Piotr Śniady
IMPAN

Lectures on random matrices and free probability theory

Lecture 3.
GUE random matrices, part II

October 29, 2019

X_N - GUE random matrix

Genus expansion for GUE

$\to$ [MS] Section 1.7
Section 1.8.

[flashback from the previous lecture...]

$$\mathbb{E} \int_{\mathbb{R}} z^{2k} \, d\mu_Y =$$

$$\mathbb{E} \operatorname{Tr} Y_N^{2k} = \sum_{\pi} \frac{1}{N^{2\,\mathrm{genus}(\pi)}} =$$

GUE random matrix

pair-partitions of $\{1, 2, \dots, 2k\}$

$$= \sum_{\pi} \frac{1}{N^{2-\chi}}$$

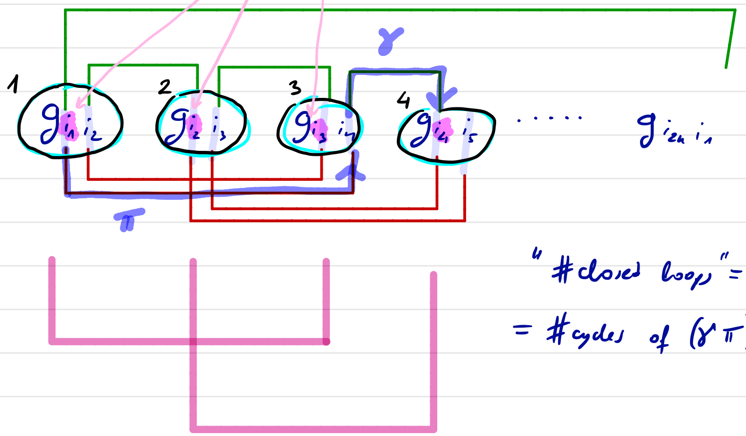
Alternative viewpoint from [MS, Section 1.8].

$$\frac{1}{N}\mathrm{Tr}\, Y_N^{2k} =$$

"Look on the left indices!"

γ

$$= \frac{1}{N}\, \mathbb{E} \sum_{i_1,\dots,i_{2k}} \overset{1}{g_{i_1 i_2}}\ \overset{2}{g_{i_2 i_3}}\ \overset{3}{g_{i_3 i_4}}\ \overset{4}{g_{i_4 i_5}} \cdots\cdots\ g_{i_{2k} i_1} =$$



pairing π

" # closed loops " =

= # cycles of $(\gamma\pi)$

$\gamma = (1, 2, \dots, 2k)$ full forward cycle

$\pi \in S_{2k}$ permutation which corresponds to the pair-partition.

$$= N^{\#\gamma\pi - k - 1}$$

$\# \gamma\pi - k - 1 \quad = ?$

① length on the group S_n

$$|\sigma| = n - \#\sigma$$

the minimal number of factors to write σ as a product of transpositions.

② $|\gamma| \leq |\gamma\pi| + |\pi^{-1}|$

$$2k - \underbrace{\#\gamma}_{1} \leq 2k - \#\gamma\pi + 2k - \underbrace{\#\pi}_{k}$$

$$\boxed{\#\gamma\pi \leq k+1.}$$

when triangle inequality becomes equality?

check: do you know what a PARTITION is?

Def.

partition π is CROSSING if

there exist indices $i < j < k < l$

such that

- $\overbrace{i \qquad k}^{\pi}$

- $\overbrace{j \qquad l}^{\pi}$

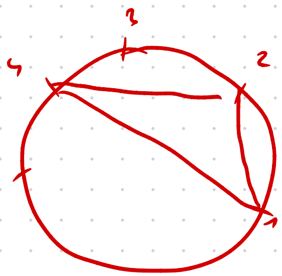
- i, j, k, l do not belong to the same single block of π.

Def.

partition is NONCROSSING

if it is not CROSSING.

Hint: model on the circle



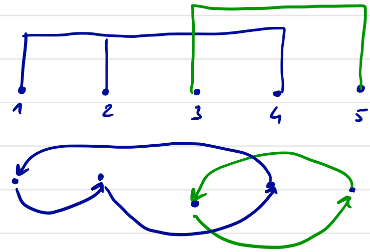
61. Robert Cori. *Un code pour les graphes planaires et ses applications*. Astérisque, No. 27. Société Mathématique de France, Paris, 1975.

Philippe Biane. Some properties of crossings and partitions. DISCRETE MATH 1997 [Section 1.3.3, Theorem 1].

Biane's result.

Given partition π of $[n] = \{1, \cdots, n\}$.

Encode π as a permutation from S_n



this is a very clever idea. We will come back to it while studying Weingarten calculus

partition π is NONCROSSING iff

$$|\gamma| = |\pi| + |\pi^{-1}\gamma|$$

COROLLARY: we count NONCROSSING PAIR PARTITIONS

$= \#$ ways to legally write $2k$ Brackets

leading contribution — genus $=0$
sphere

requires some arguments
$\to$ [MS, Section 1.3]

$$\lim_{N\to\infty} \int_{\mathbb{R}} x^{2k} \, d\mathbb{E}\mu_{Y_N}(x) =$$

$$= \underbrace{\lim_{n\to\infty} \mathbb{E}\, \mathrm{tr}\, Y_N^{2k}}_{} = \#\text{ Non-crossing partitions on } [2k]$$

$$= C_k \quad \text{Catalan number}$$

non-crossing pair partitions
$\updownarrow$
Dyck paths

$$= \frac{1}{k+1}\binom{2k}{k} =$$

exercise.

the limit exists and is finite

? how to guess this measure if you start with Catalan numbers?
$\to$ SOON!

! new

$$= \int_{-2}^{2} \boxed{\frac{1}{2\pi}\sqrt{4-x^2}}\; x^{2k}\, dx =$$

this shows that the MEAN eigenvalues distribution $\mathbb{E}\mu_X$ converges to semicircular law μ_{sc} in moments

$$= \int_{\mathbb{R}} x^{2k}\, d\mu_{sc}(x)$$

which is nice, but not exactly what we want.

Conclusion:

$$\lim_{N\to\infty} \int_{\mathbb{R}} x^{2k} \, d\mathbb{E}\, \mu_{Y_N}(x) =$$

$$= \int_{\mathbb{R}} x^{2k} \, d\mu_{sc}(x)$$

By symmetry argument

$$\int_{\mathbb{R}} x^{2k+1} \, d\mathbb{E}\, \mu_{Y_N}(x) = 0 =$$

$$= \int_{\mathbb{R}} x^{2k+1} \, d\mu_{sc}(x)$$

„measures $\mathbb{E}\mu_{Y_N}$ converge to μ_{sc}

in moments"

[flashback from your undergrad course of probability theory]

moments and convergence of measures

$\longrightarrow$ [MS] Section 2.1.

probability measures on $\mathbb{R}$

moments

convergence in moments

weak convergence of probability measures

measures determined by moments

Carleman's criterion

Hint: if $\sum \frac{1}{\sqrt[2n]{m_{2n}}} = +\infty$ then moment problem is determinate

Convergence in moments vs weak convergence

~~We say that~~

Def

- suppose X is a real-valued random variable.

  We say that a probability measure μ_X on $\mathbb{R}$ is **the distribution of X** if

$$\mathbb{E}\, f(X) = \int_{\mathbb{R}} f \, d\mu \qquad \text{for every } f \in C_c(\mathbb{R})$$

continuous, bounded function on $\mathbb{R}$

Def

- sequence of probability measures on $\mathbb{R}$ (μ_n) **converges weakly** to a prob. measure μ if

$$\lim_{n\to\infty} \int f \, d\mu_n = \int f \, d\mu$$

for any $f \in C_c(\mathbb{R})$

(Hint: $\Updownarrow$

$$\lim_{n\to\infty} \underbrace{F_{\mu_n}}_{\text{cumulative distribution function}}(x) = F_\mu(x) \text{ holds for}$$

any $x \in \mathbb{R}$ such that F_μ is continuous in x.

convergence in moments is usually disdained by probabilists. For example: the limit might be not unique.

However, in the non-commutative setup we cannot really formulate "weak convergence of probability measures" and we have to stick to moments.

Hint: the limit measure μ_{sc} has compact support, so

convergence in moments $\Longrightarrow$ weak convergence

Conclusion

$$\lim_{N \to \infty} \mathbb{E} \mu_{Y_N} = \mu_{sc}$$