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Lectures on random matrices and free probability theory

Lecture 7.
Free cumulants 2

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Plan

- free central limit theorem.
- functional relation between R-transform and moment-generating function.

→ [MN], Sect. 2.4.

Functional relation

fix $a \in \mathcal{A}$

$$M(z) = 1 + \sum_{n \geq 1} \varphi(a^n) z^n$$

formal power series

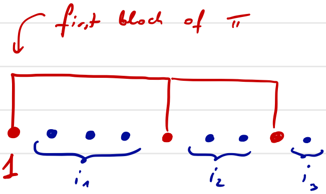
$$C(z) = 1 + \sum_{n \geq 1} K_n(\underbrace{a, \dots, a}_{n}) z^n$$

Thm.

$$M(z) = C(zM(z))$$

Proof coefficient at z^n:

$$[z^n]\, M(z) = \varphi(a^n) = \sum_{\pi \in NC(n)} K_\pi =$$



$$= \sum_{s \geq 1} \sum_{\substack{i_1, \dots, i_s \geq 0 \\ s + i_1 + \dots + i_s = n}}$$

number of elements in the first block

$$K_s \cdot \overbrace{\underbrace{\left(\sum_{\pi_1 \in NC(i_1)} K_{\pi_1} \right)}_{\varphi(a^{i_1})} \cdots \left(\cdot \right)}^{s \text{ factors}}$$

$$= [z^n] \sum_{s \geq 1} \sum_{\substack{i_1, \dots, i_s \geq 0 \\ \cancel{s} + \cancel{i_1 + \dots + i_s} = n}}$$

number of elements in the first block

s factors of this form

$$K_s \left(z \, \underbrace{z^{i_1} \sum_{\pi_1 \in NC(i_1)} K_{\pi_1}}_{z \cdot z^{i_1} \varphi(a^{i_1})} \right) \cdots \cdots$$

$$= [z^n] \quad C\big(z \quad M(z) \big)$$

Example.

Suppose $K_n(x, \dots, x) = \begin{cases} 0 & \text{otherwise.} \\ 1 & \text{if } n = 2 \end{cases}$

$$C(z) = 1 + z^2$$

$$M(z) = C(zM(z)$$

$$M(z) = 1 + (zM(z))^2$$

$$z^2 M(z)^2 - M(z) + 1 = 0$$

$$M(z) = \frac{1 \overset{\text{Minus!}}{\pm} \sqrt{1 - 4z^2}}{2z^2}$$

Hint: Catalan numbers.

Hint:

$$(1+x)^\alpha = \sum_{i \geq 0} \binom{\alpha}{i} x^i$$

$$\sqrt{1-4z^2} =$$

$$= \sum_{i \geq 0} \binom{\frac{1}{2}}{i} (-4)^i z^{2i} =$$

$$= 1 + \binom{\frac{1}{2}}{1} (-4) z^2 + \dots$$

fix $a \in \mathcal{A}$

$$M(z) = 1 + \sum_{n \geq 1} \varphi(a^n)\, z^n$$

formal power series in z

$$C(z) = 1 + \sum_{n \geq 1} \kappa(a, \dots, a)\, z^n$$

$$M(z) = C(zM(z))$$

Cauchy transform

$$G(z) = \varphi\left(\frac{1}{z-a}\right) = \sum_{n \geq 0} \frac{\varphi(a^n)}{z^{n+1}} = \frac{1}{z} \, M\left(\frac{1}{z}\right)$$

formal power series in $\frac{1}{z}$

R-transform

$$R(z) = \frac{C(z) - 1}{z} = \sum_{n \geq 0} \kappa_{n+1}(a, \dots a) \cdot z^n$$

$$K(z) = R(z) + \frac{1}{z} = \frac{C(z)}{z}$$

formal Laurent series !

except for $\frac{1}{z}$ it is a formal power series in z

Then

$$K(G(z)) = \frac{1}{G(z)} \, C(G(z)) = \frac{1}{G(z)} \, C\left(\frac{1}{z} M\left(\frac{1}{z}\right)\right) =$$

$$= \frac{1}{G(z)} \quad M\left(\frac{1}{z}\right) = z$$

$\boxed{\text{formal Laurent series}}$

strange feeling inside the belly:

we work with strange objects strange power series.

To which extent our usual calculations require new justification?

Quick check if this all makes sense

$$G(z) = \frac{1}{z} + \frac{?}{z^2} + \frac{?}{z^3} + \dots$$

$$\frac{1}{G(z)} = z + ? + \frac{?}{z} + \frac{?}{z^2} + \dots =$$

$$z \cdot \left(\text{formal series in } \frac{1}{z}\right)$$

$$K(z) = \frac{1}{z} + ? + ?\,z + ?\,z^2 + \dots$$

$$\frac{1}{K(z)} = z + ?\,z^2 + ?\,z^3 = z \cdot \left(\text{power series in } z\right)$$

$$R_a(G(z)) = ?\,G(z) + ?\,G(z)^2 + ?\,G(z)^3 + \dots$$

well defined ✓.

$$\omega_a(x) = x - R(G(x)) =$$

$$= x + ? + ?\,\frac{1}{x} + ?\,\frac{1}{x^2} + \dots$$

$$K(G(z)) = \frac{1}{G(z)} + ? + ?\, G(z) + ?\, G(z)^2 + \dots$$

$$= z + \left(\text{formal series in } \frac{1}{z} \right).$$

$$G(K(z)) = \frac{1}{K(z)} + \frac{?}{K(z)^2} + \frac{?}{K(z)^3} + \dots =$$

$$= z + ?\, z^2 + ?\, z^3 + \dots$$

Compositional inverses (left and right) of G <u>exist</u> as formal power series (upper triangular system of algebraic equations).

Left and right inverse must be equal.

$$\boxed{G(K(z)) = z}$$

R-transform can be equivalently defined by

$$G\left(R(z) + \frac{1}{z}\right) = z.$$

freeness $\Rightarrow$ $R_{a+b}(z) = R_a(z) + R_b(z)$

oops! we never proved that $K_n(a+b) =$
$= K_n(a) + K_n(b)$

MAGIC AHEAD

$$z = G_{a+b}\Big(\overbrace{R_{a+b}(z) + \tfrac{1}{z}}^{w:=}\Big) =$$

$$= G_{a+b}\Big(R_a(z) + R_b(z) + \tfrac{1}{z}\Big) =$$

$$= G_a\Big(R_a(z) + \tfrac{1}{z}\Big) =$$

$$= G_a\big(w - R_b(z)\big) =$$

$$= G_a\Big(\underbrace{w - R_b\big(G_{a+b}(w)\big)}_{\omega_a(w):=}\Big)$$

$w = w(z)$
$W : z \longmapsto w(z)$

"SUBORDINATION FUNCTION"

we proved $G_{a+b} \circ W = G_a \circ \omega_a \circ W$
but W has a compositional inverse $(= G_{a+b})$.

$$\boxed{G_{a+b}(z) = G_a(\omega_a(z))}$$

$\omega_a(x) =$
$= x - R_b(G_{a+b}(x))$

"G_{a+b} is subordinate to G_a"

CONCEPTUAL EXPLANATION
→ AHEAD

3.1 The Cauchy transform

Definition 1. Let $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denote the complex upper half-plane, and $\mathbb{C}^- = \{z \mid \mathrm{Im}(z) < 0\}$ denote the lower half-plane. Let ν be a probability measure on $\mathbb{R}$ and for $z \notin \mathbb{R}$ let

$$G(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\nu(t);$$

G is the *Cauchy transform* of the measure ν.

Let us briefly check that the integral converges to an analytic function on $\mathbb{C}^+$.

Lemma 2. *G is an analytic function on $\mathbb{C}^+$ with range contained in $\mathbb{C}^-$.*

Lemma 3. *Let G be the Cauchy transform of a probability measure ν. Then:*

$$\lim_{y\to\infty} iy\,G(iy) = 1 \quad \textit{and} \quad \sup_{y>0,x\in\mathbb{R}} y\,|G(x+iy)| = 1.$$

Theorem 6. *Suppose ν is a probability measure on $\mathbb{R}$ and G is its Cauchy transform. For $a < b$ we have*

$$-\lim_{y\to 0^+} \frac{1}{\pi}\int_a^b \mathrm{Im}(G(x+iy))\,dx = \nu((a,b)) + \frac{1}{2}\nu(\{a,b\}).$$

If ν_1 and ν_2 are probability measures with $G_{\nu_1} = G_{\nu_2}$, then $\nu_1 = \nu_2$.

$$\mathrm{Im}(G(x+iy)) = \int_{\mathbb{R}} \mathrm{Im}\left(\frac{1}{x-t+iy}\right) d\nu(t) = \int_{\mathbb{R}} \frac{-y}{(x-t)^2+y^2} d\nu(t).$$

Thus

$$\begin{aligned}\int_a^b \mathrm{Im}(G(x+iy))\,dx &= \int_{\mathbb{R}}\int_a^b \frac{-y}{(x-t)^2+y^2}\,dx\,d\nu(t)\\ &= -\int_{\mathbb{R}}\int_{(a-t)/y}^{(b-t)/y} \frac{1}{1+\tilde{x}^2}\,d\tilde{x}\,d\nu(t)\\ &= -\int_{\mathbb{R}}\left[\tan^{-1}\left(\frac{b-t}{y}\right)-\tan^{-1}\left(\frac{a-t}{y}\right)\right] d\nu(t),\end{aligned}$$

where we have let $\tilde{x} = (x-t)/y$.

In the next two exercises we need to choose a branch of $\sqrt{z^2-4}$ for z in the upper half-plane, $\mathbb{C}^+$. We write $z^2-4=(z-2)(z+2)$ and define each of $\sqrt{z-2}$ and $\sqrt{z+2}$ on $\mathbb{C}^+$. For $z \in \mathbb{C}^+$, let θ_1 be the angle between the x-axis and the line joining z to 2; and θ_2 the angle between the x-axis and the line joining z to -2. See Fig. 3.1. Then $z-2=|z-2|e^{i\theta_1}$ and $z+2=|z+2|e^{i\theta_2}$ and so we define $\sqrt{z^2-4}$ to be $|z^2-4|^{1/2}e^{i(\theta_1+\theta_2)/2}$.

Exercise 2. For $z=u+iv \in \mathbb{C}^+$ let $\sqrt{z}=\sqrt{|z|}e^{i\theta/2}$ where $0<\theta<\pi$ is the argument of z. Show that

$$\mathrm{Re}(\sqrt{z}) = \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} \quad\text{and}\quad \mathrm{Im}(\sqrt{z}) = \sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}.$$

This shows that ν_1 and ν_2 agree on all open intervals and thus are equal. ☐

Example 7 (The semi-circle distribution).

As an example of Stieltjes inversion let us take a familiar example and calculate its Cauchy transform using a generating function and then using only the Cauchy transform find the density by using Stieltjes inversion. The density of the semi-circle law $\nu := \mu_s$ is given by

$$d\nu(t) = \frac{\sqrt{4-t^2}}{2\pi}\,dt \qquad \text{on } [-2,2];$$

and the moments are given by

$$m_n = \int_{-2}^{2} t^n\,d\nu(t) = \begin{cases} 0, & n \text{ odd} \\ C_{n/2}, & n \text{ even} \end{cases},$$

where the C_n's are the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Now let $M(z)$ be the moment-generating function

$$M(z) = 1 + C_1 z^2 + C_2 z^4 + \cdots$$

then

$$M(z)^2 = \sum_{m,n\geq 0} C_m C_n z^{2(m+n)} = \sum_{k\geq 0} \big(\sum_{m+n=k} C_m C_n \big) z^{2k}.$$

Now we saw in equation (2.5) that $\sum_{m+n=k} C_m C_n = C_{k+1}$, so

$$M(z)^2 = \sum_{k\geq 0} C_{k+1} z^{2k} = \frac{1}{z^2} \sum_{k\geq 0} C_{k+1} z^{2(k+1)}$$

and therefore

$$z^2 M(z)^2 = M(z) - 1 \qquad \text{or} \qquad M(z) = 1 + z^2 M(z)^2.$$

By replacing $M(z)$ by $z^{-1}G(1/z)$ we get that G satisfies the quadratic equation $zG(z) = 1 + G(z)^2$. Solving this we find that

$$G(z) = \frac{z \pm \sqrt{z^2-4}}{2}.$$

We use the branch of $\sqrt{z^2-4}$ defined before Exercise 2, however we must choose the sign in front of the square root. By Lemma 3, we require that $\lim_{y\to\infty} iyG(iy) = 1$. Note that for $y > 0$ we have that, using our definition, $\sqrt{(iy)^2-4} = i\sqrt{y^2+4}$. Thus

$$\lim_{y\to\infty} (iy) \frac{iy - \sqrt{(iy)^2-4}}{2} = 1$$

and

$$\lim_{y\to\infty} (iy) \frac{iy + \sqrt{(iy)^2-4}}{2} = \infty.$$

Hence

$$G(z) = \frac{z - \sqrt{z^2-4}}{2}.$$

Of course, this agrees with the result in Exercise 5.

Returning to the equation $zG(z) = 1 + G(z)^2$ we see that $z = G(z) + 1/G(z)$, so $K(z) = z + 1/z$ and thus $R(z) = z$ i.e. all cumulants of the semi-circle law are 0 except κ_2, which equals 1, something we observed already in Exercise 2.9.

Now let us apply Stieltjes inversion to $G(z)$. We have

$$\operatorname{Im}\left(\sqrt{(x+iy)^2-4}\right) = \left|(x+iy)^2-4\right|^{1/2} \sin((\theta_1+\theta_2)/2)$$

$$\lim_{y\to 0^+} \operatorname{Im}\left(\sqrt{(x+iy)^2-4}\right) = \begin{cases} |x^2-4|^{1/2}\cdot 0 = 0, & |x|>2 \\ |x^2-4|^{1/2}\cdot 1 = \sqrt{4-x^2}, & |x|\le 2 \end{cases}$$

Convergence is uniform over x in a compact subset of $\mathbb{R}$.

and thus

$$\begin{aligned} \lim_{y\to 0^+} \operatorname{Im}(G(x+iy)) &= \lim_{y\to 0^+} \operatorname{Im}\left(\frac{x+iy-\sqrt{(x+iy)^2-4}}{2}\right) \\ &= \begin{cases} 0, & |x|>2 \\ \dfrac{-\sqrt{4-x^2}}{2}, & |x|\le 2 \end{cases}. \end{aligned}$$

Therefore

$$-\lim_{y\to 0^+} \frac{1}{\pi}\operatorname{Im}(G(x+iy)) = \begin{cases} 0, & |x|>2 \\ \dfrac{\sqrt{4-x^2}}{2\pi}, & |x|\le 2 \end{cases}.$$

Hence we recover our original density.

If G is the Cauchy transform of a probability measure we cannot in general expect $G(z)$ to converge as z converges to $a \in \mathbb{R}$. It might be that $|G(z)| \to \infty$ as $z \to a$ or that G behaves as if it has an essential singularity at a. However $(z-a)G(z)$ always has a limit as $z \to a$ if we take a *non-tangential* limit. Let us recall the definition. Suppose $f : \mathbb{C}^+ \to \mathbb{C}$ and $a \in \mathbb{R}$, we say $\lim_{\sphericalangle z\to a} f(z) = b$ if for every $\theta > 0$, $\lim_{z\to a} f(z) = b$ when we restrict z to be in the cone $\{x+iy \mid y>0 \text{ and } |x-a| < \theta y\} \subset \mathbb{C}^+$.

Proposition 8. *Suppose ν is a probability measure on $\mathbb{R}$ with Cauchy transform G. For all $a \in \mathbb{R}$ we have $\lim\limits_{\sphericalangle z\to a} (z-a)G(z) = \nu(\{a\})$.*

Proof: Let $\theta > 0$ be given. If $z = x+iy$ and $|x-a| < \theta y$, then for $t \in \mathbb{R}$ we have

$$\left|\frac{z-a}{z-t}\right|^2 = \frac{(x-a)^2+y^2}{(x-t)^2+y^2} = \frac{1+(\frac{x-a}{y})^2}{1+(\frac{x-t}{y})^2} \le 1 + \left(\frac{x-a}{y}\right)^2 < 1+\theta^2.$$

$$G(z) = \frac{z - \sqrt{z^2-4}}{2}.$$

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$$\mathrm{Im}\left(\sqrt{(x+iy)^2-4}\right) = \left|(x+iy)^2-4\right|^{1/2}\sin((\theta_1+\theta_2)/2)$$

$$\lim_{y\to 0^+}\mathrm{Im}\left(\sqrt{(x+iy)^2-4}\right) = \begin{cases} |x^2-4|^{1/2}\cdot 0 = 0, & |x| > 2 \\ |x^2-4|^{1/2}\cdot 1 = \sqrt{4-x^2}, & |x| \le 2 \end{cases}$$

and thus

$$\begin{aligned}\lim_{y\to 0^+}\mathrm{Im}(G(x+iy)) &= \lim_{y\to 0^+}\mathrm{Im}\left(\frac{x+iy-\sqrt{(x+iy)^2-4}}{2}\right) \\ &= \begin{cases} 0, & |x| > 2 \\ \dfrac{-\sqrt{4-x^2}}{2}, & |x| \le 2 \end{cases}.\end{aligned}$$

Therefore

$$-\lim_{y\to 0^+}\frac{1}{\pi}\mathrm{Im}(G(x+iy)) = \begin{cases} 0, & |x| > 2 \\ \dfrac{\sqrt{4-x^2}}{2\pi}, & |x| \le 2 \end{cases}.$$

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