museum of visual ART
Asymptotic Representation Theory

guided tour with Piotr Śniady

transparencies, references, homework available on
http://psniady.impan.pl/fpsac

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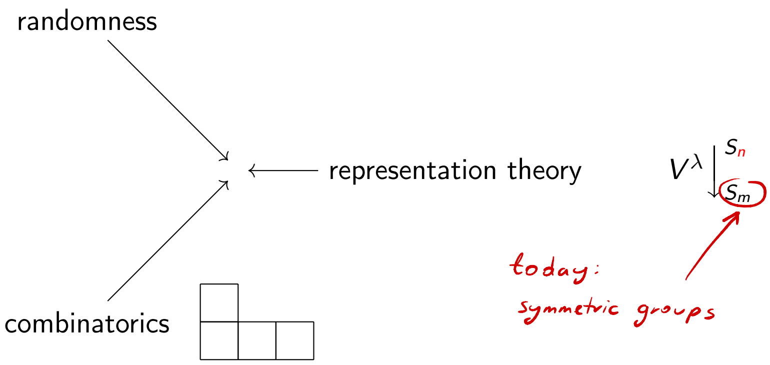
one-slide summary



visual viewpoint on algebraic combinatorics creates nice questions

one-slide summary

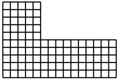
randomness

asymptotic $n \to \infty$

representation theory

$V^\lambda \big\downarrow^{S_n}_{S_m}$

combinatorics



visual viewpoint on algebraic combinatorics creates nice questions

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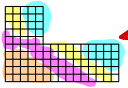


← visual

visual viewpoint on algebraic combinatorics creates nice questions

plan for today

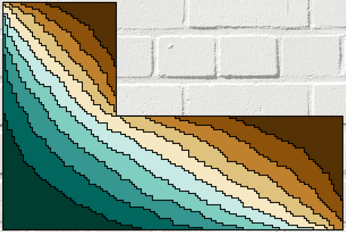


exhibit A

what can you say
about random Young diagrams?

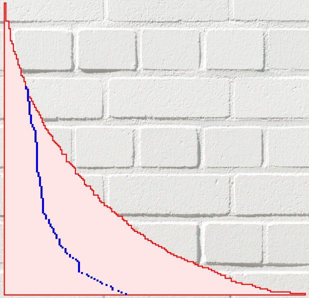
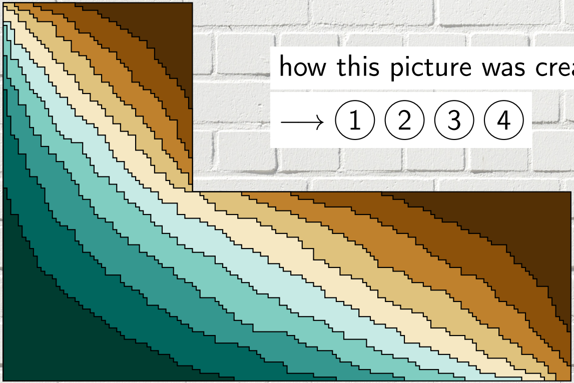


exhibit B

what can you say
about RSK
applied to random input?

exhibit A



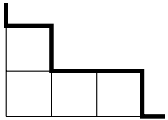
how this picture was created?

→ (1) (2) (3) (4)

what can we say

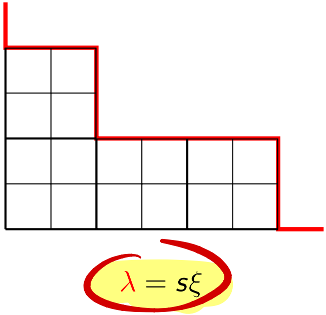
about random Young diagrams and random Young tableaux?

① scaling



ξ

start with a Young diagram ξ...



...and scale it by a factor $s \in \{1, 2, \dots\}$

---

if $s > 0$ is a real number, $s\xi$ is a *generalized Young diagram*

$\Lambda = \frac{1}{\sqrt{|\lambda|}}\lambda$ is called *asymptotic shape of λ*

② select randomly a standard tableau with shape λ

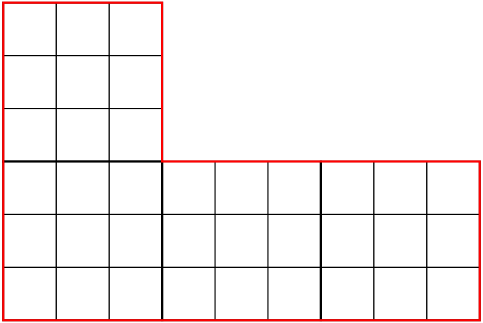


diagram λ

② select randomly a standard tableau with shape λ

20	23	31						
9	15	30						
8	14	28						
6	7	12	17	22	26	29	33	36
3	4	11	13	18	21	25	32	35
1	2	5	10	16	19	24	27	34

tableau T

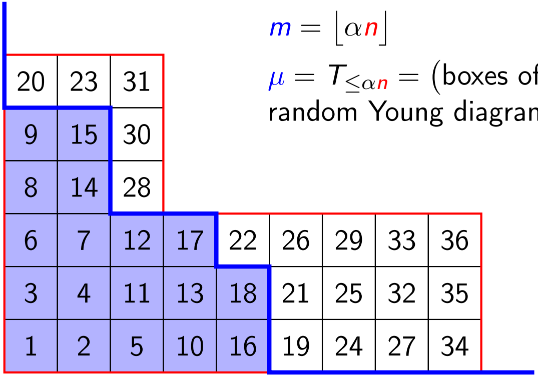
③ draw the level curves

fix a real number $0 \leq \alpha \leq 1$

$n = |\lambda|$

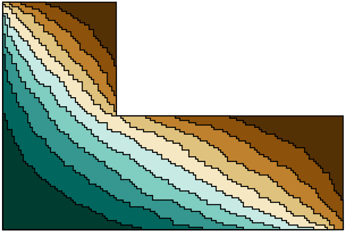
$m = \lfloor \alpha n \rfloor$

$\mu = T_{\leq \alpha n} = \big(\text{boxes of } T \text{ which are } \leq \alpha n\big)$ is a random Young diagram with m boxes



level curve $\alpha = \frac{1}{2}$

④ this is a layer tinting of a random standard tableau of fixed shape λ



hint: $\lambda = s\xi$

for best viewing experience scale the picture by $\frac{1}{s}$

then $s \to \infty$

reducible representation ⟶ random Young diagram μ

let W be a reducible representation of the symmetric group S_m;
its decomposition into irreducibles is given by

$$W = \bigoplus_{\mu \vdash m} \mathfrak{m}_\mu V^\mu$$

$\mathfrak{m}_\mu \in \{0, 1, \dots\}$ is the multiplicity of V^μ in W

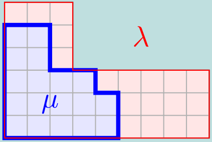
we declare the probability of sampling the Young diagram μ to be equal to

$$\mathbb{P}_W(\mu) = \frac{\mathfrak{m}_\mu \dim V^\mu}{\dim W}$$

"random irreducible component of W"

our concrete example

$$W = V^\lambda \big\downarrow^{S_n}_{S_m}$$



μ - random diagram with m boxes

framework

idea: k, ℓ are fixed while $m \to \infty$

cycles

$$[k] = (1, \dots, k) \in S_m \qquad \text{for } m \geq k,$$
$$[k, \ell] = (1, \dots, k)(k+1, \dots, k+\ell) \in S_m \qquad \text{for } m \geq k+\ell$$

---

characters

$$\chi_W(\pi) = \frac{\operatorname{Tr} \rho_W(\pi)}{\operatorname{Tr} \rho_W(\mathrm{id})} \quad \text{for } \pi \in S_m$$

---

asymptotic setting

somebody gives us some interesting sequence $W_1, W_2, \dots$

W_m is a (reducible or irreducible) representation of S_m,

$\chi_m = \chi_{W_m}$ is its character

$\lambda^{(m)}$ is a random Young diagram with m boxes,

or $\mu^{(m)}$

the random irreducible component of W_m

limit shape ⟷ characters

the following two conditions are equivalent
(if you add sufficiently many technical assumptions):

the asymptotic shape $\frac{1}{\sqrt{m}}\lambda^{(m)}$ converges to some limit shape Λ

$\Downarrow$ $\Uparrow$

the character χ_m has a specific asymptotic behavior
which depends on the limit shape Λ

⟶Philippe Biane 1998, 2001

limit shape ⟶ characters

start with a Young diagram ξ

$\lambda^{(n)} := s\xi$ if n is of the form $n = s^2|\xi|$,

in this way $\frac{1}{\sqrt{n}}\lambda^{(n)} = \frac{1}{\sqrt{|\xi|}}\xi = \Lambda$,

$W_n = V^{\lambda^{(n)}}$

Theorem (Philippe Biane 1998)

for each $k \in \{1, 2, \dots\}$ the limit

$$R_{k+1}(\Lambda) := \lim_{n\to\infty} \chi_n\big([k]\big)\, n^{\frac{k-1}{2}}$$

exists and is a nice function of the limit shape Λ

$R_2, R_3, \dots$ are called free cumulants of Λ → random matrix theory

characters ⟶ limit shape

- assume that for each $k \in \{1, 2, \dots\}$ the limit exists

$$R_{k+1} := \lim_{m\to\infty} \chi_m\big([k]\big)\, m^{\frac{k-1}{2}}$$

- assume that

$$\chi_m\big([k,l]\big) \approx \chi_m\big([k]\big)\chi_m\big([l]\big) \quad \text{for } m \to \infty;$$

let a random Young diagram $\mu^{(m)}$ be a random irreducible component of W_m

Theorem (Phlippe Biane 2001)

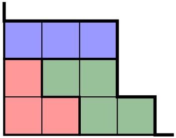
$$\frac{1}{\sqrt{m}}\mu^{(m)} \xrightarrow[m\to\infty]{\text{in probability}} \textit{generalized Young diagram}$$

with free cumulants $R_2, R_3, \dots$

exhibit A: why the limit curves exist?

	shapes	characters
$V^{\lambda^{(n)}}$	① $\lambda^{(n)} = s\,\xi$ for $n = s^2 \lvert\xi\rvert$ ⟹ (start here)	② $\lim_{n\to\infty} \chi_{V^{\lambda^{(n)}}}([k]) \cdot n^{\frac{k-1}{2}} = R_{k+1}(\Lambda)$ exists ⇓
$W^{(m)} = V^{\lambda^{(n)}} \big\downarrow^{S_n}_{S_m}$ for $m = \lfloor \alpha n \rfloor$	④ $\frac{1}{\sqrt{m}} \mu^{(m)}$ converges to the shape with free cumulants $R_2', R_3', \dots$ ⟸	③ $\lim_{m\to\infty} \chi_{W^{(m)}}([k]) \cdot m^{\frac{k-1}{2}} = R'_{k+1} = \dots$ exists

why shape of $\lambda \longrightarrow$ character of V^λ?



classic tools:
Murnaghan–Nakayama rule

$\pi = [3, 4, 3]$

$\operatorname{Tr} \rho^\lambda(\pi) = (-1)^0 \cdot (-1)^1 \cdot (-1)^1 + \cdots$

lots of summands

almost half of them +1
almost half -1

Σ ?

lesson: old combinatorics is not useful today

can algebraic combinatorics
provide new exact formulas for characters
which are useful for asymptotic questions?

dual viewpoint on characters

for a Young diagram λ with n boxes
and $k \in \{1, 2, \dots\}$ we define

$$\mathrm{Ch}_k(\lambda) = \begin{cases} \underbrace{n \cdot (n-1) \cdots (n-k+1)}_{k \text{ factors}} \ \chi_\lambda\big([k]\big) & \text{if } n \geq k, \\ 0 & \text{if } n < k, \end{cases}$$

$\longrightarrow$ Ivanov, Kerov 1999

---

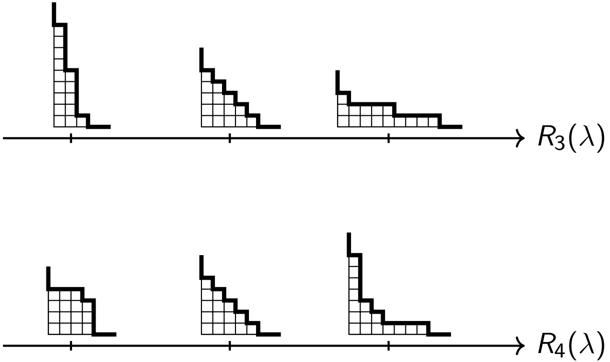
for each integer $k \geq 1$ and each Young diagram λ

$$\{1, 2, \dots\} \ni s \mapsto \mathrm{Ch}_k(s\lambda)$$

is a polynomial of degree $k+1$

free cumulants ⟷ shape

$$R_{k+1}(\lambda) = \left[s^{k+1}\right] \mathrm{Ch}_k(s\lambda) = \lim_{s\to\infty} \frac{1}{s^{k+1}} \mathrm{Ch}_k(s\lambda)$$



→ random matrix theory

“asymptotic shape=free cumulants”

Kerov positivity conjecture

Biane's results are based on

$$\mathrm{Ch}_k \approx R_{k+1}$$

$$\overbrace{\mathrm{Ch}_2}^{\text{character}} = \overbrace{R_3}^{\text{shape}},$$

$$\mathrm{Ch}_3 = R_4 + R_2,$$

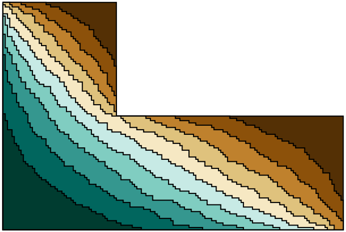
$$\mathrm{Ch}_4 = R_5 + 5R_3,$$

$$\mathrm{Ch}_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2,$$

$$\mathrm{Ch}_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3$$

why positivity? $\longrightarrow$ Stanley–Féray character formula

exhibit A: moral lessons



- Biane's machinery has many more applications! homework ⟶ http://psniady.impan.pl/fpsac

- some classical tools of algebraic combinatorics are not convenient for asymptotic questions,
- asymptotic viewpoint may create new (“dual”) tools in algebraic combinatorics,
- without asymptotic motivations you would not look for new character formulas,

outlook: Lassalle's conjecture

characters of the symmetric groups $\mathrm{Ch}_n =$
dual viewpoint on Schur polynomials

Jack characters $\mathrm{Ch}_k^{(\gamma)}$ = dual viewpoint on Jack polynomials
(toy example of Macdonald polynomials)

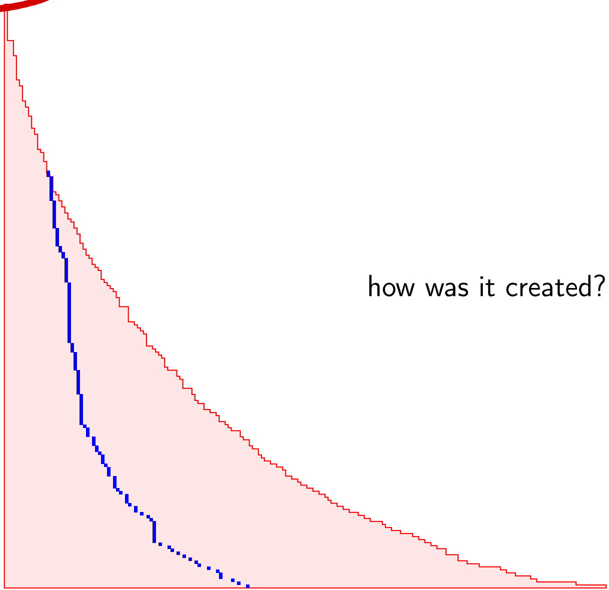
$$\begin{aligned}
\mathrm{Ch}_1^{(\gamma)} &= R_2, \\
\mathrm{Ch}_2^{(\gamma)} &= R_3 + \gamma R_2, \\
\mathrm{Ch}_3^{(\gamma)} &= R_4 + 3\gamma R_3 + (1 + 2\gamma^2) R_2, \\
\mathrm{Ch}_4^{(\gamma)} &= R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2) R_3 + (7\gamma + 6\gamma^3) R_2,
\end{aligned}$$

why positivity?

more positivity conjectures

→ psniady.impan.pl/fpsac

exhibit B



how was it created? ⟶ RSK

Robinson–Schensted–Knuth algorithm is a bijection. . .

input:

- word $\mathbf{w} = (w_1, \ldots, w_n)$

output:

- semistandard tableau P,
- standard tableau Q,

tableaux P and Q have the same shape with n boxes

---

example:

w = (23, 53, 74, 16, 99, 70, 82, 37, 41)

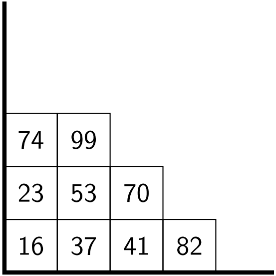
74	99		
23	53	70	
16	37	41	82

insertion tableau P(w)

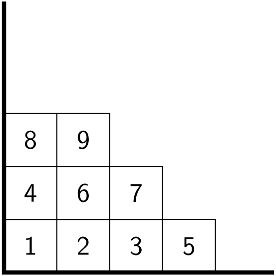
8	9		
4	6	7	
1	2	3	5

recording tableau Q(w)

Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

Robinson-Schensted-Knuth algorithm — induction step

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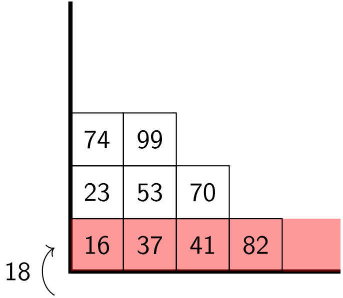

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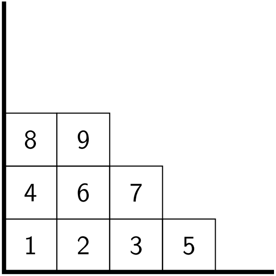

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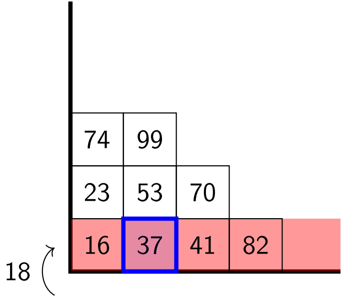

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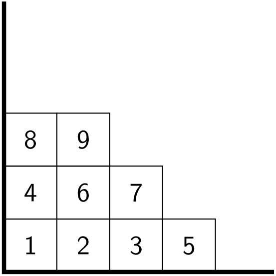

recording tableau $Q(\mathrm{w})$

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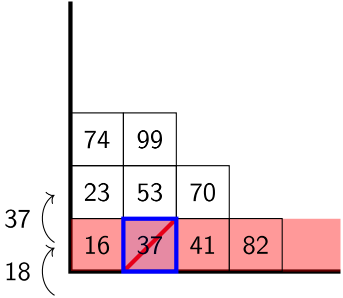
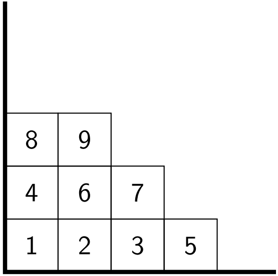


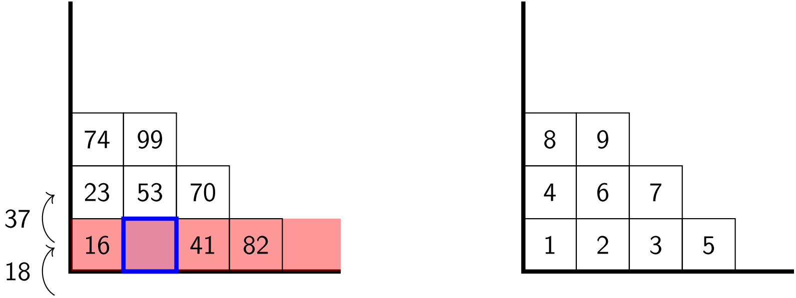
insertion tableau $P(w)$



recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

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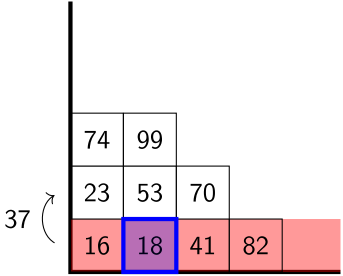
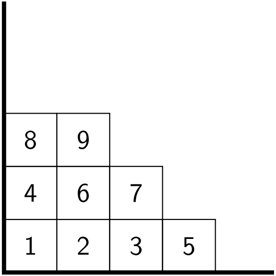
Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathrm{w})$



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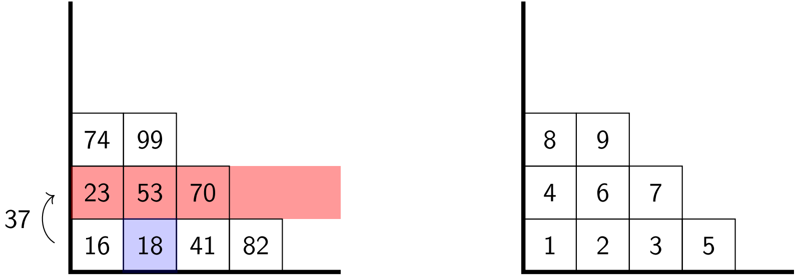

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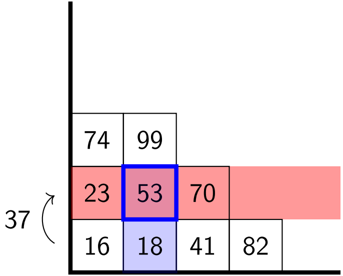
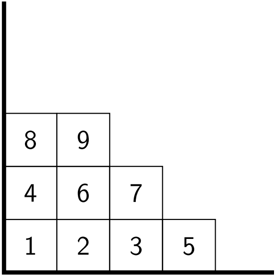


insertion tableau $P(\mathrm{w})$



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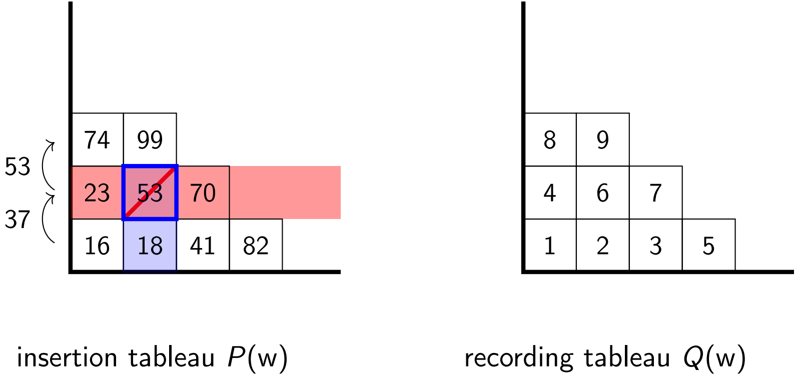

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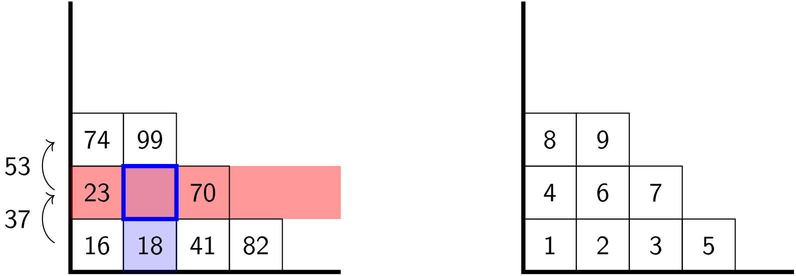

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recording tableau $Q(\mathrm{w})$

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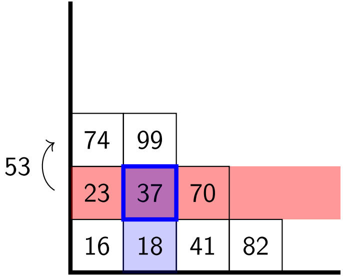

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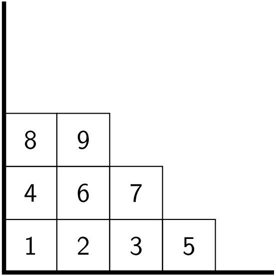

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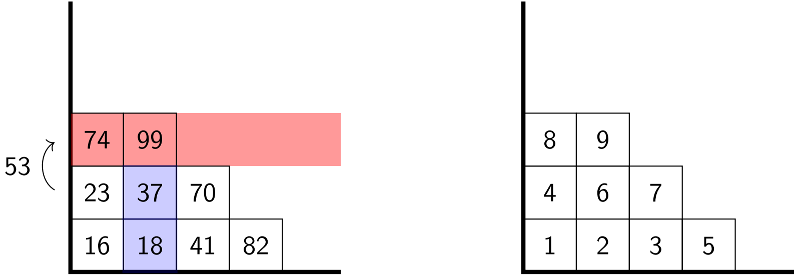

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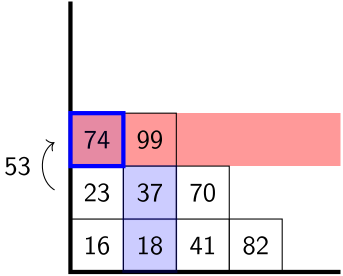

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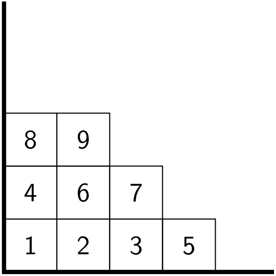

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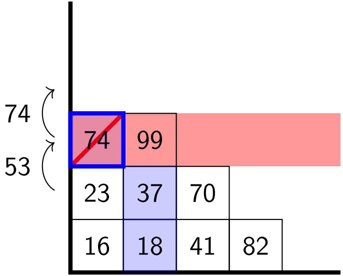

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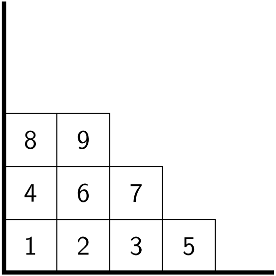

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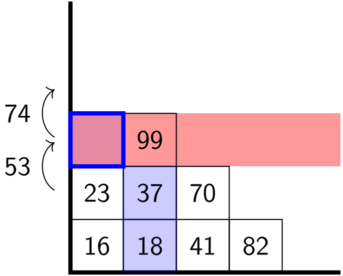
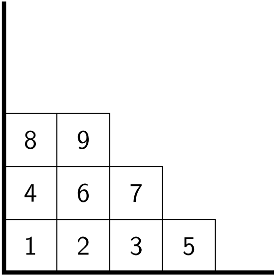
$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



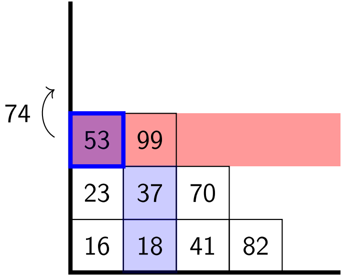
insertion tableau $P(\mathrm{w})$



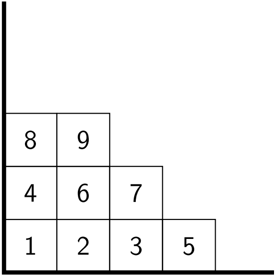
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



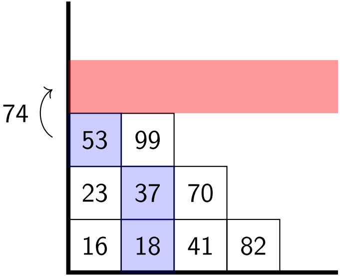
insertion tableau $P(\mathrm{w})$



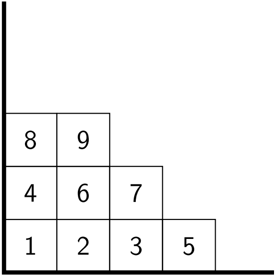
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



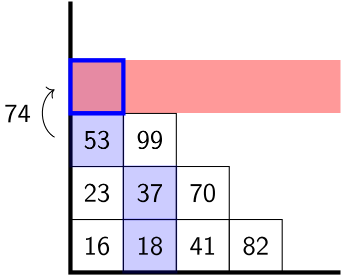
insertion tableau $P(w)$



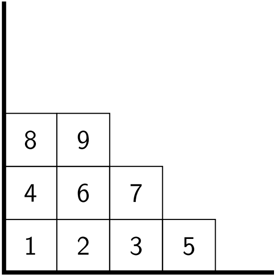
recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



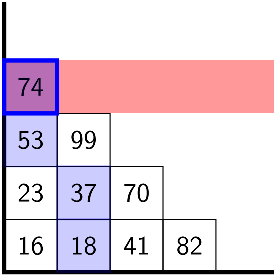
insertion tableau $P(\mathrm{w})$



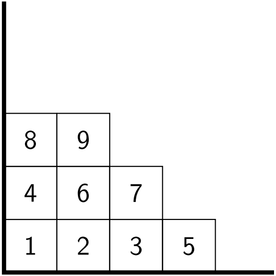
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(w)$



recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step

74			
53	99		
23	37	70	
16	18	41	82

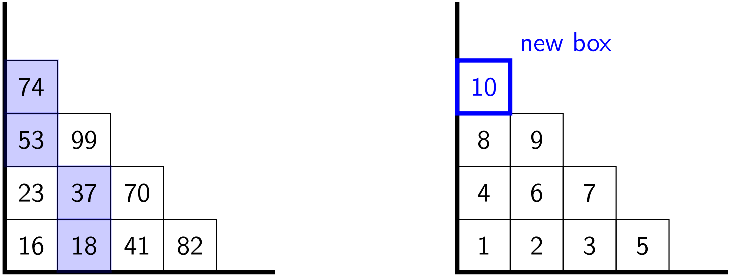
insertion tableau $P(w)$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

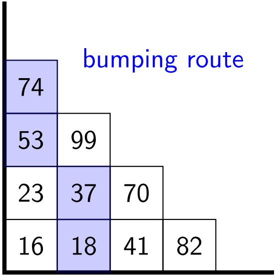
Robinson-Schensted-Knuth algorithm — induction step



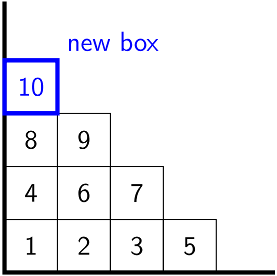
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm — induction step

74			
53	99		
23	37	70	
16	18	41	82

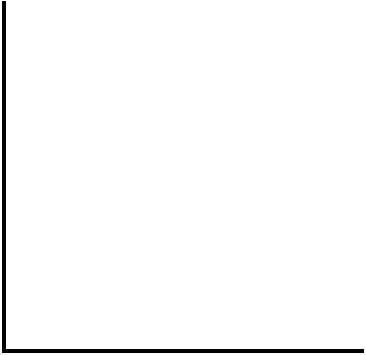
insertion tableau $P(\mathrm{w})$

10			
8	9		
4	6	7	
1	2	3	5

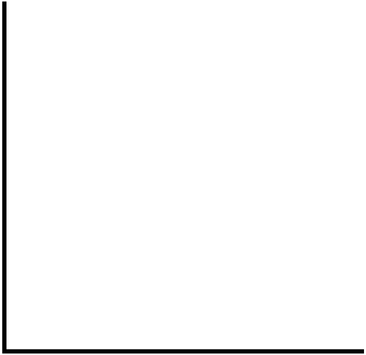
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson-Schensted-Knuth algorithm



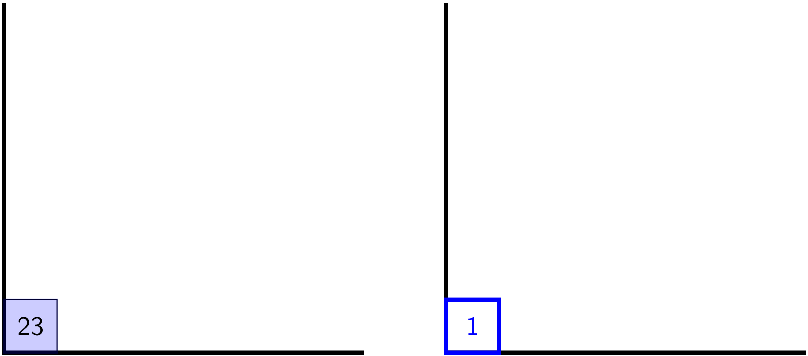
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = \emptyset$

Robinson-Schensted-Knuth algorithm

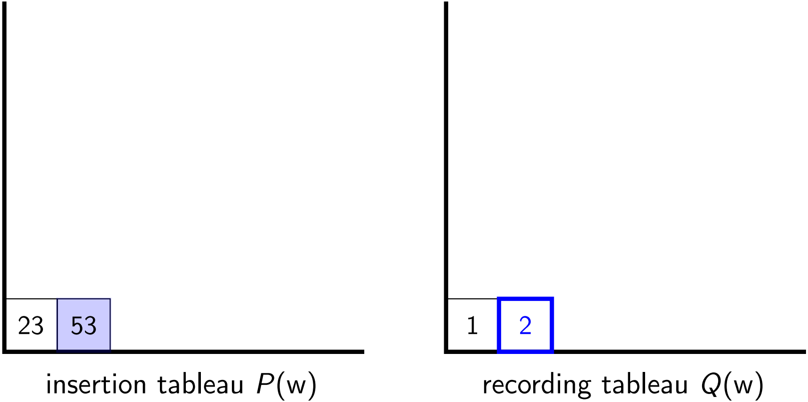


insertion tableau $P(\mathrm{w})$

recording tableau $Q(\mathrm{w})$

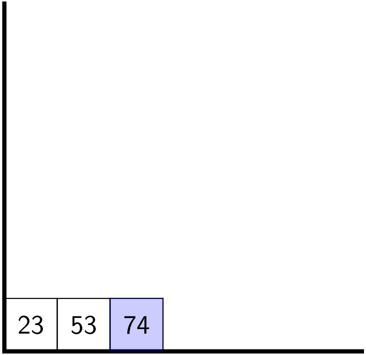
$\mathrm{w} = (23)$

Robinson-Schensted-Knuth algorithm

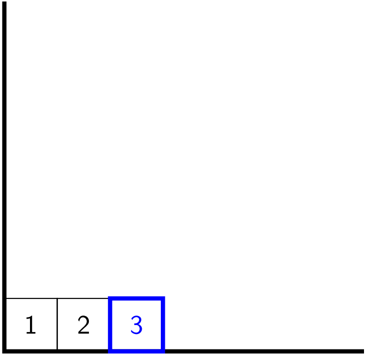


$\mathrm{w} = (23, 53)$

Robinson-Schensted-Knuth algorithm



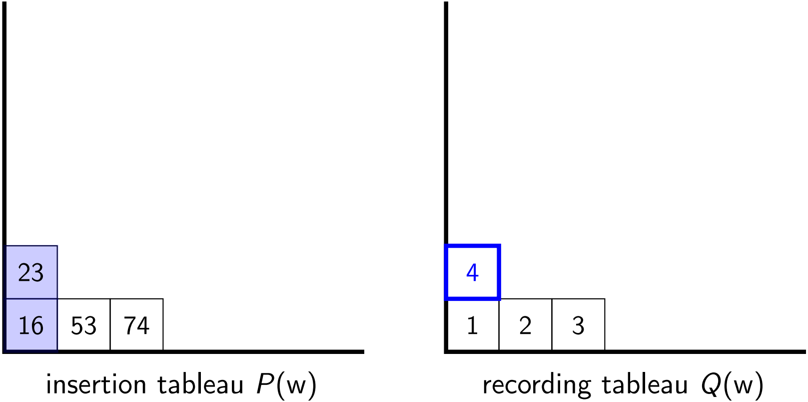
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

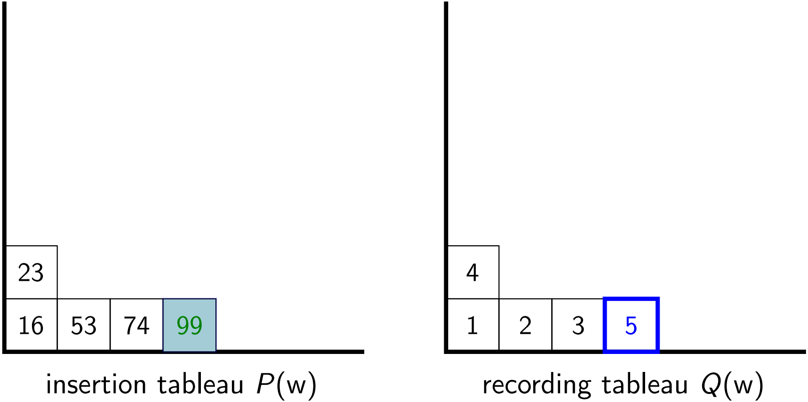
$\mathrm{w} = (23,\ 53,\ 74)$

Robinson-Schensted-Knuth algorithm



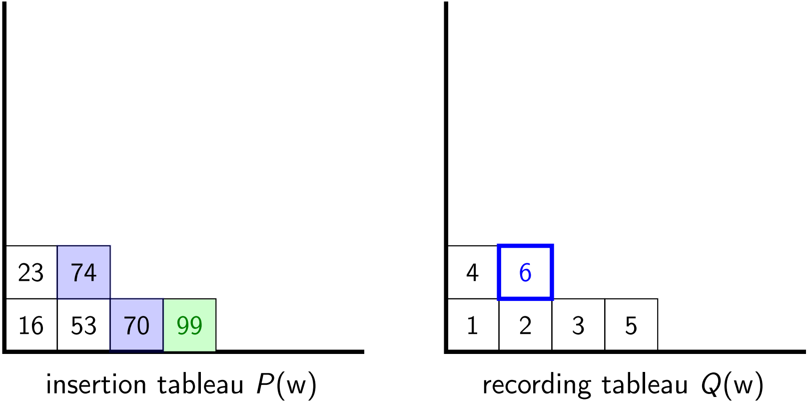
$\mathrm{w} = (23,\ 53,\ 74,\ 16)$

Robinson-Schensted-Knuth algorithm



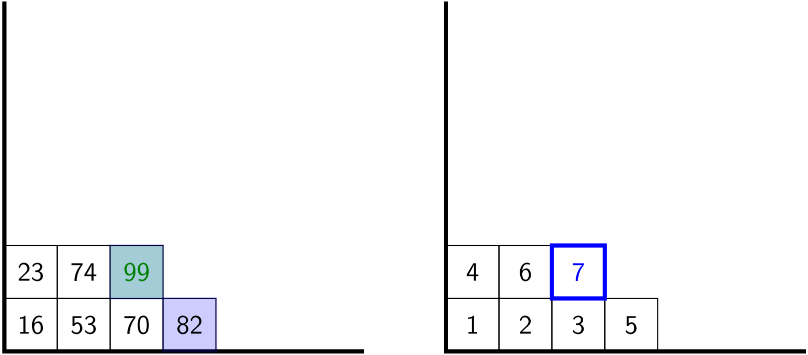
$w = (23, 53, 74, 16, 99)$

Robinson-Schensted-Knuth algorithm



$w = (23, 53, 74, 16, 99, 70)$

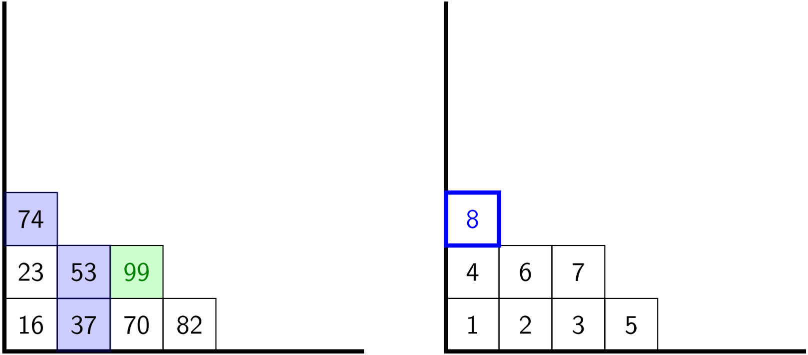
Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82)$$

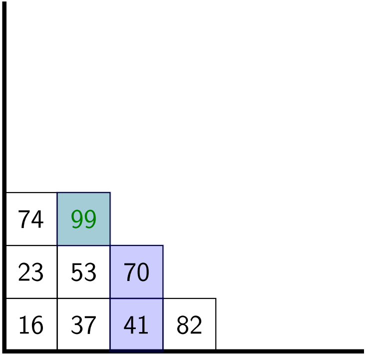
Robinson-Schensted-Knuth algorithm



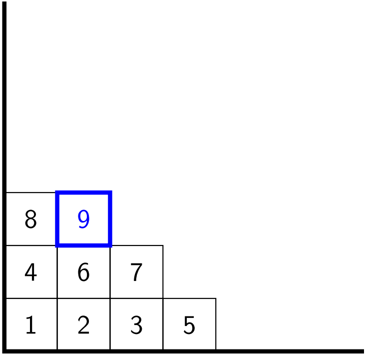
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37)$

Robinson-Schensted-Knuth algorithm



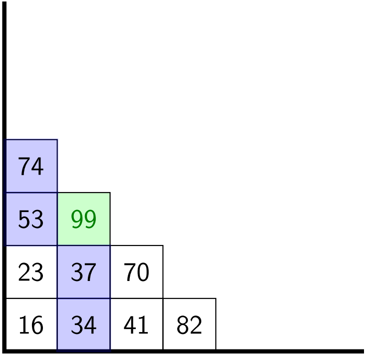
insertion tableau $P(\mathrm{w})$



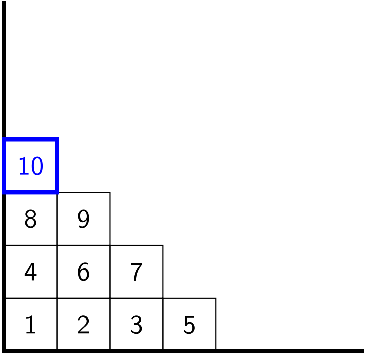
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(w)$



recording tableau $Q(w)$

$$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34)$$

Robinson-Schensted-Knuth algorithm

74			
53	99		
23	37	70	82
16	34	41	73

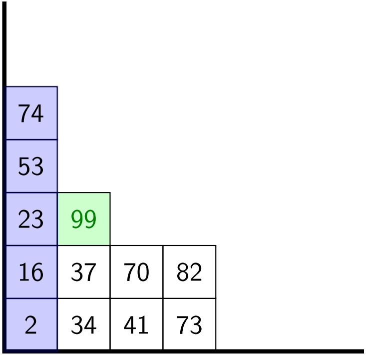
insertion tableau $P(\mathrm{w})$

10			
8	9		
4	6	7	11
1	2	3	5

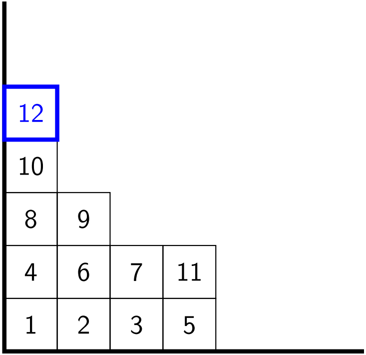
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73)$

Robinson-Schensted-Knuth algorithm



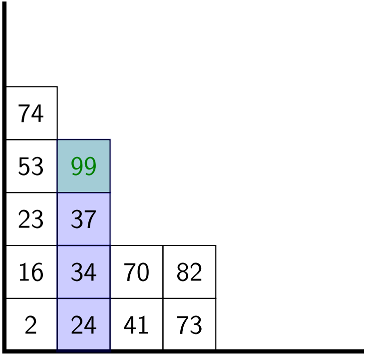
insertion tableau $P(w)$



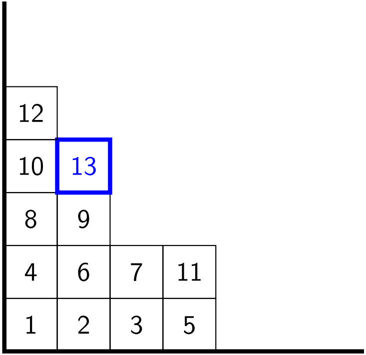
recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2)$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(w)$

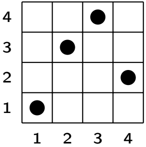


recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2, 24)$

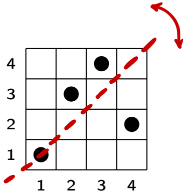
magic symmetries of RSK

$$(1,3,4,2) =$$



$$\mapsto \left(\begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right)$$

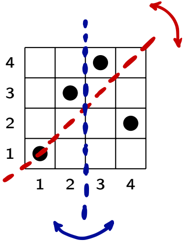
magic symmetries of RSK

$(1,3,4,2) =$ 

$$\mapsto \left(\begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \; , \; \begin{array}{|c|cc} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right)$$

magic symmetries of RSK

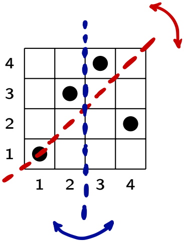
$(1, 3, 4, 2) =$



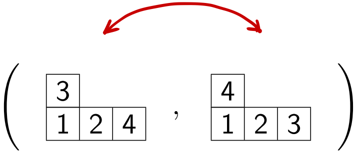
$$\mapsto \left(\begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \; , \; \begin{array}{|c|cc} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right)$$

magic symmetries of RSK

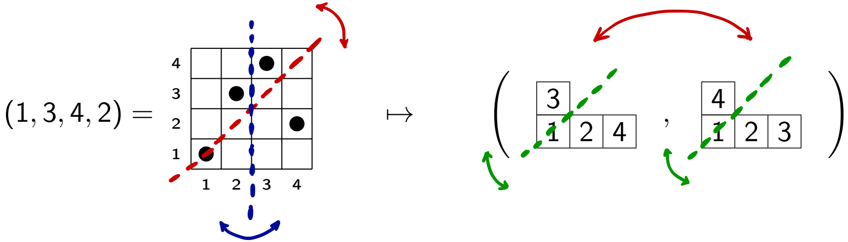
$(1, 3, 4, 2) =$



$\mapsto$



magic symmetries of RSK



magic symmetries of RSK

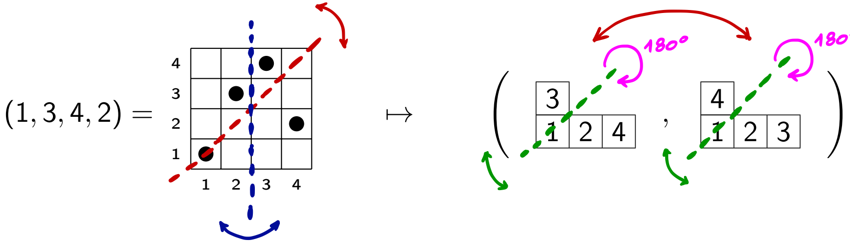
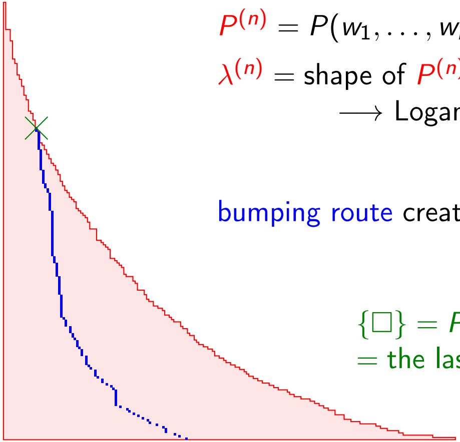


exhibit B: how it was created?

let $w_1, \dots, w_n$ be independent random variables with the uniform distribution on $[0,1]$

$0 \leq s \leq 1$



$P^{(n)} = P(w_1, \dots, w_n)$

$\lambda^{(n)} =$ shape of $P^{(n)}$

$\longrightarrow$ Logan, Shepp, Vershik, Kerov (1977)

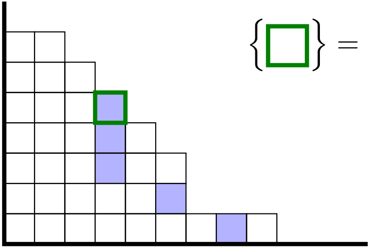
bumping route created in the insertion $P^{(n)} \leftarrow s$

(in this example $s = 0.2$)

$\{\square\} = P(w_1, \dots, w_n, s) \setminus P(w_1, \dots, w_n)$

$=$ the last box in the bumping route

the end of the bumping route

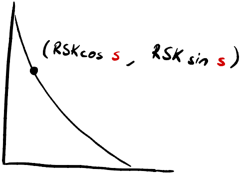


$$\{\square\} = P(w_1, \dots, w_n, s) \setminus P(w_1, \dots, w_n)$$

for best viewing scale this picture by $\frac{1}{\sqrt{n}}$

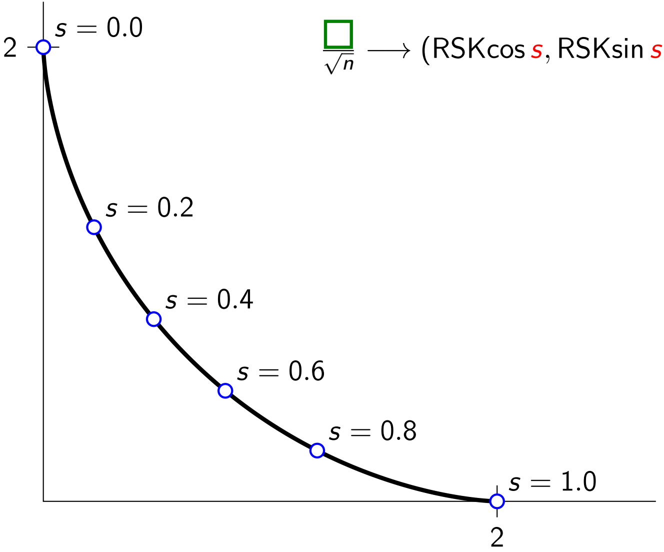
Theorem (Dan Romik, Piotr Śniady 2015)

$$\frac{\square}{\sqrt{n}} \xrightarrow[\text{in probability}]{n\to\infty} (\mathrm{RSKcos}\, s, \mathrm{RSKsin}\, s)$$

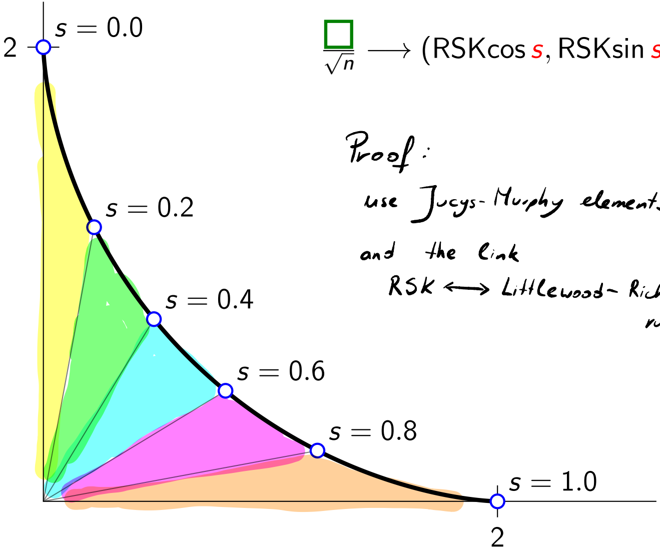


the end of the bumping route

$$\frac{\square}{\sqrt{n}} \longrightarrow (\mathrm{RSKcos}\, s, \mathrm{RSKsin}\, s)$$



the end of the bumping route



$$\frac{\square}{\sqrt{n}} \longrightarrow (\mathrm{RSKcos}\, s, \mathrm{RSKsin}\, s)$$

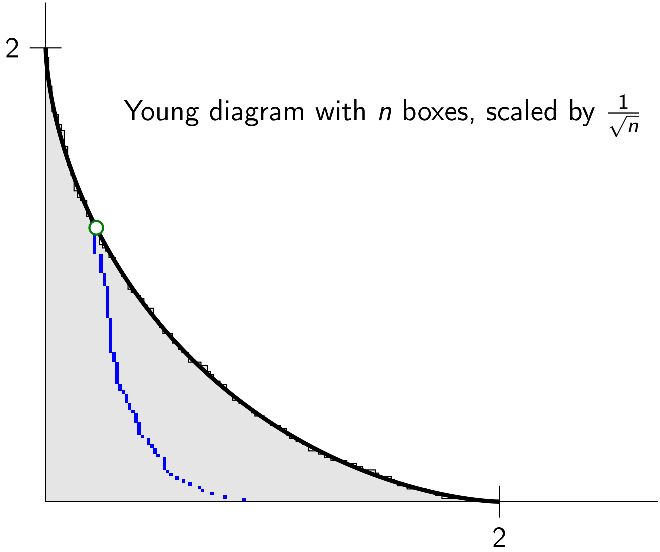
Proof:

use Jucys-Murphy elements

and the link

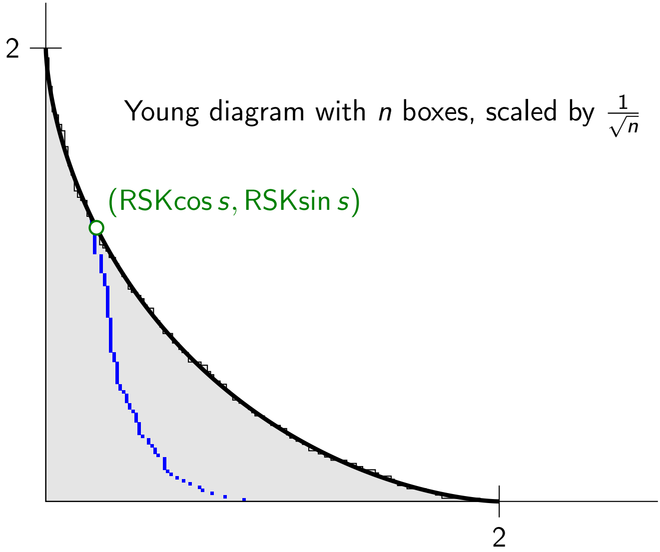
RSK ⟷ Littlewood-Richardson rule

the bumping route



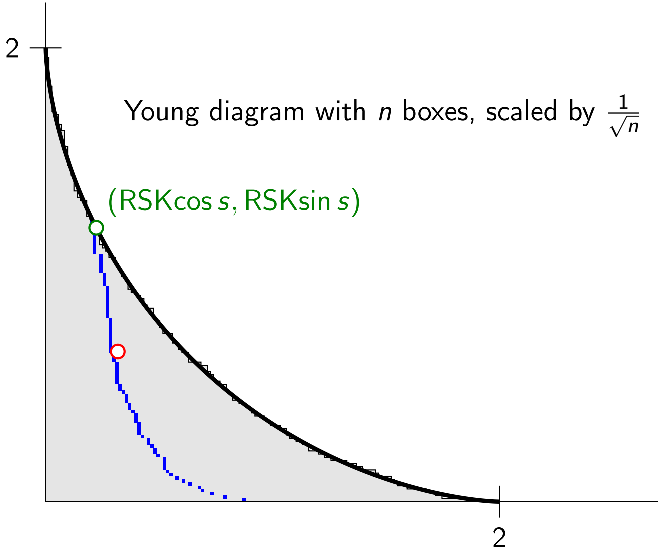
Dan Romik, Piotr Śniady 2016

the bumping route



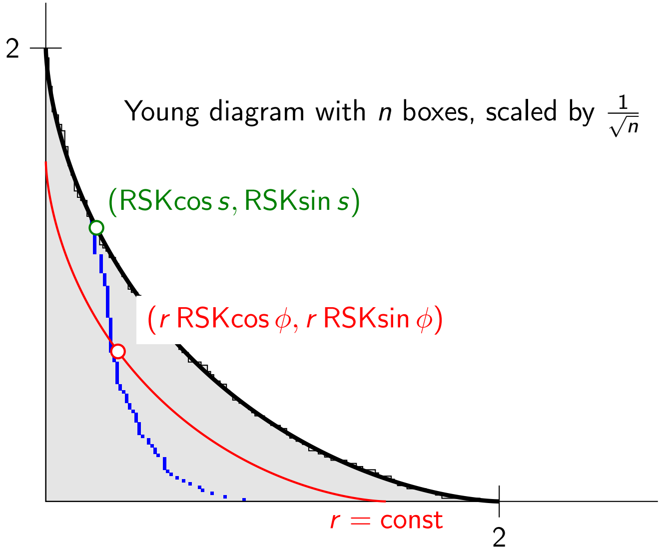
Dan Romik, Piotr Śniady 2016

the bumping route



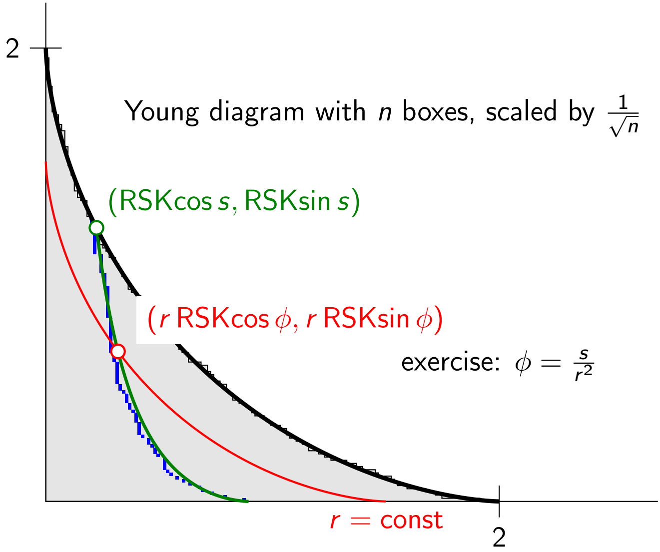
Dan Romik, Piotr Śniady 2016

the bumping route



Dan Romik, Piotr Śniady 2016

the bumping route



Dan Romik, Piotr Śniady 2016

diffusion of a box in the insertion tableau $P(w)$

⟶ Mikołaj Marciniak (2022)

[this slide contains a JavaScript animation]

this Young diagram has more and more boxes over time

and moves up and to the left

diffusion of a box in the insertion tableau $P(w)$

will this box ever reach the first column?

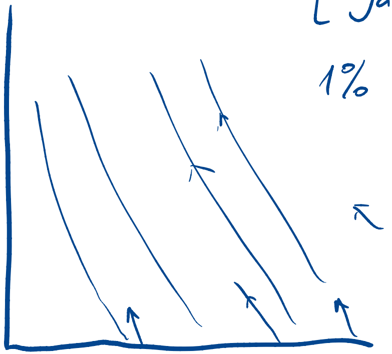
⟶ Marciniak, Maślanka, Śniady 2021

[same animation as on the previous slide]

hydrodynamics of the insertion tableau $P(w)$

[JavaScript animation]

1% of boxes is colored



boxes appear on the bottom and follow some curves representation theory of S_n

- representation:

$$\rho\colon S_n \to \mathrm{End}(V)$$

  V is finite dimensional

- irreducible representations,
- irreducible characters,

repres. theory of $S_\infty = \bigcup_{n\geq 1} S_n$

- representation:

$$\rho\colon S_\infty \to B(\mathcal{H})$$

  $\mathcal{H}$ is a *Hilbert space*

- *factorial representations* $\longrightarrow$ operator algebras
- *extremal characters*,

Vershik, Kerov:
link between

- factorial representations of S_∞,
- RSK applied to random input,
- random infinite tableaux,

infinite version of RSK

$$\Omega = [0,1]^\infty \ni (w_1, w_2, \dots) \xrightarrow{\ \Phi_\infty\ } t = Q(w_1, w_2, \dots)$$

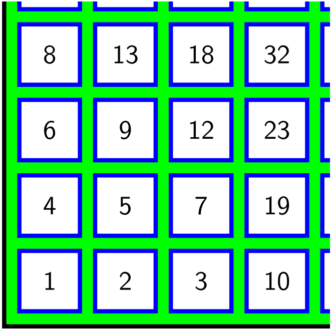
(random) infinite Young tableau

PROBLEM: inverse?

8	13	18	32
6	9	12	23
4	5	7	19
1	2	3	10

jeu de taquin

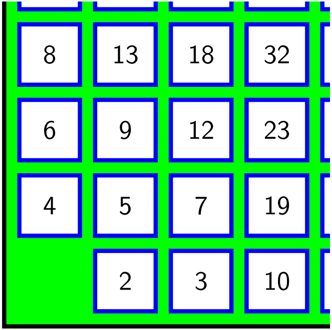
start with (infinite) tableau
$t = Q(w_1, w_2, \dots)$,



jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

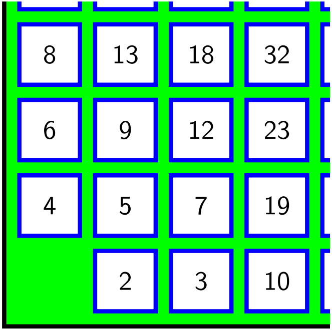
① remove corner box,



jeu de taquin

start with (infinite) tableau
$t = Q(w_1, w_2, \dots)$,

① remove corner box,

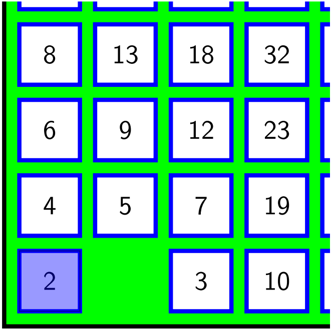


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \ldots)$,

① remove corner box,

② sliding,

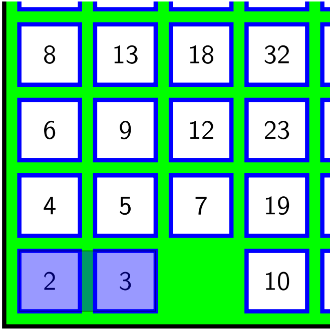


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,

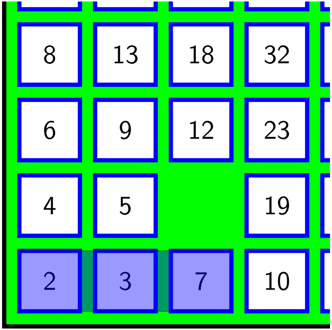


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,

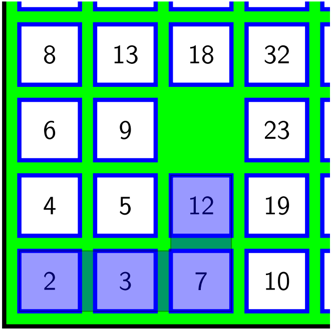


jeu de taquin

start with (infinite) tableau
$t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,

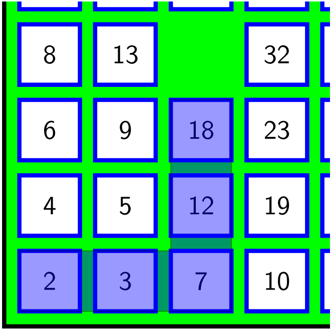


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,

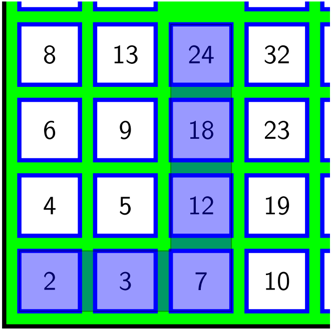


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,



jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

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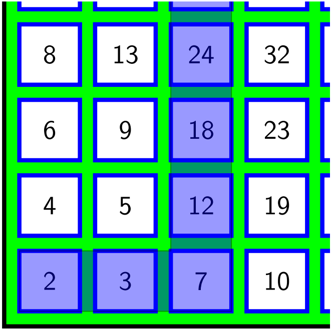
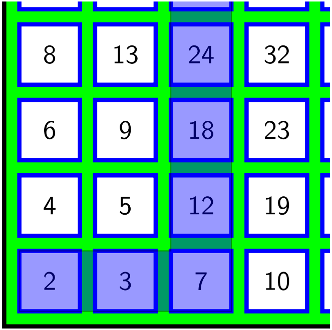


jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,



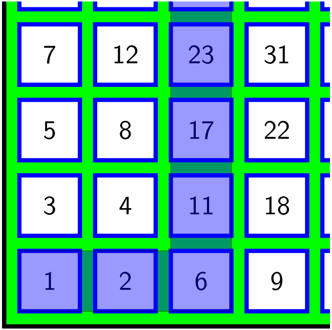
jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,

③ subtract 1 from all boxes



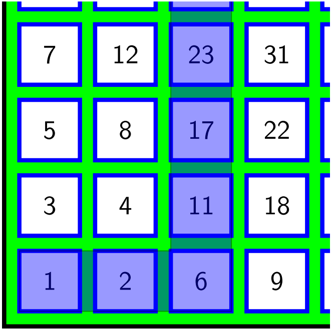
jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

② sliding,

③ subtract 1 from all boxes



jeu de taquin

start with (infinite) tableau $t = Q(w_1, w_2, \dots)$,

① remove corner box,

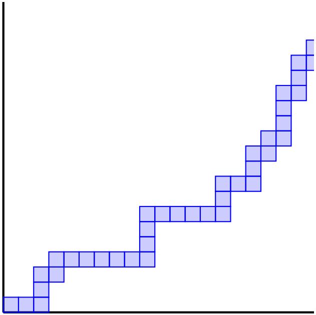
② sliding,

③ subtract 1 from all boxes

output:

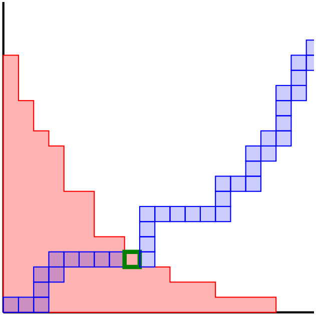
- new tableau $= Q(\cancel{w_1}, w_2, w_3, \dots)$,
- blue trajectory

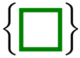
trajectory of jeu de taquin has an asymptote



if $t = Q(w_1, w_2, \dots)$ is a random infinite tableau ...

trajectory of jeu de taquin has an asymptote

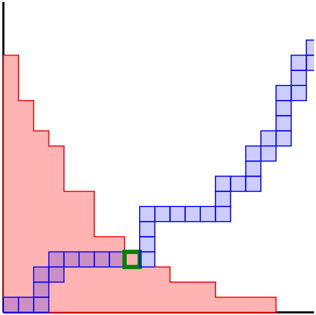




if $t = Q(w_1, w_2, \dots)$ is a random infinite tableau …

$\lambda^{(n)} = \{\text{boxes} \leq n\}$

trajectory of jeu de taquin has an asymptote

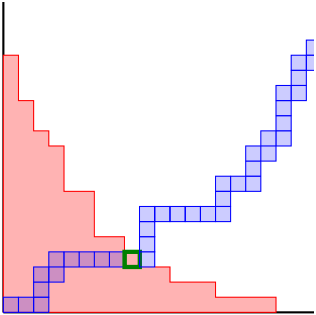


if $t = Q(w_1, w_2, \dots)$ is a random infinite tableau …

$\lambda^{(n)} = \{\text{boxes} \le n\}$

$\{\square\} = Q(w_1, w_2, \dots, w_n) \setminus Q(w_2, \dots, w_n) =$

trajectory of jeu de taquin has an asymptote



if $t = Q(w_1, w_2, \dots)$ is a random infinite tableau ...

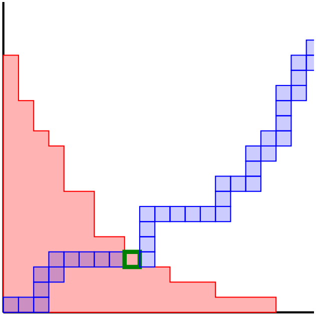
$\lambda^{(n)} = \{\text{boxes} \leq n\}$

"magic symmetries of RSK"

$$\{\square\} = Q(w_1, w_2, \dots, w_n) \setminus Q(w_2, \dots, w_n) =$$

$$Q(1-w_n, \dots, 1-w_2, 1-w_1) \setminus Q(1-w_n, \dots, 1-w_2)$$

trajectory of jeu de taquin has an asymptote

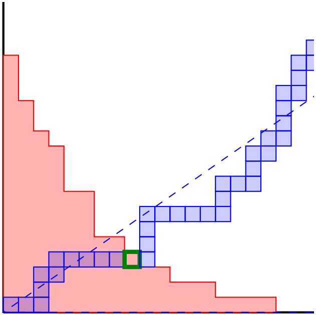


if $t = Q(w_1, w_2, \dots)$ is a random infinite tableau ...

$\lambda^{(n)} = \{\text{boxes} \leq n\}$

$$\{\square\} = Q(w_1, w_2, \dots, w_n) \setminus Q(w_2, \dots, w_n) =$$
$$Q(1-w_n, \dots, 1-w_2, 1-w_1) \setminus Q(1-w_n, \dots, 1-w_2)$$
$$\approx \sqrt{n}\big(\,\mathrm{RSKcos}(1-w_1), \mathrm{RSKsin}(1-w_1)\big)$$

trajectory of jeu de taquin has an asymptote



if $t = Q(w_1, w_2, \dots)$ is a random infinite tableau . . .

$\lambda^{(n)} = \{\text{boxes} \leq n\}$

$$\left\{ \square \right\} = Q(w_1, w_2, \dots, w_n) \setminus Q(w_2, \dots, w_n) =$$

$$Q(1 - w_n, \dots, 1 - w_2, 1 - w_1) \setminus Q(1 - w_n, \dots, 1 - w_2)$$

$$\approx \sqrt{n}\big(\mathrm{RSKcos}(1 - w_1), \mathrm{RSKsin}(1 - w_1)\big)$$

infinite version of RSK

$$\Omega = [0,1]^\infty \ni$$

$$(w_1, w_2, \dots) \overset{Q_\infty}{\longmapsto} t = Q(w_1, w_2, \dots)$$

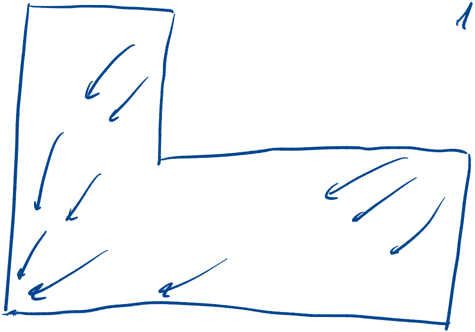
(random) infinite Young tableau

Yes, inverse exists!

jeu de taquin in action

⟶ Łukasz Maślanka, Piotr Śniady 2022

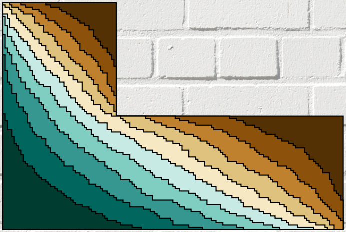
[this is a JavaScript animation]

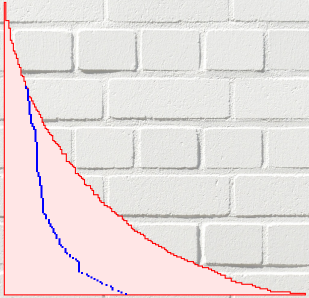


1% of boxes is colored

boxes follow some curves.

conclusions





asymptotic / visual viewpoint may give new questions,

interesting from the algebraic combinatorics viewpoint

during coffee break, look for my coauthors!

Mikołaj Marciniak

Łukasz Maślanka

transparencies, references, homework available on
`http://psniady.impan.pl/fpsac`

Museum of visual ART

references, extra notes, exercises

Piotr Śniady

how the pictures were created?

write SageMath code which will generate output that can be copy-pasted into your TikZ / LaTeX code

for example, the following SageMath code

```
size=10;
diagram=Partition([3*size]*size + [size] *
    size)
boxes=diagram.size()
tabl=StandardTableaux(diagram).random_element
    ()
for number in reversed(range(1,6)):
    threshold = number / 5
    color="blue!{}!red".format(20*(6-number))
    limit = boxes * threshold
    restricted= Partition([ sum(1 for i in
        line if i <= limit) for line in tabl ])
    a=[ (x+1,y+1) for (y,x) in restricted.
        inside_corners()] + [(0,0)]
    b=[ (x,y) for (y,x) in restricted.
        outside_corners()]
    contour= [ j  for i in zip(b,a) for j in i
        ]
    print "\draw[draw=black, fill={}] ".format
```

```
    (color)
print ' — '.join([str(point) for point in
    contour])
print " — cycle;"
```

generates a part of the following LaTeXfile

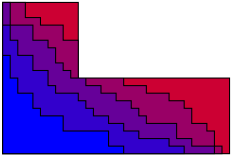
```
\documentclass[tikz]{standalone}


\begin{document}
\begin{tikzpicture}[scale=0.1]
    \draw[draw=black, fill=blue!20!red]
    (30, 0) -- (30, 10) -- (10, 10) -- (10, 20)
        -- (0, 20) -- (0, 0)
    -- cycle;
    \draw[draw=black, fill=blue!40!red]
    (29, 0) -- (29, 1) -- (28, 1) -- (28, 3) --
        (27, 3) -- (27, 4) -- (25, 4) -- (25,
        6) -- (24, 6) -- (24, 7) -- (22, 7) --
        (22, 8) -- (19, 8) -- (19, 9) -- (16, 9)
        -- (16, 10) -- (10, 10) -- (10, 15) --
        (8, 15) -- (8, 17) -- (5, 17) -- (5, 18)
        -- (3, 18) -- (3, 20) -- (0, 20) -- (0,
        0)
    -- cycle;
```

```
\draw[draw=black, fill=blue!60!red]
(28, 0) -- (28, 1) -- (25, 1) -- (25, 2) --
    (23, 2) -- (23, 4) -- (19, 4) -- (19,
   5) -- (18, 5) -- (18, 6) -- (17, 6) --
   (17, 7) -- (15, 7) -- (15, 8) -- (13, 8)
    -- (13, 9) -- (11, 9) -- (11, 10) --
   (9, 10) -- (9, 11) -- (8, 11) -- (8, 12)
    -- (7, 12) -- (7, 14) -- (6, 14) -- (6,
    15) -- (4, 15) -- (4, 17) -- (1, 17) --
    (1, 20) -- (0, 20) -- (0, 0)
-- cycle;
\draw[draw=black, fill=blue!80!red]
(24, 0) -- (24, 1) -- (22, 1) -- (22, 2) --
    (18, 2) -- (18, 3) -- (16, 3) -- (16,
   4) -- (14, 4) -- (14, 5) -- (12, 5) --
   (12, 6) -- (11, 6) -- (11, 7) -- (10, 7)
    -- (10, 8) -- (7, 8) -- (7, 9) -- (6,
   9) -- (6, 11) -- (4, 11) -- (4, 13) --
   (2, 13) -- (2, 15) -- (1, 15) -- (1, 17)
```

```
        -- (0, 17) -- (0, 0)
    -- cycle;
    \draw[draw=black, fill=blue!100!red]
    (16, 0) -- (16, 1) -- (14, 1) -- (14, 3) --
        (8, 3) -- (8, 5) -- (5, 5) -- (5, 6) --
        (4, 6) -- (4, 8) -- (2, 8) -- (2, 10)
      -- (1, 10) -- (1, 13) -- (0, 13) -- (0,
      0)
    -- cycle;
\end{tikzpicture}
\end{document}
```

which gives the following output picture



how the animations were created?

the LaTeX package `animate` embeds JavaScript into the PDF file;
the output PDF file must be opened in Acrobat Reader in order to show the animation
minimalist sample code on the next pagex

```
\documentclass{beamer}
\usepackage{animate}
\begin{document}

    \begin{frame}
        \begin{animateinline}[autoplay,loop]{5}
            % 5 frames per second
            frame a
            \newframe

            frame b
            \newframe

            frame c
        \end{animateinline}
    \end{frame}

\end{document}
```

random Young diagrams

Philippe Biane.
Free cumulants and representations of large symmetric groups.
XIIIth International Congress on Mathematical Physics (London, 2000), 321-–326, Int. Press, Boston, MA, 2001.
http://igm.univ-mlv.fr/~biane/ICMP.pdf

Philippe Biane.
Approximate factorization and concentration for characters of symmetric groups.
Internat. Math. Res. Notices 2001, no. 4, 179—192.
https://doi.org/10.48550/arXiv.math/0006111

nobody has done this experiment before

run the following experiment in SageMath:
let $w = (w_1, \ldots, w_n)$ be a very long random sequence of length $n = 10^4$;

entries are independent random variables,
sampled from $\{1, \ldots, d\}$ with $d = 10^2$

- visualize the insertion tableau $P(w)$
- visualize the insertion tableau $Q(w)$
- now find the theoretical explanation

$\longrightarrow$ `https://doi.org/10.1007/s00029-020-0535-2`
Remark 1.6

Stanley–Féray character formula, Kerov positivity conjecture

short reading

Piotr Śniady.
Stanley character polynomials.
The mathematical legacy of Richard P. Stanley, 323–334, Amer. Math. Soc., Providence, RI, 2016
https://doi.org/10.48550/arXiv.1409.7533

Piotr Śniady.
Combinatorics of asymptotic representation theory.
European Congress of Mathematics, 531-–545, Eur. Math. Soc., Zürich, 2013
https://doi.org/10.48550/arXiv.1203.6509

Stanley–Féray character formula, Kerov positivity conjecture

Maciej Dołęga, Valentin Féray, Piotr Śniady.
Characters of symmetric groups in terms of free cumulants and Frobenius coordinates.
21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), 337–348, Discrete Math. Theor. Comput. Sci. Proc., AK, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009.
https://doi.org/10.48550/arXiv.1105.2549

long reading

Pierre-Loïc Méliot.
Representation theory of symmetric groups.
Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2017.
https://doi.org/10.1201/9781315371016

open positivity conjectures

- Ian P. Goulden, Amarpreet Rattan.
  An explicit form for Kerov's character polynomials.
  Trans. Amer. Math. Soc. 359 (2007), no. 8, 3669–3685
  https://doi.org/10.1090/S0002-9947-07-04311-5
- Michel Lassalle.
  Two positivity conjectures for Kerov polynomials.
  Adv. in Appl. Math. 41 (2008), no. 3, 407–422.
  https://doi.org/10.1016/j.aam.2008.01.001
- Michel Lassalle.
  Jack polynomials and free cumulants.
  Adv. Math. 222 (2009), no. 6, 2227–2269
  https://doi.org/10.1016/j.aim.2009.07.007

bumping routes

- Dan Romik, Piotr Śniady.
  Limit shapes of bumping routes in the Robinson–Schensted correspondence.
  Random Structures Algorithms 48 (2016), no. 1, 171–182
  https://doi.org/10.48550/arXiv.1304.7589
- Mikołaj Marciniak, Łukasz Maślanka, Piotr Śniady.
  Poisson limit of bumping routes in the Robinson–Schensted correspondence.
  Probab. Theory Related Fields 181 (2021), no. 4, 1053–1103
  https://doi.org/10.1007/s00440-021-01084-y
- Mikołaj Marciniak.
  Hydrodynamic limit of the Robinson–Schensted–Knuth algorithm.
  Random Structures Algorithms 60 (2022), no. 1, 106—116
  https://doi.org/10.48550/arXiv.2005.03147

jeu de taquin

- Dan Romik, Piotr Śniady.
  Jeu de taquin dynamics on infinite Young tableaux and second class particles
  Ann. Probab, Volume 43, Number 2 (2015), 682-737
  https://doi.org/10.1214/13-AOP873

- Łukasz Maślanka, Piotr Śniady.
  Limit shapes of evacuation and jeu de taquin paths in random square tableaux.
  Sém. Lothar. Combin. 84B (2020), Art. 8, 12 pp.
  https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2020/8.html

- Łukasz Maślanka, Piotr Śniady.
  Second class particles and limit shapes of evacuation and sliding paths for random tableaux.
  https://doi.org/10.48550/arXiv.1911.08143

a bit off-topic but you will enjoy this book

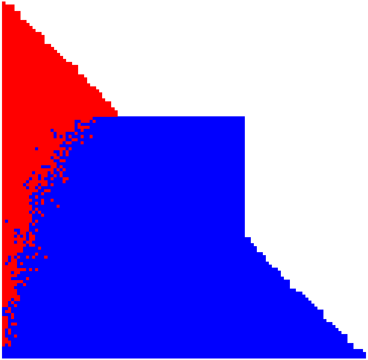
Dan Romik.
The surprising mathematics of longest increasing subsequences.
Institute of Mathematical Statistics Textbooks, 4. Cambridge University Press, New York, 2015. xi+353 pp
https://www.math.ucdavis.edu/~romik/book/

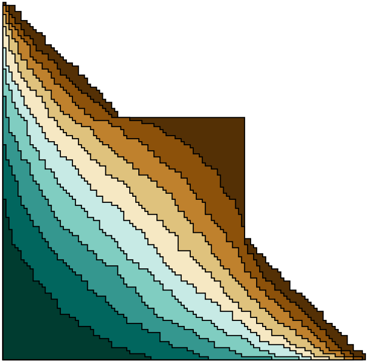
experiment: rectification of large random tableaux

computer experiment with SageMath:
the following code generates a random skew tableau with a shape which consists of two squares, touching in the corner;
the 'red' small square is in north-west, the 'blue' big square is in south-east;
then we rectify the the tableau

```
import random
sizeA=40
sizeB=80
mylist = [0]* sizeA^2 + [1]*sizeB^2
random.shuffle(mylist)
indicesA=[ index+1 for index,entry in enumerate(
    mylist) if entry==0]
indicesB=[ index+1 for index,entry in enumerate(
    mylist) if entry==1]
tabA =StandardTableaux([sizeA]*sizeA).
    random_element()
tabB =StandardTableaux([sizeB]*sizeB).
    random_element()
newtabA= [ [ indicesA[entry-1] for entry in row ]
    for row in tabA]
newtabB= [ [None] * sizeA + [ indicesB[entry-1]
    for entry in row ] for row in tabB]
skew=SkewTableau(newtabB+newtabA)
rectified=skew.rectify()
```

the following two pictures visualize the outcome of the rectification;
the left picture shows which boxes of the rectified tableau originate from the red and which from the blue part
the right picture shows the usual level curves for the rectified tableau





easy homework: find theoretical explanation for the right picture
hard problem: find theoretical explanation for the left picture
what other questions can we ask?