Piotr Śniady
IMPAN

Lectures on random matrices and free probability theory

Lecture 5. Classical cumulants

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Lecture 5 table of contents.

* classical cumulants, moment-cumulant formula,
  Wick formula revisited,
  lattice of (all) partitions,
  Möbius function
  independence $\approx$ vanishing cumulants.

* free cumulants $\to$ 2.2.
  freeness = vanishing free cumulants

classical cumulants

→ [MS] Section 1.1

- single random variable / probability measure on $\mathbb{R}$

log of Laplace transform

$$\log \mathbb{E}\, e^{tX} = \sum_{n\geq 0} k_n \frac{t^n}{n!}$$

k_n ← n-th classical cumulant.

$k_1 = \mathbb{E} X$
$k_2 = \operatorname{Var} X$

NICE!

LINK: $k_n = k(\underbrace{X, \ldots, X}_{n \text{ times}})$

BETTER IDEA!

- multiple random variables

$$k(X_1, \ldots, X_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \log \mathbb{E}\, e^{t_1 X_1 + \cdots + t_n X_n} \Big|_{t_1 = \cdots = t_n = 0} =$$

$$= [t_1 \cdots t_n] \log \mathbb{E}\, e^{t_1 X_1 + \cdots + t_n X_n}$$

each cumulant k is linear with respect to each of its arguments.

$k(X, Y) = \operatorname{Cov}(X, Y)$

NICE!

INDEPENDENCE ⟹ VANISHING OF CUMULANTS.

if $\{A_1, A_2, \ldots\}$ and $\{B_1, B_2, \ldots\}$ are INDEPENDENT then

$$k(A_{i_1}, \ldots, A_{i_r}, B_{j_1}, \ldots, B_{j_s}) = 0 \quad \text{IF } r \geq 1,\ s \geq 1$$

HEURISTICS:

- "cumulants quantify violation of the naive guess

$$\mathbb{E}\, X_1 \cdots X_n = \mathbb{E}\, X_1 \cdots \mathbb{E}\, X_n$$
"

- know the mean value? naive guess

$$\mathbb{E}\, X_1 X_2 \approx \mathbb{E} X_1 \cdot \mathbb{E} X_2$$

Covariance measures the violation of this guess.

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}\, X_1 X_2 - \mathbb{E} X_1 \cdot \mathbb{E} X_2$$

- know the mean value <u>and</u> covariance?

naive guess:

$$\begin{aligned} \mathbb{E}\, X_1 X_2 X_3 = {} & \mathbb{E} X_1 \cdot \mathbb{E} X_2 \cdot \mathbb{E} X_3 + \\ & + \mathbb{E} X_1 \ \mathrm{Cov}(X_2, X_3) + \\ & + \mathbb{E} X_2 \ \mathrm{Cov}(X_1, X_3) + \\ & + \mathbb{E} X_3 \ \mathrm{Cov}(X_1, X_2) \end{aligned}$$

$$\kappa(X_1, X_2, X_3) = \ldots$$

$\to$ [MS]
Section 1.2

Exercise:

- find cumulants of the normal distribution $N(0,1)$

Hint: $$\mathbb{E}\, e^{tX} = \int_{-\infty}^{\infty} e^{ts} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, ds =$$

linear change of variables
$$=$$

$$= e^{\frac{1}{2}t^2}$$

$$\log \mathbb{E}\, e^{tX} = \frac{1}{2} t^2$$

- find cumulants of multidimensional Gaussian distribution.

Gaussian $\Rightarrow$ cumulants ZERO (except for $\mathbb{E}$ and Cov).

"cumulants measure deviation from Gaussianity"

CLASSICAL MOMENT-CUMULANT FORMULA

→ example NEXT PAGE

$$\mathbb{E}\, X_1 \cdots X_n = [t_1 \cdots t_n]\; \mathbb{E}\, e^{t_1 X_1 + \cdots + t_n X_n} =$$

$$= [t_1 \cdots t_n]\; \exp \boxed{\log \mathbb{E}\, e^{t_1 X_1 + \cdots + t_n X_n}} =$$

t-free term = 0

NICE.

"set-partitions to k ordered non-empty parts"

$$= [t_1 \cdots t_n] \sum_{k \geq 0} \frac{1}{k!} \underbrace{\boxed{} \cdots \boxed{}}_{k \text{ factors.}} =$$

Hint: if $i_1, i_2, \ldots$ are all different

$$[t_{i_1} t_{i_2} \cdots t_{i_k}] \boxed{} = \kappa(X_{i_1}, \ldots, X_{i_k})$$

$$= \sum_{\pi \in P(n)} \underbrace{\prod_{b \in \pi} \kappa(X_i : i \in b)}$$

we may denote it by $\kappa_\pi(X_1, \ldots, X_n)$ ← !

Exercise: revisit Wick formula for moments of multidimensional Gaussian distribution.

Example

$$\mathbb{E} X_1 = \kappa(X_1)$$

$$\mathbb{E} X_1 X_2 = \kappa(X_1, X_2) + \kappa(X_1)\, \kappa(X_2)$$

$$\begin{aligned}\mathbb{E} X_1 X_2 X_3 = {} & \kappa(X_1, X_2, X_3) + \\ & + \kappa(X_1)\, \kappa(X_2, X_3) + \\ & + \kappa(X_2)\, \kappa(X_1, X_3) + \\ & + \kappa(X_3)\, \kappa(X_1, X_2) + \\ & \quad + \kappa(X_1)\, \kappa(X_2)\, \kappa(X_3)\end{aligned}$$

Central limit theorem via cumulants

[ASSUMPTION]

Let $X_1, X_2, \dots$ be a sequence of independent random variables with the same distribution,

$\mathbb{E} X_i = 0$

$\mathbb{E} X_i^2 = \operatorname{Var} X_i = 1$

$\mathbb{E}|X_i^\alpha| < \infty$ (all moments exist)

for this method of proof we need this assumption, but there are alternative proofs which work for weaker assumptions.

Define $$Y_n = \frac{X_1 + \dots + X_n}{\sqrt{n}}$$

$$K_k(Y_n) = K_k\Big(\overbrace{\frac{X_1 + \dots + X_n}{\sqrt{n}}, \dots, \frac{X_1 + \dots + X_n}{\sqrt{n}}}^{k \text{ times}}\Big) =$$

[cumulant is linear with respect to each argument

$$= \frac{1}{n^{k/2}} \sum_{1 \le i_1, \dots, i_k \le n} \underbrace{K_k(X_{i_1}, \dots, X_{i_k})}_{=0 \text{ unless } i_1 = \dots = i_k} =$$

$$= \frac{1}{n^{k/2}} \sum_i K_k(X_i, \dots, X_i) = \frac{1}{n^{k/2}}\, n \, K_k(X_1, \dots, X_1)$$

$X_1, X_2, \dots$ have the same distribution.

① case $k=1$

$$K_1(Y_n) = \frac{\mathbb{E} X_1 \cdot n}{\sqrt{n}} = 0$$

② case $k=2$

$$K_2(Y_n) = K_2(X_1) = \operatorname{Var} X_1 = 1$$

③ case $k \geq 3$

$$\lim_{n\to\infty} \kappa_k(Y_n) = \lim_{n\to\infty} \frac{1}{n^{\frac{k-2}{2}}} \kappa_k(X_1,\dots,X_1) = 0$$

Conclusion:

Cumulants of Y_n converge for $n \to \infty$
to the cumulants of $N(0,1)$ normal distribution.

THEREFORE

moments of Y_n converge for $n \longrightarrow \infty$
to the moments of $N(0,1)$ normal distribution

THEREFORE [Carleman's criterion]

distribution of Y_n converges weakly
to $N(0,1)$

Lattice of patitions

→ theory of POSETs
= Partially Ordered SETs
is a business of its
own

- partition of an ordered set
- blocks
- partial order $\pi \leq \mu$ if
  * each block of π is contained in some block of μ

    [equivalent to : $a \overset{\pi}{\sim} b \Rightarrow a \overset{\mu}{\sim} b$]

- Lattice = supremum / infimum of any two elements exists.

- meet $\wedge$ and join $\vee$

$\pi \wedge \delta$ = maximum of elements which are smaller than both π and δ.

Hint: take intersections of all blocks of π and δ.

Hint: $\pi \vee \delta = ?$
take all partitions bigger than π AND δ, then calculate their MEET $\wedge$

- maximal / minimal element
  $\mathbb{1}$ and $\mathbb{O}$

[flashback]

moment-cumulant formula

$$\mathbb{E}\, X_1 \cdots X_n = \sum_{\pi \in \mathcal{P}(n)} \underbrace{\kappa_\pi (X_1, \dots, X_n)}$$

$$:= \prod_{B \in \pi} \kappa (X_i \; : i \in B)$$

"multiplicative extension of cumulants"

Example:

$$\kappa_{\vert\,\vert\,\vert} (X_1, X_2, X_3) = \kappa(X_1)\, \kappa(X_2)\, \kappa(X_3)$$

$$\kappa_{\vert\,\sqcup} (X_1, X_2, X_3) = \kappa(X_1) \cdot \kappa(X_2, X_3)$$

MULTIPLICATIVE EXTENSION OF MOMENTS

fix random variables $a_1, \dots, a_n$

μ - partition of $\{1, \dots, n\}$

function on the set of partitions of $\{1, \dots, n\}$

$$\underbrace{\mathbb{E}_\mu(a_1, a_2 \cdots a_n)}_{:= \prod_{b \in \mu} \mathbb{E}\left(\prod_{i \in b} a_i\right)} = \sum_{\substack{\pi \\ \pi \le \mu}} \kappa_\pi(a_1, \dots, a_n) \qquad (☆)$$

$\pi \le \mu$ ← "refinement order"

$$:= \prod_{b \in \mu} \mathbb{E}\left(\prod_{i \in b} a_i\right)$$

„multiplicative extension of moments"

IMPORTANT

IMPORTANT

if some function $\tilde{\kappa}_\pi$ fulfills the system of equations (☆) THEN $\tilde{\kappa}_\pi$ is equal to the true cumulant κ_π

Hint: an upper-triangular system of equations has a unique solution.

convenient trick for proving Leonov–Shiryaev

$$\mathbb{E}_\mu(a_1, a_2, \cdots, a_n) = \sum_{\substack{\pi \\ \pi \leq \mu}} K_\pi(a_1, \ldots, a_n)$$

Lemma ( ). If $\pi \longmapsto \widetilde{K_\pi}$ is a function on the set $P(n)$ such that for each $\mu \in P(n)$

$$E_\mu(a_1, \cdots, a_n) = \sum_{\substack{\pi \\ \pi \leq n}} \widetilde{K_\pi}$$

THEN for each $\pi \in P(n)$

$$\widetilde{K_\pi} = K_\pi(X_1, \ldots, X_n)$$

Hint: an upper-triangular system of equations has a unique solution.

convenient trick for proving Leonov–Shiryaev

Möbius inversion formula

$$\mathbb{E}_\mu = \sum_{\pi \leq \mu} \boxed{K_\pi} \quad \text{"unknowns"}$$

„solving upper triangular system of equations"

$$\Updownarrow$$

"solution"

$$K_\pi = \sum_{\mu \leq \pi} \underbrace{\operatorname{Moeb}(\mu, \pi)}_{= \prod_{b \in \pi} (-1)^{\left(\underbrace{\#\text{blocks of } \mu \text{ which are in } b}_{\#\mu\downarrow_b} - 1\right)} \left(\underbrace{\#\text{blocks of } \mu \text{ which are in } b}_{\#\mu\downarrow_b} - 1\right)!} \mathbb{E}_\mu$$

the most interesting case if $\pi = \mathbb{1}$ is the maximal partition.

Example

$$K_{\sqcup\!\sqcup}(X_1, X_2, X_3) = \mathbb{E} X_1 X_2 X_3 - \mathbb{E} X_1 \cdot \mathbb{E} X_2 X_3 - \mathbb{E} X_2\, \mathbb{E} X_1 X_3$$
$$- \mathbb{E} X_3\, \mathbb{E} X_1 X_2 + 2 \cdot \mathbb{E} X_1 \cdot \mathbb{E} X_2 \cdot \mathbb{E} X_3$$

Proof by "guess and check correctness".
We plug "solution" of the linear system of equations to the system and check if we get correct values.

Let us fix μ.

$$\sum_{\pi \leq \mu} \sum_{\mu' \leq \pi} \mathrm{Moeb}(\mu', \pi) \, \mathbb{E}_{\mu'} =$$

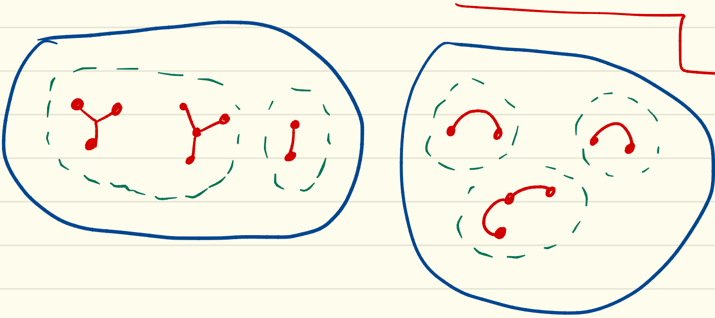
$$= \sum_{\mu' \leq \mu} \mathbb{E}_{\mu'} \underbrace{\left[\sum_{\substack{\pi \\ \mu' \leq \pi \leq \mu}} \mathrm{Moeb}(\mu', \pi) \right]}_{?} \overset{?}{=} \mathbb{E}_{\mu}$$

goal: $\begin{cases} 1 & \text{if } \mu' = \mu \quad \checkmark \text{ ok!} \\ 0 & \text{otherwise.} \quad ? \end{cases}$

$\mu' \leq \pi \leq \mu$

If you find this calculation too hard you may start with the case when $\mu = \mathbb{1}$ is the maximal partition.

You can even start with $\mu' = \mathbb{0}$.



fix μ' and μ $\quad \mu' \leq \mu$

$$\sum_{\substack{\pi \\ \mu' \leq \pi \leq \mu}} \mathrm{Moeb}(\mu', \pi)$$

sum over all PERMUTATIONS with CYCLES prescribed by set-partition π_b.

$$= \prod_{b \in \mu} \sum_{\pi_b} \prod_{c \in \pi_b} (-1)^{\#\mu'|_c - 1} \underbrace{(\#\mu'|_c - 1)!}_{\text{number of ways to arrange the elements of } \mu'|_c \text{ to a cycle}} =$$

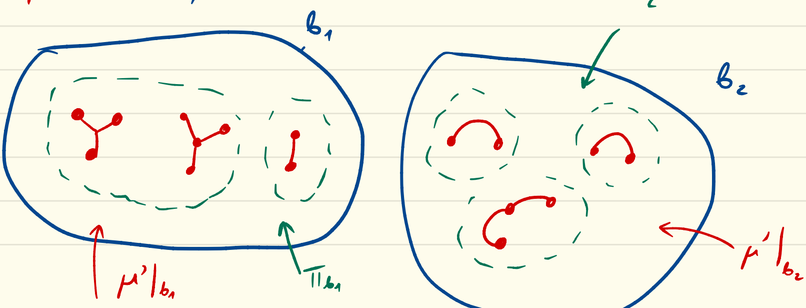
π_b - partition on blocks of $\mu'|_b$

$$= \prod_{b \in \mu} \underbrace{\left[\sum_{\hat{\pi}_b} (-1)^{\#\mu'|_b - \#\text{cycles of } \hat{\pi}_b} \right]}_{= 0 \text{ if } \#\text{blocks in } \mu'|_b \geq 2.}$$

$\hat{\pi}_b$ - permutation of blocks of $\mu'|_b$

$(-\text{length of } \hat{\pi}_b)$

$\mu' \leq \pi \leq \mu$



→[MS] Section 22 Thm 13.

LEONOV-SHIRYAEV

"how to calculate cumulants of products"?

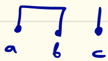
Theorem $a_1, \dots, a_n \in \mathcal{A}$, τ is an interval partition

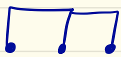
$$\kappa\left(\prod_{i \in b} a_i \; : \; b \in \tau\right) = \sum_{\substack{\sigma \\ \sigma \vee \tau = 1_n}} \kappa_\sigma(a_1, \dots, a_n)$$

cumulant of PRODUCTS

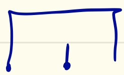
USUAL cumulants.

Example

$\tau =$ 

$$\kappa(ab, c) = \kappa(a, b, c)$$ 

$$+ \kappa(a)\,\kappa(b, c)$$ 

$$+ \kappa(b)\,\kappa(a, c)$$ 

Proof. use ()

philosophy: "guess and prove."

DEFINITION

$$\tilde{K}_{\pi} := \sum_{\substack{\sigma : \\ \sigma \vee \tau = \hat{\pi}}} K_{\sigma}(a_1, \dots, a_n)$$

USUAL cumulants.

$\sigma, \tau, \hat{\pi}$ are partitions of $\{1, \dots, n\}$

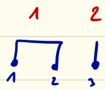
π: partition of $\{1, \dots, \# \tau\} =$
$=$ blocks of τ

passage from π to $\hat{\pi}$:
replace each block of τ by its entries.
"PARTITION OF COUNTRIES
$\mapsto$ PARTITION OF CITIES"

$$\mathbb{E}_{\mu} \overset{?}{=} \sum_{\pi \le \mu} \tilde{K}_{\pi}$$

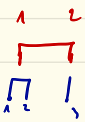
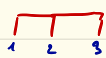
$\longrightarrow$ PROOF next page

Example

for $\tau =$ 

two choices:

a) $\pi =$  $\hat{\pi} =$ 

b) $\pi =$  $\hat{\pi} =$ 

μ, π - partitions of $\{1, \dots, \#\tau\}$

$\hat{\pi}, \delta$ - partition of $\{1, \dots, n\}$

$$\mathbb{E}_\mu = \sum_{\pi \leq \mu} \tilde{K}_\pi$$

$$L = \sum_{\delta \leq \hat{\mu}} K_\delta$$

$$R = \sum_{\pi \leq \mu} \sum_{\delta : \, \delta \vee \tau = \hat{\pi}} K_\delta =$$

$$= \sum_{\delta} \sum_{\pi \leq \hat{\mu}, \; \delta \vee \tau = \hat{\pi}} K_\delta$$

if $\delta \not\leq \hat{\mu}$ then such π does not exist

if $\delta \leq \hat{\mu}$ then such π is unique

$[\delta \leq \hat{\mu}]$

$\pi := \delta \vee \tau \Big|$ glue elements of τ.

"TURN PARTITION OF CITIES TO A PARTITION OF COUNTRIES"

Resources which might be useful for the lecturer only = IGNORE THIS PAGE

- Sandrine Dudoit

https://www.stat.berkeley.edu/~sandrine/Docs/TerrySelectedWorksSpringer/Version1/McCullagh/McCullagh.pdf

interesting personal insight into the combinatorics behind classical cumulants

- Terence Paul Speed
  „Cumulants and partition lattices"
  Australian J. Statist 25 (1983), no. 2
  378-388

legendary paper which shows the link between cumulants and the lattices, Möbius functions, etc.

- → [NS] p.144 Corollary 9.13

$$\#\{\pi \in NC(n) \mid \pi \text{ has } k \text{ blocks}\} =$$

$$= \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

NARAYANA NUMBERS.

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