* \cong [su(2)]^* \cong su(2)$$

real antisymmetric matrices 3×3

trace zero antihermitian matrices 2×2

(*) continued. Example of an isomorphism of $su(2)$ modules

$$\mathbb{R}^3 \longleftrightarrow (\text{traceless } 2\times 2 \text{ antihermitian matrices})$$

is given by Pauli matrices (multiplied by $i=\sqrt{-1}$)

$\vec{J} \in \mathbb{R}^3$ corresponds to a traceless antihermitian matrix with eigenvalues $\pm i\,|\vec{J}|$!

$\vec{J} \in \mathbb{R}^3$ uniformly random on a sphere with radius $|\vec{J}|$

$\longleftrightarrow$ uniformly random antihermitian matrix with eigenvalues $\pm i\,|\vec{J}|$.

random matrix theory shows up!

$$= U \begin{bmatrix} +i\,|\vec{J}| & \\ & -i\,|\vec{J}| \end{bmatrix} U^{-1}$$

normalization constants depend on the details of the isomorphism $\mathbb{R}^3 = [su(2)]^*$

Quantum system = Hilbert space $\mathcal{H}$

<u>unitary</u> representation of $G = SO(3)$ or $SU(2)$

"how symmetries of the physical space are implemented by a quantum system?"

What is angular momentum in quantum mechanics?

$$x \in \mathfrak{g}$$

$$\Downarrow$$

$$\mathbb{R} \ni t \longmapsto e^{tx} \in G$$

one-parameter group of symmetries of the physical space

$$\Downarrow \text{ apply representation}$$

symmetries of the physical system

$$\mathbb{R} \ni t \longmapsto \rho(e^{tx}) = e^{t\rho(x)} \in \mathcal{U}(\mathcal{H})$$

unitary transformations of the Hilbert space.

$$\Downarrow$$

$$\text{generator} = \underbrace{\pm i}\, \rho(x)$$

"Emma Noether in quantum set-up": eigenvalues of this generator have a physical meaning.

physicists prefer HERMITIAN operators over antihermitian.

$$\boxed{\begin{array}{l}\text{Lesson learned: angular momentum is an element of} \\ \overset{[su(2)]^*}{[so(3)]^*} \otimes B(\mathcal{H}) \quad \text{given by} \quad \overset{su(2)}{\overset{\wr}{so(3)}} \ni x \longmapsto \rho(x)\end{array}}$$

if you are a real physicist, at some places you should add the Planck constant $\hbar$.

related to the details of how "Quantum Emma Noether" works.

still don't like dual spaces?

matrix, the entries of which are matrices

angular momentum is an element of

$$[su(2)]^* \otimes B(\mathcal{H}) \cong \underline{su(2) \otimes B(\mathcal{H})}$$

given by

$\boxed{\text{PERELOMOV-POPOV MATRIX}}$

$$J = \big[\, \varrho(e_{ij}) \,\big]_{1 \le i,j \le 2}$$

for $\langle x, y \rangle = \mathrm{Tr}\, xy$ you get the transpose of this matrix.

$\boxed{\text{good tool for studying } \mathcal{U}(d)}$

2×2 matrix <u>with non-commuting entries</u>

HERMITIAN.

physicists on previous page would add a factor of $\pm i\hbar$ $\rightarrow$ NONHERMITIAN MATRIX.

$e_{ij} \notin su(2)$

why $\varrho(e_{ij})$ is well-defined?

⚠ ① how to convert to antihermitian matrices?

$\rightarrow$ COMPLEXIFICATION!

② how to deal with trace-zero?

QUANTUM MECHANICS

irreducible representations V^j of G are indexed by $j \in \{0, 1, 2, \dots\}$

"angular momentum = spin = $j\hbar$ is quantized"

"Quantum addition of angular momenta"

$$V^j \otimes V^k = \bigoplus_{l} c_{jk}^{l} \, V^l$$

"we have two quantum sytems with well-defined angular momenta.

we combine them and put to a single box.

what can be possible values for the angular momentum of the sytem?"

TODO: maximally mixed state

probability distribution of l

Newtonian limit ← 'what happens to a quantum system when it becomes so large that quantum effects disappear?'

(ρ_N) - sequence of irreducible representations of $SU(2)$

ρ_N corresponds to $j_N \in \{0, 1, 2, \dots\}$

$\hbar_N$ - "Planck constant"

$$\boxed{\begin{array}{c} \hbar_N j_N \longrightarrow j \\ \text{and} \quad \hbar_N \longrightarrow 0. \end{array}}$$

rescaled PP matrix

depends on N

$$\left[\hbar_N \, \rho_N(e_{ij}) \right]_{1 \leq i,j \leq 2} = \left[A_{ij} \right]_{1 \leq i,j \leq 2}$$

non commutative random variables with respect to $\mathbb{E} = \mathrm{tr}$ = normalized trace

more generally: representations of $U(d)$

$\cancel{j_N} \Rightarrow (\lambda_{N,1} \geq \dots \geq \lambda_{N,d})$

$\cancel{j} \Rightarrow (\lambda_1 \geq \dots \geq \lambda_d)$

$\boxed{\text{Theorem}}$

convergence of (non-commutative) moments

Noncommutative joint distribution of $(A_{ij})_{1\le i,j\le 2}$

converges (as $N \longrightarrow \infty$) to the noncommutative distribution

of $(x_{ij})_{1\le i,j\le 2}$ where

random Haar unitary in $SU(2)$
(more generally: G)

(more generally: the highest weight)

$$X = (x_{ij}) = \underbrace{V \begin{bmatrix} j & \\ & -j \end{bmatrix}}_{\text{uniformly random hermitian matrix with eigenvalues } \pm j} V^*$$

commuting random variables

The same result remains true for ANY compact Lie group G

the eigenvalues carry more information in this case

$\cancel{\pm j} \longrightarrow (\lambda_1, \dots \lambda_n)$

https://arxiv.org/abs/math/0610285

Collins & Śniady TAMS

"scaling in which the Lie group is fixed and the representation tends to infinity"

"PROOF" $\longrightarrow$ NEXT PAGE

THIS IS THE ONLY RIGOROUS PART OF THE PROOF

non-asymptotic result

no passing to the limit

CLAIM

The matrix $[A_{ij}]$ is unitarily invariant,

i.e. the joint noncommutative distributions of (A_{ij}) and (A'_{ij}) are equal

works fine also for $G = (S)U(d)$

for $[A'_{ij}] = V \, A \, V^{-1} = \underbrace{(V \otimes 1)}_{\in M_2(\mathbb{C})} \, A \, (V^{-1} \otimes 1)$ ($V \otimes 1 \in \mathrm{End}(V^?)$)

V as you like:

V is a deterministic / random / random Haar unitary matrix

Proof

for fixed $g \in G$

interesting map on the Lie group "adjoint action"

$$G \ni h \longmapsto g h g^{-1} = Ad_g \, h \in G \quad \leftarrow 1 \in G \text{ is a fixpoint}$$

$\Downarrow$

derivative = transformation on the tangent space

$$\mathfrak{g} \ni x \longmapsto Ad_g \, x \in \mathfrak{g}$$

$$Ad_g \, x = g \, x \, g^{-1}$$

even though $g, g^{-1} \in G$ and $x \in \mathfrak{g}$ are different kinds of objects this product makes sense due to the matrix structure

makes sense only if G has a matrix structure

representation of Lie ALGEBRA $\mathfrak{g}$

$$\rho(g x g^{-1}) = \rho(g) \, \rho(x) \, \rho(g^{-1})$$

representation of LIE GROUP G.

Hint: $\rho(g h g^{-1}) = \rho(g) \, \rho(h) \, \rho(g^{-1})$
take derivative for $h \to 1$

representation of Lie ALGEBRA $\mathfrak{g}$

$$\rho(gxg^{-1}) = \rho(g)\,\rho(x)\,\rho(g^{-1})$$

representation of LIE GROUP G.

the same in coordinates: $g = \mathcal{U} = (\mathcal{U}_{ij})_{ij}$

$x = e_{kl}$ "the matrix unit" COMPLEXIFICATION

$$\mathrm{tr}_N\,\rho\Big(\sum_{ij}\underbrace{\mathcal{U}_{ik}\,(\mathcal{U}^{-1})_{lj}}_{\text{numbers}}\,\underbrace{e_{ij}}_{\in\mathfrak{g}}\Big) = \mathrm{tr}_N\,\rho(\underset{\in G}{\mathcal{U}})\,\rho(\underset{\in\mathfrak{g}}{e_{kl}})\,\rho(\underset{\in G}{\mathcal{U}^{-1}})$$

$$= \sum_{ij}\mathcal{U}_{ik}\,(\mathcal{U}^{-1})_{lj}\,\underbrace{\mathrm{tr}_N\,\rho(e_{ij})}_{A_{ij}} = \big[\mathcal{U}^T A\,(\mathcal{U}^{-1})^T\big]_{kl} = A'_{kl}$$

$$= \rho(\mathcal{U})\,A_{kl}\,\rho(\mathcal{U}^{-1})$$

CLAIM follows immediately.

$$\boxed{\text{Conclusion: } A' = \rho(\mathcal{U})\,A\,\rho(\mathcal{U})^{-1}}$$

Conjugated by $(V\otimes 1)$

$\underset{\in M_?(\mathbb{C})}{1} \otimes \underset{\in \mathrm{End}(V^\lambda)}{\rho(u)}$

conjugation of PP matrix

is equivalent to entrywise conjugation of each entry.

$$A'_{ij} = \mathrm{tr}_N \sum_{kl}(\mathcal{U}^T)_{ik}\,\rho(e_{kl})\,(\mathcal{U}^{-1\,T})_{lj}$$

$$\boxed{\mathcal{U} = V^T}$$

– this transpose is kind of funny but it seems this is the right way to do it. Don't like transpose – change definition of PP matrix!

Conclusion, with bigger fonts.

$$\boxed{(V \otimes 1)\, A\, (V^{-1} \otimes 1) = \big(1 \otimes \rho(u)\big)\, A\, \big(1 \otimes \rho(u)^{-1}\big)}$$

$\boxed{\text{This is NOT a proof}}$

for this part you need to know the relationship between j_N ("the highest weight") and the representation

- $\hbar_N j_N = O(1)$ implies that

"the norm of A is uniformly bounded"

for best results you MAY use the operator norm but there are some other norms available as well (think $d \to \infty$)

- $[A_{ij}, A_{kl}] = [\hbar_N \, \rho_N(e_{ij}), \hbar_N \, \rho_N(e_{kl})] =$

$$\hbar_N^2 \, \rho_N([e_{ij}, e_{kl}]) = \hbar_N^2 \left([j=k] \, \rho(e_{il}) - [l=i] \, \rho(e_{kj}) \right)$$

$$= \hbar_N \, [j=k] \, A_{il} - \hbar_N \, [l=i] \, A_{kl}$$

"noncommutativity converges to zero"

the limit distribution of (A_{ij}) – if exists – is commutative and G-invariant.

remaining difficulty: study the eigenvalues distribution of the limiting random matrix.

Application: Littlewood-Richardson

TWO irreducible representations of $SU(2)$

$$\varrho^{(1)} : SU(2) \longrightarrow \operatorname{End} V^{(1)}$$
$$\varrho^{(2)} : SU(2) \longrightarrow \operatorname{End} V^{(2)}$$

$$\varrho^{(3)} = \varrho^{(1)} \otimes \varrho^{(2)} : SU(2) \longrightarrow \operatorname{End} V^{(1)} \otimes V^{(2)}$$

do not commute among themselves

PP matrix

$$\left[\varrho^{(3)}(e_{ij})\right]_{ij} = \left[\varrho^{(1)}(e_{ij}) \otimes 1\right]_{ij} + \left[1 \otimes \varrho^{(2)}(e_{ij})\right]_{ij}$$

$\longrightarrow$ Leibnitz rule

entries commute

with respect to tr –
– classically independent

"sum of two independent random matrices (with non-commuting entries)"

not VERY useful if we do not have extra information

Application: asymptotics of Littlewood-Richardson

$i \in \{1,2\}$

TWO sequences of irreducible representations of $SU(2)$

$(\rho_N^{(i)})$ - corresponds to $j_N^{(i)} \in \{0,1,2,\dots\}$

$$\hbar_N j_N^{(i)} \longrightarrow j^{(i)} \quad \text{and} \quad \hbar_N \longrightarrow 0.$$

Converge in distribution to entries of a random matrix with prescribed eigenvalues.

PP matrix

$$\left[\hbar_N \rho_N^{(3)}(e_{ij})\right]_{ij} = \left[\hbar_N \rho_N^{(1)}(e_{ij}) \otimes 1\right]_{ij} + \left[1 \otimes \hbar_N \rho_N^{(2)}(e_{ij})\right]_{ij}$$

$\longrightarrow$ Leibnitz rule

entries commute

with respect to tr -

- classically independent

Theorem

convergence of (non-commutative) moments

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Application: asymptotics of Littlewood-Richardson

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