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IMPAN

Lectures on random matrices and free probability theory

Lecture 6.
Free cumulants

November 26, 2019

→ MS, Section 1.12

→ philosophical insight:

Tao, Section 2.5.
pages 183–191, 194, 201
RECOMMENDED.

non-commutative probability space

TODO

* non-commutative distribution

Q:
can we do better than just moments?

* convergence in distribution.

"there is no weak convergence"

philosophy:
what is
and what is NOT
captured by the limit.

Definition 12. In general we refer to a pair $(\mathcal{A}, \varphi)$, consisting of a unital algebra $\mathcal{A}$ and a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(1) = 1$, as a *non-commutative probability space*. If $\mathcal{A}$ is a $*$-algebra and φ is a *state*, i.e., in addition to $\varphi(1) = 1$ also positive (which means: $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$), then we call $(\mathcal{A}, \varphi)$ a $*$-probability space. If $\mathcal{A}$ is a C^*-algebra and φ a state, $(\mathcal{A}, \varphi)$ is a *C^*-probability space*. Elements of $\mathcal{A}$ are called *non-commutative random variables* or just random variables.

If $(\mathcal{A}, \varphi)$ is a $*$-probability space and $\varphi(x^*x) = 0$ only when $x = 0$ we say that φ is *faithful*. If $(\mathcal{A}, \varphi)$ is a non-commutative probability space, we say that φ is *non-degenerate* if we have: $\varphi(yx) = 0$ for all $y \in \mathcal{A}$ implies that $x = 0$; and $\varphi(xy) = 0$ for all $y \in \mathcal{A}$ implies that $x = 0$. By the Cauchy-Schwarz inequality, for a state on a $*$-probability space "non-degenerate" and "faithful" are equivalent. If $\mathcal{A}$ is a von Neumann algebra and φ is a faithful normal state, i.e. continuous with respect to the weak-* topology, $(\mathcal{A}, \varphi)$ is called a *W^*-probability space*. If φ is also a trace, i.e., $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$, then it is a *tracial W^*-probability space*. For a tracial W^*-probability space we will usually write (M, τ) instead of $(\mathcal{A}, \varphi)$.

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Example? random matrices

$\longrightarrow$ SOON

10-minutes summary of lecture 2

* Ginibre ensemble:

$X = (f_{ij})$ is $N \times N$ random matrix

$(\mathrm{Re}\, f_{ij}, \mathrm{Im}\, f_{ij})$ - iid $N(0, \sigma^2)$ random variables

* GUE random matrix

$Y = (g_{ij})$ is $N \times N$ random matrix

$$Y = \underset{\substack{\uparrow \\ \text{Ginibre.}}}{X} + X^*$$

our favorite normalization: $\mathbb{E}\, |g_{ij}|^2 = \frac{1}{N}$

* $$\lim_{N \to \infty} \mathbb{E}\, \underbrace{\frac{1}{N} \mathrm{Tr}}_{\substack{tr = \\ \text{normalized} \\ \text{trace}}} Y^k = \sum_{\pi \in NC_2(k)} 1$$

"noncrossing 2-partitions of $[k] = \{1, \dots, k\}$"

IMPORTANT TODAY!

PLAN FOR TODAY: use GUE as a motivating example for (asymptotic) freeness.

One good example is a good thing :)

→ MS, Section 1.11

NON-COMMUTATIVE PROBABILITY SPACE – THE KEY EXAMPLE

let $Y_{N,1}, \dots, Y_{N,s}$ be independent $N\times N$ GUE random matrices

(we often skip it)

$\mathcal{A}_{N,i} = \mathbb{C}[Y_{N,i}]$ polynomials in $Y_{N,i}$ (one variable!)

$\mathcal{A}_N = \mathbb{C}\langle Y_{N,1}, \dots \rangle$ (non-commutative) polynomials in $Y_{N,1}, \dots$

commutative polynomials $\mathbb{C}[\cdots]$

non-commutative polynomials $\mathbb{C}\langle\cdots\rangle$

$\varphi_N : \mathcal{A}_N \longrightarrow \mathbb{C}$ functional

$\varphi_N(A) := \mathbb{E}\, \mathrm{tr}\, A$

"each N is a separate world".

---

the limiting object.

"Convergence in the sense of (noncommutative) mixed moments"

$\mathcal{A}$ = algebra of non-commutative polynomials in (abstract) variables $Y_1, \dots, Y_s$ $= \mathbb{C}\langle Y_1, \dots, Y_s\rangle$

(✡)

$$\varphi\big(p(Y_1, \dots, Y_s)\big) := \lim_{N\to\infty} \varphi_N\big(p(Y_{N,1}, \dots, Y_{N,s})\big)$$

THE LIMIT EXISTS

→ NEXT PAGE!

Many GUE random matrices.

M & S for long time denoted GUE random matrices by the symbol Y. At page 23 they start to use the symbol X. We have to live with this.

Assume $Y_1, \dots, Y_s$ are independent $N \times N$ GUE matrices.

We proved that $\rightarrow$ [MS, Sect. 1, Lemma 9]

(kind of, one has to revisit the proof)

$$\lim_{N\to\infty} \mathbb{E}\, \mathrm{tr}\, Y_{i_1} \cdots Y_{i_k} = \underbrace{\sum_{\pi \in NC_2} \prod_j \left[i_{\pi_{j,1}} = i_{\pi_{j,2}} \right]}_{\text{M\&S say that } \pi \text{ RESPECTS the COLORING } (i_1, \dots, i_k)} \qquad (*)$$

$$\boxed{\varphi(Y_{i_1} \cdots Y_{i_k}) :=}$$

$\pi = \{ \{\pi_{1,1}, \pi_{1,2}\}, \dots$

Plan for today

find a better understanding of this formula via "FREE CUMULANTS"

almost* like Wick formula for independent $N(0,1)$ Gaussian random variables

only non-crossing pairings.

NON-CROSSING PARTITIONS, revisited

→ MS, Section 2.2

→ NS, Lecture 9

non-crossing partitions appeared already in Lecture 1 (at the very end)

- non-crossing partition of an ordered set
- blocks
- partial order on NC "reverse refinement order"
- meet $\wedge$ and join $\vee$

$\pi \wedge \sigma$ = maximum of elements which are smaller than both π and σ.

Hint: take intersections of all blocks of π and σ.

Hint: $\pi \vee \sigma = ?$ take all partitions bigger than π AND σ, then calculate their MEET $\wedge$

JOIN in $\mathcal{P}$ and NC are the same.

MEET in NC and $\mathcal{P}$ are NOT the same.

- maximal / minimal element 1 and 0
- lattice = unique supremum, unique infimum

? why [NS] Proposition 9.17 does not show existence of meet $\vee$ for non-crossing partitions ?

⚠ MEET in NC and $\mathcal{P}$ are NOT the same. ← Example?

BUT! if τ – INTERVAL partition

π – NC-partition

THEN

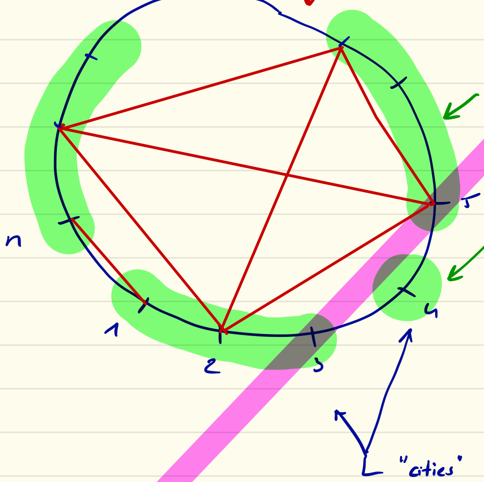
$$\tau \vee_{NC} \pi = \tau \vee_{\mathcal{P}} \pi$$

Hint: calculating $\tau \vee_{\mathcal{P}} \pi$ in a few simple steps...

1

STARTING POINT:

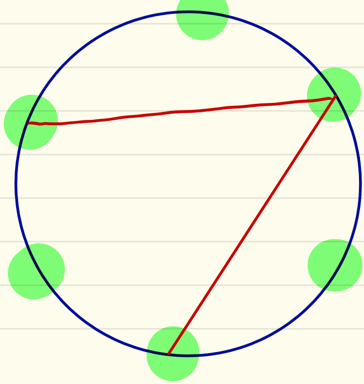
partition π defines an equivalence relation

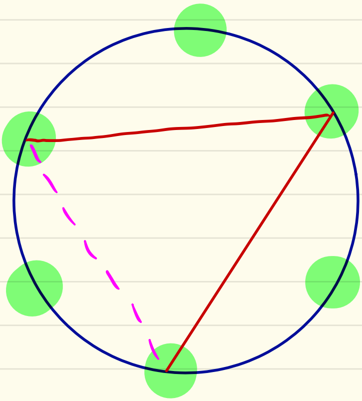


2

two equivalence classes of τ are connected by new relation $\tilde{\pi}$ if some pair of their elements is connected.

PASS TO EQUIVALENCE CLASSES OF τ





③ not transitive?

MAKE IT TRANSITIVE by iterative closing the TRIANGLES.

at each step property "NC" is preserved.

$\tau \vee_{\beta} \delta$ is NC ✓.

FREE CUMULANTS

→ MS, Section 2.2

→ NS, lecture 11

inspired by the classical moment-cumulant formula...

$\boxed{\text{WE DECLARE}}$

$$\varphi(a_1 a_2 \cdots a_n) = \sum_{\pi \in NC(n)} \underbrace{K_\pi(a_1, \dots, a_n)}$$

NEW! →

"multiplicative extension" $:= \prod_{b \in \pi} K(a_i : i \in b)$

we like multiplicative extensions so much that we will define multiplicative extension of φ as well → NEXT PAGE.

INSIGHT: this is an upper-triangular system of equations which CAN be inductively solved. Gives a DEFINITION of free cumulants.

Example:

$$\varphi(a_1) = K(a_1)$$

Really interesting things start to happen for 4 factors.

$$\varphi(a_1 a_2) = K(a_1, a_2) + K(a_1)\, K(a_2)$$

$$\varphi(a_1 a_2 a_3) = K(a_1, a_2, a_3) + \cdots$$

⚠ each free cumulant is LINEAR with respect to each of its arguments

GUE and free cumulants. (*) revisited

$$\varphi(Y_{i_1} \cdots Y_{i_n}) = \sum_{\pi \in NC_2} \prod_j [i_{\pi_{j,1}} = i_{\pi_{j,2}}]$$

$$\pi = \{ \{\pi_{1,1}, \pi_{1,2}\}, \\ \vdots \\ \}$$

$$\Downarrow$$

$$\kappa(Y_{\ell_1}, \ldots, Y_{\ell_r}) = \begin{cases} 0 & \text{if } r \neq 2 \\ [\ell_1 = \ell_2] & \text{if } r = 2 \end{cases}$$

$\boxed{\text{MORE GENERAL SETUP}}$

Fix $a_1, \dots, a_n \in \mathcal{A}$

φ and K are now functions on $NC(n)$

$$\underbrace{\varphi_\sigma(a_1 a_2 \cdots a_n)}_{:= \prod_{b \in \sigma} \varphi\left(\prod_{i \in b} a_i\right)} = \sum_{\substack{\pi \in NC(n) \\ \pi \leq \sigma}} K_\pi(a_1, \dots, a_n) \qquad (\star)$$

fast forward. you have seen it before.

> this is **NOT** a definition;
> this is a COROLLARY.

However ⚠ TWISTED LOGIC AHEAD

if some function $\tilde{K}_\pi$ fulfills the system of equations $(\star)$ THEN $\tilde{K}_\pi$ is equal to the true free cumulant K_π

Hint: an upper-triangular system of equations has a unique solution.

convenient trick for proving Leonov–Shiryaev

IMPORTANT

IMPORTANT

Möbius inversion formula.

→ [NS] Lecture 10.

if P is a finite poset...

$$P^{(2)} := \{ (\pi, \sigma) : \pi \leq \sigma, \ \pi, \sigma \in P \}$$

think: $P = NC(n)$
or $P =$ all partitions of $[n]$

we are interested in the class of functions from $P^{(2)}$ to $\mathbb{C}$

CONVOLUTIONS:

- for $F, G : P^{(2)} \to \mathbb{C}$

$$(F * G)(\pi, \sigma) := \sum_{\pi \leq \rho \leq \sigma} F(\pi, \rho)\, G(\rho, \sigma)$$

this convolution can be interpreted as a matrix multiplication.
$\implies$ associativity

Homework: is it true that $F * G = G * F$?

- for $f : P \to \mathbb{C}$, $G : P^{(2)} \to \mathbb{C}$

$$(f * G)(\sigma) := \sum_{\rho \leq \sigma} f(\rho)\, G(\rho, \sigma)$$

ABSTRACT NONSENSE AHEAD. FEEL FREE TO SKIP THIS PART

- $\delta : P^{(2)} \longrightarrow \mathbb{C}$

$$\delta(\pi, \vartheta) = [\pi = \vartheta]$$

is the unit of this convolution:

$F * \delta = \delta * F = F$

$f * \delta = f$

- $\zeta : P^{(2)} \longrightarrow \mathbb{C}$ zeta function

$$\zeta(\pi, \vartheta) = 1 \qquad (\text{for } \pi \leq \vartheta)$$

- $\mu : P^{(2)} \longrightarrow \mathbb{C}$ Möbius function is the inverse of ζ

$$\mu * \zeta = \zeta * \mu = \delta$$

left- and right-inverse (if they exist) must be equal

$$\Updownarrow$$

$\forall_{\pi \leq \vartheta}$

$$\sum_{\pi \leq \tau \leq \vartheta} \mu(\pi, \tau) = [\pi = \vartheta]$$

fix π.
use induction over ϑ to show existence and uniqueness of $\mu(\pi, \vartheta)$

END OF ABSTRACT NONSENSE

Magic fact: Möbius function for NC is given by

$$\boxed{\mu(\rho,\sigma) = \prod_{b\in\sigma} (-1)^{\#\rho|_b - 1}\, C_{\#\rho|_b - 1}}$$

Catalan numbers

#blocks of ρ which are sitting inside b.

any nice, conceptual proof?

Back to free cumulants

functions on $NC(n)$

$$\varphi = \kappa * \zeta$$

moments   free cumulants.

$$\kappa = \varphi * \mu$$

→ [MN], Section 2.2

the most interesting case is $\sigma = 1$

$$\boxed{\kappa(\sigma) = \sum_{\substack{\rho\leq\sigma \\ \rho\in NC}} \varphi(\rho)\ \mu(\rho,\sigma)}$$

THIS IS THE ONLY OUTCOME OF THE ABSTRACT NONSENSE THAT WE CARE.

more concrete version: take $\sigma = 1_n$

$$\kappa(a_1,\dots,a_n) = \sum_{\rho\in NC(n)} \varphi_\rho(a_1,\dots,a_n)\ \mu(\rho, 1_n)$$

Example:

$$\kappa(a_1) = \varphi(a_1)$$

$$\kappa(a_1, a_2) = \varphi(a_1, a_2) - \varphi(a_1)\varphi(a_2)$$

$$\begin{aligned}\kappa(a_1, a_2, a_3) = \varphi(a_1, a_2, a_3) & - \\ & - \varphi(a_1)\varphi(a_2 a_3) \\ & - \varphi(a_2)\varphi(a_1 a_3) \\ & - \varphi(a_3)\varphi(a_1 a_2) \\ & + 2\varphi(a_1)\varphi(a_2)\varphi(a_3)\end{aligned}$$

$$\kappa(a_1, \dots, a_n) = \sum_{\rho \in NC(1,\dots,n)} (-1)^{(\#\rho)-1} \underbrace{C_{(\#\rho)-1}} \varphi_\rho(a_1, \dots, a_n)$$

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

Catalan numbers.

→ [MS] Section 22 Thm 13.

LEONOV-SHIRYAEV-KRAWCZYK-SPEICHER

"how to calculate cumulants of products"?

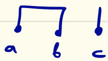
Theorem $a_1, \dots, a_n \in \mathcal{A}$, τ is an <u>interval partition</u>

$$\kappa\Big(\prod_{i \in b} a_i \ : \ b \in \tau \Big) = \sum_{\substack{\delta : NC(n) \\ \delta \vee \tau = 1_n}} \kappa_\delta (a_1, \dots, a_n)$$

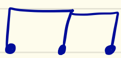
cumulant of PRODUCTS

USUAL free cumulants.

Example

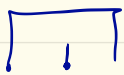
$\tau =$ 

$$\kappa(ab, c) = \kappa(a, b, c)$$



$$+ \kappa(a)\ \kappa(b,c)$$

$$+ \kappa(b)\ \kappa(a,c)$$



<u>Proof.</u> use ()

$\boxed{\text{DEFINITION}}$

$$\widetilde{K}_\pi := \sum_{\substack{\delta :\\ \delta \vee \tau = \hat{\pi}}} K_\delta(a_1, \dots, a_n)$$

USUAL free cumulants.

$\delta, \tau, \hat{\pi}$ are NC partitions of $\{1, \dots, n\}$

NC partition of $\{1, \dots, \# \tau\} =$
$=$ blocks of τ

passage from π to $\hat{\pi}$:

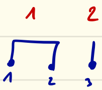
replace each block of τ by its entries.

"PARTITION OF COUNTRIES
$\mapsto$ PARTITION OF CITIES"

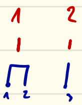
$$\varphi_\mu \overset{?}{=} \sum_{\pi \leq \mu} \widetilde{K}_\pi$$

$\longrightarrow$ PROOF next page

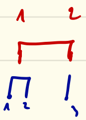
Example

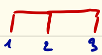
for $\tau =$ 

two choices:

a) $\pi =$ 

$\hat{\pi} =$ 

b) $\pi =$ 

$\hat{\pi} =$ 

μ, π – partitions of $\{1, \dots, \#\tau\}$
$\hat{\pi}, \delta$ – partition of $\{1, \dots, n\}$

$$\varphi_\mu = \sum_{\pi \leq \mu} \tilde{\kappa}_\pi$$

$$L = \sum_{\delta \leq \hat{\mu}} \kappa_\delta$$

$$R = \sum_{\pi \leq \mu} \sum_{\delta : \delta \vee \tau = \hat{\pi}} \kappa_\delta =$$

$$= \sum_{\delta} \sum_{\substack{\pi \leq \hat{\mu}, \\ \delta \vee \tau = \hat{\pi}}} \kappa_\delta$$

if $\delta \not\leq \hat{\mu}$ then such π does not exist

if $\delta \leq \hat{\mu}$ then such π is unique

$[\delta \leq \hat{\mu}]$

$\pi := \delta \vee \tau$ | glue elements of τ.

"TURN PARTITION OF CITIES TO A PARTITION OF COUNTRIES"

Cumulant-oriented definition of freeness.

Definition

assume

$(\mathcal{A}, \varphi)$ - noncommutative probability space

$A_1, A_2, \dots \subseteq \mathcal{A}$

finite or infinite collection of sets.

We say that $A_1, A_2, \dots$ are **free** if

for any $x_1 \in A_{i_1}, x_2 \in A_{i_2}, \dots, x_m \in A_{i_m}$

$$\kappa(x_1, \dots, x_m) \neq 0 \implies i_1 = \dots = i_m$$

„all mixed cumulants vanish"

[or, equivalently,

if $i_1, \dots, i_m$ are NOT all equal

$$\implies \kappa(x_1, \dots, x_m) = 0 \quad]$$

idea: for classical cumulants

independence $\implies$
$\implies$ vanishing of cumulants.

opposite implication ALMOST true.
[measures for which moment problem is not determinate]

Example

$\{Y_1\}, \dots, \{Y_m\}$ are free.

Theorem

if $A_1, A_2, \ldots$ are free

and $\mathcal{A}_i = \mathrm{Alg}(1, A_i)$ = unital algebra generated by A_i

THEN

$\mathcal{A}_1, \mathcal{A}_2, \ldots$ are free.

Proof.

$i_1, \cdots i_m$ NOT all equal.

$$x_1 = y_{1,1}\, y_{1,2} \cdots y_{1,l_1} \in \mathcal{A}_{i_1}$$

$y_{1,1}, \ldots, y_{1,l_1} \in A_{i_1}$

$$x_2 = y_{2,1}\, y_{2,2} \cdots y_{2,l_2} \in \mathcal{A}_{i_2}$$

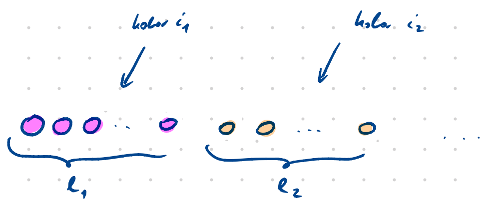
$y_{2,1}, \ldots, y_{2,l_2} \in A_{i_2}$

$$\vdots$$

$$x_m = \ldots$$

Leonov-Shiryaev

$$K(y_{1,1}\, y_{1,2} \cdots y_{1,l_1}, \ldots\ldots) = \sum_{\substack{\pi \in NC(l_1 + \cdots + l_m) \\ \pi \vee \tau = 1}} K_\pi(y_{1,1}, \ldots, y_{m,l_m}) = 0.$$



π - does not connect cities of different colors [freeness]

τ - [Block structure]

useful in the proof: sets $A_1, A_2, \dots$ are free $\Rightarrow$
$\Rightarrow$ unital algebras $\mathcal{A}_1, \mathcal{A}_2, \dots$ which they generate are free.

$\rightarrow$ [MS] Section 22. Proposition 15.

Theorem.

If $r \geq 2$!

and $a_i \in \mathbb{C}$ for some i

THEN

$$\kappa(a_1, a_2, \dots, a_r) = 0.$$

Proof.

use trick

remove singleton i

Define

$$\widetilde{\kappa_\pi} := \begin{cases} \underbrace{\kappa_\pi(a_1, \dots, \cancel{a_i}, \dots)}_{\text{the usual free cumulant.}} \cdot a_i & \text{if } i \text{ is a singleton in } \pi \\ 0 & \text{otherwise.} \end{cases}$$

? $$\varphi_\sigma \overset{?}{=} \sum_{\pi \leq \sigma} \widetilde{\kappa_\pi} = \varphi_{\sigma|_{1,2,\dots,\cancel{i},\dots}}(a_1, \dots, \cancel{a_i}, \dots) \cdot a_i =$$

because $a_i \in \mathbb{C}$ is a scalar

$$= \varphi_\sigma(a_1, \dots, a_i, \dots)$$

nonzero contribution only if i is a singleton in π.

like a sum over partitions of $1, 2, \dots, \cancel{i}, \dots$ smaller than σ [restricted to $1, 2, \dots, \cancel{i}, \dots$]

✓ Yes!

So $\widetilde{\kappa_\pi} = \kappa_\pi$ gives free cumulants.

$\square$

φ - oriented definition of freeness.

$\boxed{\text{freeness}}$

abstract framework
inspired by GUE
random matrices.

$\longrightarrow$ [MS] Section 1.11

ORIGINAL DEFINITION OF FREENESS

DEFINITION

Let $(\mathcal{A}, \varphi)$ be a unital algebra with a unital linear functional.

Suppose $\mathcal{A}_1, \mathcal{A}_2, \dots$ are <u>unital</u> subalgebras.

„φ-oriented definition"

We say that $\mathcal{A}_1, \mathcal{A}_2$ are freely independent (or, shortly, free) with respect to φ

if for each $r \geq 2$
and $a_1, \dots, a_r \in \mathcal{A}$ st:

- $\varphi(a_i) = 0$ for $i \in [r]$
- $a_i \in \mathcal{A}_{j_i}$
- $j_1 \neq j_2, j_2 \neq j_3, \dots, j_{r-1} \neq j_r$

we must have

$$\varphi(a_1 \cdots a_r) = 0$$

„alternating product of centered elements is centered"

THIS IS JUST A DEFINITION.
WHETHER IT IS USEFUL OR NOT
DEPENDS ON <u>EXAMPLES</u>

Note to myself:
sometimes we use indices $i_1, \dots, i_r$
sometimes $j_1, \dots, j_r$

DEFINITION

elements $y_1, \dots, y_r \in \mathcal{A}$ are free if

the unital algebras which they generate are free (in the above sense).

$= \mathbb{C}[y_1], \mathbb{C}[y_2], \dots$ algebras of polynomials!

DEFINITION

sets $\subseteq \mathcal{A}$ are free if

the unital algebras which they generate are free

Free cumulants and freeness

"what free cumulants are good for?"

→ [MS] Section 2.2, Thm 16
→ [NS] lecture 11
Theorem 11.16

important and FANTASTIC Theorem

Sets $A_1, A_2, \ldots \subseteq \mathcal{A}$ are free
(= unital algebras which they generate are free)
if and only if 'all mixed cumulants vanish'

„φ-oriented definition of freeness"

i.e.

$$\kappa(x_1, \ldots, x_n) = 0 \quad \text{whenever} \quad x_k \in A_{i_k}$$

„cumulant-oriented definition of freeness"

and $i_1, \ldots, i_n$ are not all equal.

FANTASTIC!

NO assumption that neighbors different

NO assumption on centeredness.

enough to take generators.

[the two definitions of freeness are equivalent]

this characterization of freeness is more convenient than "vanishing of state on alternating product of centered elements" proof

PART $\Leftarrow$

"if mixed cumulants involving generators vanish, more complex cumulants vanish as well"

1 show that

$$\kappa(x_1, \dots, x_n) = 0 \quad \text{whenever} \quad x_k \in A_{i_k}$$

and $i_1, \dots, i_n$ are <u>not</u> all equal.

$\Downarrow$

done.

$\mathcal{A}_{i_k} :=$

$$\kappa(x_1, \dots, x_n) = 0 \quad \text{whenever} \quad x_k \in \mathrm{Alg}(1, A_{i_k})$$

and $i_1, \dots, i_n$ are <u>not</u> all equal.

HINT:

use $\rightarrow$ ⓐ free cumulants involving $\mathbb{C}$ are ZERO.
ⓑ Leonov - Shiryaev - Krawczyk - Speicher

2 alternating product of centered random variables $x_k \in \mathcal{A}_{i_k}$

$$\varphi(x_1 x_2 \cdots x_n) = \sum_{\pi \in NC(n)} \kappa_\pi(x_1, \dots, x_n) = 0$$

BECAUSE...

"each NC partition contains a block which is an interval"

- centered $\Rightarrow$ singletons forbidden
- mixed cumulants vanish $\Rightarrow$ π respects i
- alternating $\Rightarrow$ no connections to neighbors

proof [NS] Thm 11.16

PART $\Rightarrow$

$$\kappa(a_1, a_2, \dots, a_n) \overset{?}{=} 0 \qquad \boxed{\text{if } n \geq 2}$$

- if $a_1, \dots, a_n$ centered and alternating

$$\kappa(a_1, \dots, a_n) = \sum_{\pi \in NC} \mu(\pi, 1_n) \underbrace{\varphi_\pi(a_1, \dots, a_n)}_{=0}$$

Hint: the minimal block which is an interval

- if $a_1, \dots, a_n$ ~~centered and~~ alternating

Hint: $\kappa(\dots, 1, \dots) = 0$ [$n \geq 2$!]

- if $a_1, \dots, a_n$ ~~alternating~~

$i_1, i_2, \dots, i_n$ not all equal

Hint: $\tau :=$ interval partition

$a < b$ are connected by τ if

$i_a = i_{a+1} = \dots = i_b$

$\boxed{\text{INDUCTION OVER } n.}$

Leonov & Shiryaev

$$0 = \kappa\left(\prod_{i \in b} a_i : b \in \tau\right) = \kappa(a_1, \dots, a_n) + \underbrace{(\text{remaining terms})}_{=0 \text{ by inductive hypothesis}}$$

alternating product

! inductive hypothesis does NOT apply to the MINIMAL partition.

this partition appears?

$\to$ all $a_1, \dots$ from the same $A_{i_1} \dots$

products of free random variables
&
Kreweras complement

→ [MS] Section 2.3.

if $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ are free...

$$\varphi(a_1 b_1 a_2 b_2 \cdots a_r b_r) = \sum_{\pi \in NC(2r)} \kappa_\pi (\overbrace{a_1, b_1, \dots, a_r, b_r}^{\pi_A})_{\pi_B} =$$

$$= \sum_{\pi_A \in NC(r)} \kappa_{\pi_A}(a_1, \dots, a_r) \boxed{\sum_{\pi_B \in NC(r)} \kappa_{\pi_B}(b_1, \dots, b_r)}$$

⚠ $\pi_A \cup \pi_B$ is non-crossing

$$= \varphi_{K(\pi_A)}(b_1, \dots, b_r)$$

π_A - partition on $1, \quad 2, \quad 3$

π_B - partition on $\bar{1}, \quad \bar{2}, \quad \bar{3} \quad \dots$

Wow: there exists a MAXIMAL non-crossing partition π_B such that $\pi_A \cup \pi_B$ is non-crossing.

We call it Kreweras complement of π_A $\quad K(\pi_A)$

→ Lecture 4 part B.

→ [MN], Sect. 2.4.

Functional relation

fix $a \in \mathcal{A}$

$$M(z) = 1 + \sum_{n \geq 1} \varphi(a^n)\, z^n$$

formal power series

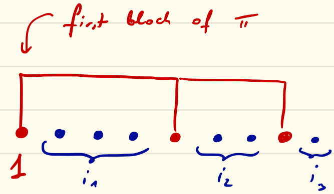
$$C(z) = 1 + \sum_{n \geq 1} \kappa_n(\underbrace{a, \ldots, a}_{n})\, z^n$$

Thm.

$$M(z) = C(zM(z))$$

Proof coefficient at z^n:

$$[z^n]\, M(z) = \varphi(a^n) = \sum_{\pi \in NC(n)} \kappa_\pi =$$



$$= \sum_{s \geq 1} \sum_{\substack{i_1, \ldots, i_s \geq 0 \\ s + i_1 + \cdots + i_s = n}}$$

number of elements in the first block

$$\kappa_s \cdot \underbrace{\sum_{\pi_1 \in NC(i_1)} \kappa_{\pi_1}}_{\varphi(a^{i_1})} \cdot \ldots \ldots \cdot$$

$$= [z^n] \sum_{s \geq 1} \sum_{\substack{i_1, \ldots, i_s \geq 0 \\ \cancel{s} + \cancel{i_1 + \cdots + i_s} = n}}$$

number of elements in the first block

$$\underbrace{K_s \left(z\, z^{i_1} \underbrace{\sum_{\pi_1 \in NC(i_1)} K_{\pi_1}}_{z \cdot z^{i_1} \varphi(a^{i_1})} \right) \left(z\, z^{i_2} \cdots\cdots \right) \cdots}_{s \text{ factors}}$$

$$= [z^n] \quad C\big(z\, M(z)\big)$$