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Lectures on random matrices and free probability theory

Lecture 8. Free cumulants 3

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Example. What is the limit eigenvalues distribution of GUE random matrices?

$Y^{(n)}$ - $n \times n$ GUE random matrix $\longrightarrow$ Lecture 3.

$$\underbrace{\lim_{n\to\infty} \mu_{Y^{(n)}}}_{\text{limit in the sense of}} = \mu$$

limit in the sense of
a) mean value, or
b) almost sure convergence

μ: mysterious measure on $\mathbb{R}$

$$\int_{-\infty}^{\infty} z^k \, d\mu(z) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{1}{\ell+1}\binom{2\ell}{\ell} & \text{if } k = 2\ell \text{ is even} \end{cases}$$

Catalan numbers.

$\mu = ?$

Alternative viewpoint:

we have some noncommutative probability space $(\mathcal{A}, \varphi)$

and an abstract element $x \in \mathcal{A}$

with distribution given by the limit of GUE

random matrices

$$\varphi(x^k) := \lim_{n\to\infty} \mathbb{E}\, \frac{1}{n} \operatorname{Tr}\left[Y^{(n)}\right]^k =$$

$$= \begin{cases} 0 & \dots \\ \frac{1}{\ell+1}\binom{2\ell}{\ell} & \dots \end{cases}$$

Equivalently,

$$K_n(x, \dots, x) = \begin{cases} 0 & \text{otherwise.} \\ 1 & \text{if } n = 2 \end{cases}$$

What is the spectral measure of x?

we look for a measure μ_x on $\mathbb{R}$ s.t.

$$\varphi(x^k) = \int z^k \, d\mu_x(z)$$

For some subtle issues moments are NOT enough.

Hint: use functional analysis, C^*-algebras!

[EVEN BETTER:

$$\varphi(F(x)) = \int F(z) \, d\mu_x(z)$$

for a suitable class of functions F.

PROBLEMS

$$\left.\begin{array}{l}\text{Cumulant generating function...}\\ \text{Moment generating function...}\\ \text{Cauchy transform ....}\end{array}\right\} \text{ of } \left\{\begin{array}{l}\text{our favorite measure } \mu \\ \\ \text{our favorite element } x.\end{array}\right.$$

Cauchy transform

$$\underset{\substack{\uparrow\\ \text{measure}}}{G_\mu}(z) = \int_{\mathbb{R}} dt \, \frac{1}{z-t} \, d\mu$$

$$\underset{\substack{\uparrow\\ \text{abstract element}}}{G_x}(z) = \int_{\mathbb{R}} dt \, \frac{1}{z-t} \underbrace{d\mu_x}_{\text{spectral measure of } x.} = \underbrace{\varphi\left(\frac{1}{z-x}\right)}_{\text{possibly a problematic object.}} =$$

works best if x is an element of a C^*-algebra and $x = x^*$.

$$= \frac{1}{z} \varphi\left(\frac{1}{1-\frac{x}{z}}\right) =$$

works best in a C^* algebra, $|z| > \|x\|$

$$= \sum_{k \geq 0} \frac{\varphi(x^k)}{z^{k+1}} = \frac{1}{z} M\left(\frac{1}{z}\right)$$

$$M(z) = 1 + \sum_{n \geq 1} \varphi(a^n) z^n$$

moment-generating function.

Until now we regarded M as a <u>formal</u> power series.
But if we are lucky: $|z| < \|a\|$ the series is convergent.

FLASHBACK

Example.

Suppose $\quad K_n(x, \dots, x) = \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } n = 2 \end{cases}$

$$C(z) = 1 + z^2$$

$$M(z) = C(z M(z))$$

$$M(z) = 1 + (z M(z))^2$$

$$z^2 M(z)^2 - M(z) + 1 = 0$$

Minus!

$$M(z) = \frac{1 \pm \sqrt{1 - 4z^2}}{2z^2}$$

Hint: Catalan numbers.

Hint:

$$(1+x)^\alpha = \sum_{i \geq 0} \binom{\alpha}{i} x^i$$

$$\sqrt{1-4z^2} =$$

$$= \sum_{i \geq 0} \binom{\frac{1}{2}}{i} (-4)^i z^{2i} =$$

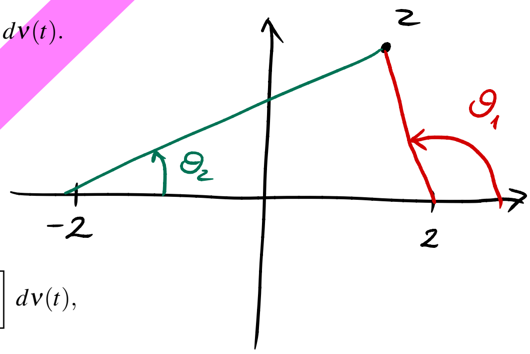
$$= 1 + \binom{\frac{1}{2}}{1} (-4) z^2 + \dots$$

$$G(z) = \frac{1}{z} M\left(\frac{1}{z}\right) = \frac{1}{z} \frac{1 - \sqrt{1 - \frac{4}{z^2}}}{2 \cdot \frac{1}{z^2}} =$$

$$= \frac{1}{2} \left(z - \sqrt{z^2 - 4} \right) =$$

$$= \frac{1}{2} \frac{z^2 - (z^2 - 4)}{z + \sqrt{z^2 - 4}} = \frac{2}{z + \sqrt{z^2 - 4}}$$

$$\mathrm{Im}(G(x+iy)) = \int_{\mathbb{R}} \mathrm{Im}\Big(\frac{1}{x-t+iy}\Big)\,d\nu(t) = \int_{\mathbb{R}} \frac{-y}{(x-t)^2+y^2}\,d\nu(t).$$



Thus

$$\begin{aligned}\int_a^b \mathrm{Im}(G(x+iy))\,dx &= \int_{\mathbb{R}}\int_a^b \frac{-y}{(x-t)^2+y^2}\,dx\,d\nu(t)\\ &= -\int_{\mathbb{R}}\int_{(a-t)/y}^{(b-t)/y} \frac{1}{1+\tilde{x}^2}\,d\tilde{x}\,d\nu(t)\\ &= -\int_{\mathbb{R}}\Big[\tan^{-1}\Big(\frac{b-t}{y}\Big) - \tan^{-1}\Big(\frac{a-t}{y}\Big)\Big]\,d\nu(t),\end{aligned}$$

where we have let $\tilde{x} = (x-t)/y$.

In the next two exercises we need to choose a branch of $\sqrt{z^2-4}$ for z in the upper half-plane, $\mathbb{C}^+$. We write $z^2-4 = (z-2)(z+2)$ and define each of $\sqrt{z-2}$ and $\sqrt{z+2}$ on $\mathbb{C}^+$. For $z \in \mathbb{C}^+$, let θ_1 be the angle between the x-axis and the line joining z to 2; and θ_2 the angle between the x-axis and the line joining z to -2. See Fig. 3.1. Then $z-2 = |z-2|e^{i\theta_1}$ and $z+2 = |z+2|e^{i\theta_2}$ and so we define $\sqrt{z^2-4}$ to be $|z^2-4|^{1/2}e^{i(\theta_1+\theta_2)/2}$.

Exercise 2. For $z = u+iv \in \mathbb{C}^+$ let $\sqrt{z} = \sqrt{|z|}e^{i\theta/2}$ where $0 < \theta < \pi$ is the argument of z. Show that

$$\mathrm{Re}(\sqrt{z}) = \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} \quad \text{and} \quad \mathrm{Im}(\sqrt{z}) = \sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}.$$

$\varepsilon_n \to 0^+$ $\varepsilon_n \to 0^+$

This shows that ν_1 and ν_2 agree on all open intervals and thus are equal. □

Example 7 (The semi-circle distribution).

As an example of Stieltjes inversion let us take a familiar example and calculate its Cauchy transform using a generating function and then using only the Cauchy transform find the density by using Stieltjes inversion. The density of the semi-circle law $\nu := \mu_s$ is given by

3.1 The Cauchy transform

Definition 1. Let $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denote the complex upper half-plane, and $\mathbb{C}^- = \{z \mid \mathrm{Im}(z) < 0\}$ denote the lower half-plane. Let ν be a probability measure on $\mathbb{R}$ and for $z \notin \mathbb{R}$ let

$$G(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\nu(t);$$

G is the *Cauchy transform* of the measure ν.

Let us briefly check that the integral converges to an analytic function on $\mathbb{C}^+$.

Lemma 2. *G is an analytic function on $\mathbb{C}^+$ with range contained in $\mathbb{C}^-$.*

Lemma 3. *Let G be the Cauchy transform of a probability measure ν. Then:*

$$\lim_{y\to\infty} iy\,G(iy) = 1 \quad \textit{and} \quad \sup_{y>0, x\in\mathbb{R}} y\,|G(x+iy)| = 1.$$

Theorem 6. *Suppose ν is a probability measure on $\mathbb{R}$ and G is its Cauchy transform. For $a < b$ we have*

$$-\lim_{y\to 0^+} \frac{1}{\pi} \int_a^b \mathrm{Im}(G(x+iy))\, dx = \nu((a,b)) + \frac{1}{2}\nu(\{a,b\}).$$

If ν_1 and ν_2 are probability measures with $G_{\nu_1} = G_{\nu_2}$, then $\nu_1 = \nu_2$.

$-\frac{1}{\pi}$!

$$G(z) = \frac{z - \sqrt{z^2-4}}{2}.$$

Of course, this agrees with the result in Exercise 5.

Returning to the equation $zG(z) = 1 + G(z)^2$ we see that $z = G(z) + 1/G(z)$, so $K(z) = z + 1/z$ and thus $R(z) = z$ i.e. all cumulants of the semi-circle law are 0 except κ_2, which equals 1, something we observed already in Exercise 2.9.

Now let us apply Stieltjes inversion to $G(z)$. We have

$$\operatorname{Im}\left(\sqrt{(x+iy)^2-4}\right) = \left|(x+iy)^2-4\right|^{1/2} \sin((\theta_1+\theta_2)/2)$$

$$\lim_{y\to 0^+} \operatorname{Im}\left(\sqrt{(x+iy)^2-4}\right) = \begin{cases} |x^2-4|^{1/2}\cdot 0 = 0, & |x|>2 \\ |x^2-4|^{1/2}\cdot 1 = \sqrt{4-x^2}, & |x|\le 2 \end{cases}$$

Convergence is uniform over x in a compact subset of $\mathbb{R}$.

and thus

$$\begin{aligned} \lim_{y\to 0^+} \operatorname{Im}(G(x+iy)) &= \lim_{y\to 0^+} \operatorname{Im}\left(\frac{x+iy-\sqrt{(x+iy)^2-4}}{2}\right) \\ &= \begin{cases} 0, & |x|>2 \\ \dfrac{-\sqrt{4-x^2}}{2}, & |x|\le 2 \end{cases}. \end{aligned}$$

Therefore

$$-\lim_{y\to 0^+} \frac{1}{\pi}\operatorname{Im}(G(x+iy)) = \begin{cases} 0, & |x|>2 \\ \dfrac{\sqrt{4-x^2}}{2\pi}, & |x|\le 2 \end{cases}.$$

Hence we recover our original density.

If G is the Cauchy transform of a probability measure we cannot in general expect $G(z)$ to converge as z converges to $a\in\mathbb{R}$. It might be that $|G(z)|\to\infty$ as $z\to a$ or that G behaves as if it has an essential singularity at a. However $(z-a)G(z)$ always has a limit as $z\to a$ if we take a *non-tangential* limit. Let us recall the definition. Suppose $f:\mathbb{C}^+\to\mathbb{C}$ and $a\in\mathbb{R}$, we say $\lim_{\sphericalangle z\to a} f(z) = b$ if for every $\theta>0$, $\lim_{z\to a} f(z) = b$ when we restrict z to be in the cone $\{x+iy \mid y>0 \text{ and } |x-a|<\theta y\}\subset\mathbb{C}^+$.

Proposition 8. *Suppose ν is a probability measure on $\mathbb{R}$ with Cauchy transform G. For all $a\in\mathbb{R}$ we have $\lim\limits_{\sphericalangle z\to a}(z-a)G(z) = \nu(\{a\})$.*

Proof: Let $\theta>0$ be given. If $z=x+iy$ and $|x-a|<\theta y$, then for $t\in\mathbb{R}$ we have

$$\left|\frac{z-a}{z-t}\right|^2 = \frac{(x-a)^2+y^2}{(x-t)^2+y^2} = \frac{1+(\frac{x-a}{y})^2}{1+(\frac{x-t}{y})^2} \le 1+\left(\frac{x-a}{y}\right)^2 < 1+\theta^2.$$