Jeu de taquin forests
and the inverse infinite RSK correspondence

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Abstract

The *Plancherel-random infinite Young tableau* arises from applying the infinite Robinson–Schensted–Knuth (RSK) correspondence to a sequence of i.i.d. random variables distributed uniformly on the unit interval $[0, 1]$. Building on the isomorphism of dynamical systems established in prior work, we provide an explicit geometric characterization of the inverse map. Our approach makes use of a previously unexplored structure, the *jeu de taquin forest* on the infinite tableau, where edges connect boxes according to local comparison rules. We prove that each tree in this forest almost surely extends toward infinity with a well-defined asymptotic direction, establishing a canonical bijection between trees and values in the i.i.d. input sequence. The ordering is recovered through tree lifetimes under iterated jeu de taquin transformations.

1 Introduction

1.1 The RSK and jeu de taquin maps on infinite tableaux

We recall the main definitions and results from [RŚ15]. An *Infinite Young Tableau* (IYT) is an array

$$T = (T_{x,y})_{x,y\geq 0}$$

where the entries $T_{x,y}$ are increasing along rows and columns and take positive integer values, such that T contains each positive integer exactly once. We visualize T graphically as an infinite array of *boxes*, where each pair $(x, y) \in \mathbb{N}_0^2$ of indices is referred to as a "box" and is associated with (and treated as synonymous with) the square $[x, x + 1] \times [y, y + 1]$ that is depicted as containing the entry $T_{x,y}$. See Figure 1 for an example.

We denote by Ω the space of infinite Young tableaux. This space (viewed as a measurable space in the usual way by equipping it with the minimal σ-algebra $\mathcal{F}$ that turns all coordinate functions $T \mapsto T_{x,y}$ into measurable functions) is equipped with a natural probability measure P, known as *(infinite) Plancherel measure*, that is characterized by the property that for each (finite) Young diagram λ and standard Young tableau $Q = (q_{x,y})_{(x,y)\in\lambda}$ of shape λ, we have

$$P\Big(T_{x,y} = q_{x,y} \text{ for all } (x, y) \in \lambda\Big) = \frac{f^\lambda}{n!},$$

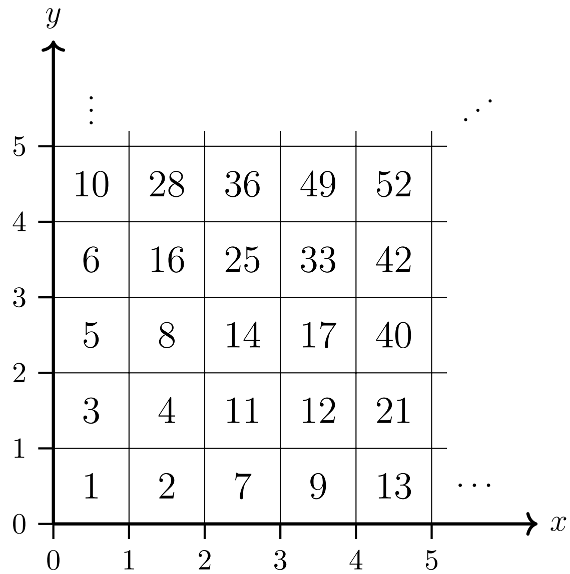


Figure 1: An infinite Young tableau.

where $n = |\lambda|$ (the number of boxes in λ) and f^λ denotes the number of standard Young tableaux of shape λ.

The Plancherel measure space $(\Omega, \mathcal{F}, P)$ has a deep connection to two fundamental operations from algebraic combinatorics and representation theory: the *Robinson–Schensted–Knuth (RSK) map* [Rob38; Sch61; Knu70], and the *jeu de taquin (JDT) map* of Schützenberger [Sch77]. An infinite version of the RSK correspondence was first considered, in a more general setup, by Kerov and Vershik [KV86] in their study of the characters of the infinite symmetric group. In the infinite tableau setting, the two operations are defined as follows:

- The RSK map is a map $\mathrm{RSK} : [0,1]^{\mathbb{N}} \to \Omega$ defined as the projective limit of the RSK maps on finite sequences of real numbers, in the following sense: let $\mathbf{z} = (z_n)_{n=1}^{\infty} \in [0,1]^{\mathbb{N}}$. For each $n \geq 1$, let $(\lambda^{(n)}, P_n, Q_n)$ be the triple associated to the finite sequence $z_1, \ldots, z_n$ by the usual RSK correspondence; this means that $\lambda^{(n)}$ is a Young diagram with n boxes (the "RSK shape"), P_n is a semistandard Young tableau of shape $\lambda^{(n)}$ with the entries $z_1, \ldots, z_n$ (the "insertion tableau"), and Q_n is a standard Young tableau of shape λ (the "recording tableau"). Then $\mathrm{RSK}(\mathbf{z})$ is defined as the (necessarily unique) IYT $T = (T_{x,y})_{x,y \geq 0}$ whose restriction to the boxes of the Young diagram $\lambda^{(n)}$ is equal to Q_n for any $n \geq 1$, if such an IYT exists.

  The value $\mathrm{RSK}(\mathbf{z})$ is well-defined for *almost every* (but not every) input $\mathbf{z}$, with respect to the product of Lebesgue measures on $[0,1]^{\mathbb{N}}$.

- The jeu de taquin map is a map $J : \Omega \to \Omega$ that acts on an IYT through the following sequence of operations (see Figure 2a):

  1. remove the box with entry $T_{0,0} = 1$ at position $(0,0)$.
  2. Perform Schützenberger promotion, consisting of a sequence of sliding moves involv-

ing the chain of boxes with positions

$$(x_0, y_0) \leftarrow (x_1, y_1) \leftarrow (x_2, y_2) \leftarrow \dots \tag{1}$$

defined by the recurrence

$$(x_n, y_n) = \begin{cases} (x_{n-1} + 1, y_{n-1}) & \text{if } T_{x_{n-1}+1,y_{n-1}} < T_{x_{n-1},y_{n-1}+1} \\ (x_{n-1}, y_{n-1} + 1) & \text{if } T_{x_{n-1}+1,y_{n-1}} > T_{x_{n-1},y_{n-1}+1} \end{cases} \qquad (n \geq 1),$$

with the initial condition $(x_0, y_0) = (0, 0)$; the box (x_n, y_n) with entry T_{x_n,y_n} slides into position (x_{n-1}, y_{n-1}) for each $n \geq 1$. (The sequence of boxes $((x_n, y_n))_{n=0}^{\infty}$ is called the *jeu de taquin path* of T; it is shown in blue in Figure 2a.) In words, at each step the path moves to whichever of the two neighbours $(x_{n-1} + 1, y_{n-1})$ and $(x_{n-1}, y_{n-1} + 1)$ holds the smaller entry.

3. Decrease all entries by 1.

Below, we denote by $\mathcal{B}$ the Borel σ-algebra on $[0, 1]^{\mathbb{N}}$, and by Λ the product of Lebesgue measures on $[0, 1]^{\mathbb{N}}$. We summarize the main results of [RŚ15]:

Theorem 1.1. *1. The Plancherel measure P is the push-forward of the infinite-dimensional Lebesgue measure Λ on $[0, 1]^{\mathbb{N}}$ under the map $\operatorname{RSK}(\cdot)$.*

2. *The jeu de taquin map J is a measure-preserving transformation on the Plancherel measure space $(\Omega, \mathcal{F}, P)$.*

3. *The jeu de taquin map is intertwined with the shift map S on $[0, 1]^{\mathbb{N}}$ (defined by $S(z_1, z_2, \dots) = (z_2, z_3, \dots)$) by the map $\operatorname{RSK}(\cdot)$, in the sense that the relation*

$$J \circ \operatorname{RSK} = \operatorname{RSK} \circ S$$

holds Λ-almost surely on $[0, 1]^{\mathbb{N}}$.

4. *The RSK map $\operatorname{RSK}(\cdot)$ is an isomorphism between the two measure-preserving dynamical systems $([0, 1]^{\mathbb{N}}, \mathcal{B}, \Lambda, S)$ and $(\Omega, \mathcal{F}, P, J)$.*

5. *The inverse map $\operatorname{RSK}^{-1}(\cdot)$ is given explicitly by*

$$\operatorname{RSK}^{-1}(T) = (\zeta(T), \zeta \circ J(T), \zeta \circ J^2(T), \dots) \quad \textit{for } P\textit{-almost every } T \in \Omega, \tag{2}$$

where

$$\zeta(T) = 1 - F\left(\lim_{n \to \infty} \frac{x_n - y_n}{\sqrt{T_{x_n,y_n}}}\right), \tag{3}$$

where $((x_n, y_n))_{n=0}^{\infty}$ denotes the jeu de taquin path associated with T as in (1) and $F(u)$ is given by

$$F(u) = \frac{1}{2\pi} \int_{-2}^{u} \sqrt{4 - s^2}\, ds \qquad (-2 \leq u \leq 2). \tag{4}$$

(This statement includes the assertion that the limit in (3) exists and takes a value in $[-2, 2]$ P-almost surely.)

A single application of J is illustrated in Figure 3 in Section 3.

1.2 The jeu de taquin forest and key results

Note that $F(\cdot)$ in (4) is the cumulative distribution function of the semicircle distribution on $[-2,2]$. The fact that the limit in the definition of $\zeta(T)$ exists almost surely for a Plancherel-random IYT T is equivalent to the statement that the jeu de taquin path of T is almost surely asymptotically a straight line, with the formulas (3)–(4) being a convenient way to describe the distribution of the random direction of this line. (See Theorem 1.1 of [RŚ15] for an alternative description of the law of this random variable.)

One surprising aspect of these results is that, while in the finite setting the usual RSK correspondence is a combinatorial bijection that requires both the insertion tableau and recording tableau in order to invert the map and recover the original sequence $z_1, \ldots, z_n$, in the infinite setting the insertion tableau is no longer needed. (Moreover, in the infinite setting it is not even meaningful anymore to talk about an insertion tableau.) Furthermore, remarkably, the recovery process for the infinite sequence consists of iterating the JDT map J and calculating the asymptotic directions of the jeu de taquin paths arising with each iteration. Each of these asymptotic directions produces, via the linearizing map $u \mapsto 1 - F(u)$, one additional value in the input sequence $z_1, z_2, \ldots$.

In other words, if one "looks" at the tableau $T = \mathrm{RSK}(\mathbf{z})$, one can "see" the initial value z_1 of the input sequence in a relatively direct way—it is given by $\zeta(T)$, as stated in (2). However, the inversion formula (2) suggests that it ought to be difficult or impossible to "see" the next value z_2, or any of the other values in the sequence $\mathbf{z}$, without first calculating the tableau $J(T)$.

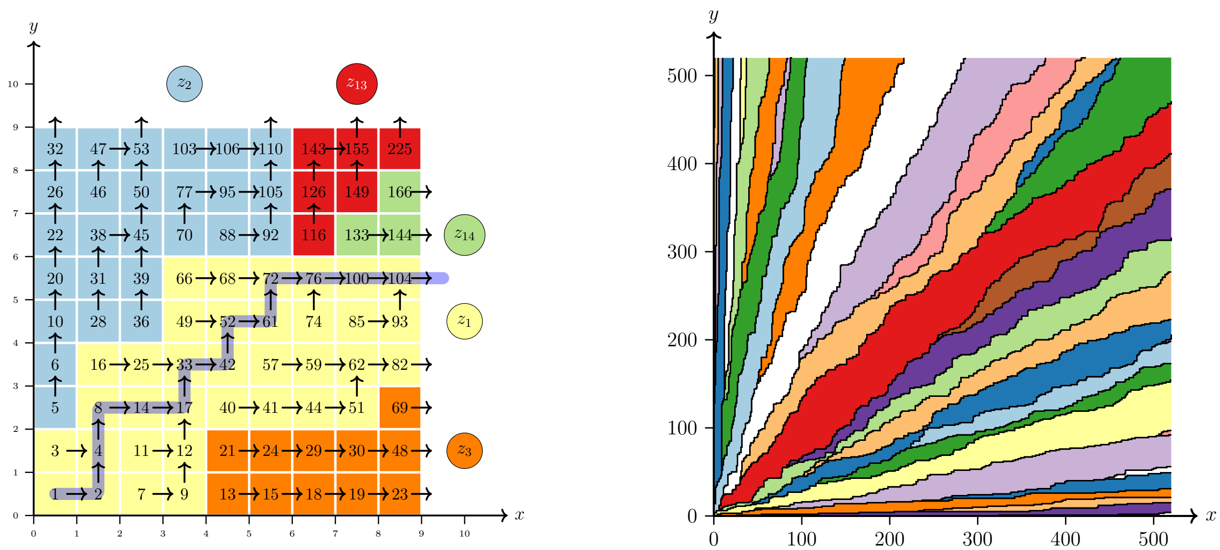
One of the goals of this paper is to show that this intuition is false, in the following sense: the set of numbers $z_1, z_2, \ldots$, with some partial information about their ordering, *can* in fact be seen directly by looking at the tableau T, without applying the JDT map J. In particular, the values $z_2, z_3, \ldots$ can be calculated from the asymptotic directions of certain tree substructures embedded within T, in the same way that z_1 is related to the asymptotic direction of the jeu de taquin path.

To make this idea precise, we introduce the notion of the *jeu de taquin forest*. Given an IYT $T = (T_{x,y})_{x,y \geq 0}$, we associate with each box (x,y) of T an oriented edge pointing either towards $(x+1,y)$ or $(x,y+1)$, according to whether $T_{x+1,y} < T_{x,y+1}$ or $T_{x+1,y} > T_{x,y+1}$. This is the same local rule that defines the jeu de taquin path. This turns $\mathbb{N}_0^2$ into a directed graph in which every box has exactly one outgoing edge. The underlying undirected graph is acyclic: the entries strictly increase along each directed edge, so the box with the smallest entry on a putative cycle would have both of its incident cycle edges outgoing, contradicting the uniqueness of the outgoing edge. The graph is therefore a (directed) forest. We call this the jeu de taquin forest of T, and denote it by $\Phi(T)$; see Figure 2a.

The JDT forest of T is made up of a disjoint union of directed trees corresponding to the connected components of the underlying undirected forest. Since every box has an outgoing edge, each tree contains an infinite directed path; in particular, every tree of the forest is infinite. One of our key results states that these trees are in a bijective correspondence with the numbers $z_1, z_2, \ldots$ produced by the inverse RSK map $\mathrm{RSK}^{-1}(T)$. The statement is as follows.

Theorem 1.2 (Partial inverse of the RSK map)**.** *Let T be a Plancherel-distributed random IYT, and let $(z_1, z_2, \ldots) = \mathrm{RSK}^{-1}(T)$ be the associated sequence of i.i.d. $U(0,1)$-distributed random variables. The following statements hold almost surely:*

1. *Each of the trees τ in $\Phi(T)$ possesses an asymptotic direction $\alpha_{\text{asym}}(\tau)$ (see Section 4.2 for the definition).*



(a) (b)

Figure 2: The jeu de taquin forest of a Plancherel-distributed IYT T, shown at two scales. (a) A microscopic window: a portion of T together with its forest $\Phi(T)$, whose oriented edges are drawn as arrows. The background colors indicate the trees of the forest, each associated with an entry z_i through the bijection described in Theorem 1.3 (boundary annotations); the jeu de taquin path of T is highlighted in blue. (b) A macroscopic view of the same realization: each tree occupies a thin wedge-shaped region extending toward its asymptotic direction, and the wedges crowd together near the two coordinate axes, reflecting the concentration of asymptotic directions. The same realization recurs throughout the paper's illustrations; see also Figures 3, 6 and 10 and the permutation example of Section 6.4.

2. *We have the equality of sets*

$$\left\{\alpha_{\text{asym}}(\tau) \,:\, \tau \textit{ is a tree in } \Phi(T)\right\} = \{z_1, z_2, \ldots\}.$$

3. *Distinct trees have distinct asymptotic directions; in particular, the map $\tau \mapsto \alpha_{\text{asym}}(\tau)$ is a bijection between the set of trees of $\Phi(T)$ and the set $\{z_1, z_2, \ldots\}$.*

Theorem 1.2 shows that, for a Plancherel-random IYT T, the preimage $(z_1, z_2, \ldots)$ of T under the inverse RSK map is determined *up to the ordering of the z_n's* by the asymptotic directions of the trees in the jeu de taquin forest of T. Moreover, some partial information about the ordering of these numbers can also be inferred by inspecting the forest: for example, z_1 is the asymptotic direction of the unique tree in the forest containing the box $(0, 0)$ at the origin (see Lemma 5.6).

The full ordering can be recovered as well, at the price of bringing back the jeu de taquin map J. As we show in Section 3, under each application of J exactly one tree of the forest — the one containing the origin — dissolves into the neighbouring trees, while all the other trees survive, only grow, and can be tracked through the transformation. The *lifetime* $n(\tau)$ of a tree τ is the number of applications of J after which τ is consumed in this way (see Section 3 for the precise definition). The following result upgrades Theorem 1.2 to a complete description of the inverse RSK map.

Theorem 1.3 (Complete inverse of the RSK map via lifetimes)**.** *Let T be a Plancherel-distributed random IYT, and let $(z_1, z_2, \ldots) = \mathrm{RSK}^{-1}(T)$ be the associated sequence of i.i.d. $U(0,1)$-distributed random variables. Then, almost surely, the lifetime map $\tau \mapsto n(\tau)$ is a bijection between the set of trees of $\Phi(T)$ and the positive integers $\mathbb{N}$, and the numbers z_n are given by*

$$z_n = \alpha_{\text{asym}}(\tau_n) \qquad (n \geq 1),$$

where τ_n denotes the unique tree with lifetime $n(\tau_n) = n$.

1.3 Related constructions of directed graphs

Directed graph structures defined by local comparison rules on tableau entries have appeared in related contexts. Fang [Fan15] studied a *trace forest of jeu de taquin* for finite skew tableaux with equivalent local rules (after rotation), but focused on Littlewood–Richardson coefficients rather than infinite tableaux or asymptotic directions. In the study of last passage percolation, the notion of *geodesic coalescence* generates similar tree structures [CP13; FP05; GRAS17].

1.4 Structure of the paper

In the process of proving Theorems 1.2 and 1.3, we will prove many more interesting facts that reveal subtle aspects of the structure of jeu de taquin forests and the way they interact with the jeu de taquin map, and are of independent interest. Thus, the paper does not have a clearly defined "main result".

The outline is as follows: in Section 2 we develop the basic combinatorics of jeu de taquin trajectories, trees, and their cusps, and in Section 3 we study how the forest evolves under the jeu de taquin map; both sections concern an arbitrary IYT and use no probabilistic (or deterministic) assumptions. Section 4 recalls some standard facts from the theory of Plancherel measure, and recasts them in geometric language by defining a coordinate system that relates in a useful way to the geometry of jeu de taquin paths — a coordinate system which we label *RSK-polar coordinates*. Section 5 proves additional results on jeu de taquin forests that are specific to a deterministic class of IYTs we call *Plancherel-normal*, which can be interpreted as "typical" random tableaux sampled from Plancherel measure. Theorems 1.2 and 1.3 are proved in Section 5.7.

The final section, Section 6, is more speculative in nature, and contains a detailed discussion of several open problems and conjectures that arise from the ideas introduced in the current paper. This makes the point that jeu de taquin forests are an important new direction for the study of infinite Young tableaux, and that there is much more still to study and understand about them.

2 Jeu de taquin trajectories, trees, and cusps

In this section we work in a purely combinatorial setting, without any probability and without invoking the jeu de taquin map. Throughout the section, T denotes a fixed (deterministic) IYT.

2.1 Trajectories and trees

We generalize the jeu de taquin trajectories studied in [RŚ15], which focused on paths starting from the origin, by considering trajectories starting at arbitrary initial boxes.

Definition 2.1 (Jeu de taquin trajectory)**.** Let T be an IYT and $(x, y) \in \mathbb{N}_0^2$ a box. The *jeu de taquin trajectory* $\mathbf{p}_{x,y}^T$ is the unique infinite path in the jeu de taquin forest of T that starts at (x, y) and follows the directed edges.

Since the jeu de taquin forest consists of trees with edges pointing away from the origin, each trajectory extends toward infinity. The trajectory from the origin, $\mathbf{p}_{0,0}^T$, coincides with the classical jeu de taquin path studied in [RŚ15].

Trajectories are governed by an elementary confluence property.

Lemma 2.2 (Trajectories in a common tree coincide)**.** *Two boxes lie in the same tree of* $\Phi(T)$ *if and only if their jeu de taquin trajectories eventually coincide.*

Proof. If two trajectories share a box then, every box having a single outgoing edge, they coincide from that box onward; in particular their starting boxes lie in the same tree. Conversely, "eventually coinciding" is a transitive relation on boxes, so it suffices to verify it for two boxes u, v joined by an edge of $\Phi(T)$. That edge is oriented one way, say from u to v; then v is the second box of the trajectory of u, so the trajectory of u coincides with that of v from v onward. Applying transitivity along the finite path joining any two boxes of a common tree, their trajectories eventually coincide. □

2.2 The lazy parametrization

We parametrize trajectories using the *lazy parametrization* of [RŚ15], adapted to our slightly more general setting. Let $t_0 = T_{x,y}$ be the entry at the starting box (x, y); since entries increase along the trajectory, t_0 is the smallest entry occurring on it. For each integer $t \geq 0$ we define $\mathbf{p}_{x,y}^T[t] \in \mathbb{N}_0^2$ to be the last box along the trajectory whose entry is at most $\max(t, t_0)$: the box rests at its starting position (x, y) until time t_0, and thereafter advances one step each time t reaches the entry of the next box to the north or east along the path. We write the parameter in square brackets to distinguish the lazy parametrization $t \mapsto \mathbf{p}_{x,y}^T[t]$ from the trajectory $\mathbf{p}_{x,y}^T$ itself, defined in Definition 2.1 as its underlying set of boxes.

The point of the parametrization is that it lets us compare several trajectories at a common *time* t by recording their positions on the boundary of the Young diagram

$$\lambda^{(t)} = \{(x', y') : T_{x',y'} \leq t\}.$$

2.3 Comparison of trajectories

2.3.1 Partial orders on the quarter-plane

We introduce a partial order that will be essential for comparing trajectories and their asymptotic directions.

Define a partial order on $\mathbb{N}_0^2 \setminus \{(0,0)\}$ by

$$(x_1, y_1) \preceq (x_2, y_2) \iff x_1 \geq x_2 \text{ and } y_1 \leq y_2. \tag{5}$$

This order has a natural geometric interpretation: $(x_1, y_1) \preceq (x_2, y_2)$ means that (x_1, y_1) is weakly to the northwest of (x_2, y_2).

We will also use a separate coordinatewise order on $\mathbb{N}_0^2$, defined by

$$(x_1, y_1) \trianglelefteq (x_2, y_2) \iff x_1 \leq x_2 \text{ and } y_1 \leq y_2,$$

under which (x_2, y_2) lies weakly to the northeast of (x_1, y_1).

2.3.2 Convex corners

A box in a Young diagram is called a *convex corner* (also called a *removable box* or *inner corner*) if it can be removed while still leaving a valid Young diagram. The next lemma states two elementary properties of convex corners; since the claims are geometrically obvious, we do not include a proof.

Lemma 2.3 (Convex corners)**.**

(i) Any two convex corners of a Young diagram are comparable with respect to the partial order $\preceq$ defined in (5).

(ii) If (x_1, y_1) and (x_2, y_2) are two distinct convex corners of a Young diagram, then their Russian coordinates $u_i = x_i - y_i$ differ by at least 2: $|u_1 - u_2| \geq 2$.

The lazy parametrization reveals important geometric properties of trajectories, particularly their relationship to convex corners of Young diagrams.

Lemma 2.4 (Trajectories and convex corners)**.** *For any trajectory $\mathbf{p}_{x,y}^T$ and any $t \geq T_{x,y}$, the position $\mathbf{p}_{x,y}^T[t]$ is a convex corner of the Young diagram $\lambda^{(t)}$.*

Proof. Let $(x', y') = \mathbf{p}_{x,y}^T[t]$. By definition of the lazy parametrization, (x', y') is the last box along the trajectory with entry at most t, hence $(x', y') \in \lambda^{(t)}$. The next box in the jeu de taquin path is either the eastern neighbor $(x'+1, y')$ or the northern neighbor $(x', y'+1)$, whichever contains the smaller entry in T. It follows that the entry in the next box is

$$m = \min\left\{T_{x'+1,y'},\ T_{x',y'+1}\right\}.$$

Since (x', y') is the *last* box with entry at most t, the next entry on the trajectory satisfies $m > t$. Therefore, both $T_{x'+1,y'} > t$ and $T_{x',y'+1} > t$, which means both $(x'+1, y')$ and $(x', y'+1)$ lie outside $\lambda^{(t)}$; that is an equivalent description of what it means for (x', y') to be a convex corner. $\square$

2.3.3 Order preservation under the lazy dynamics

The next lemma states the useful fact that the partial order between two lazy trajectories, once present, is never lost.

Lemma 2.5 (Order preservation)**.** *Let T be an IYT and let $\mathbf{p}_a^T, \mathbf{p}_b^T$ be the lazy trajectories starting at boxes $a, b \in \mathbb{N}_0^2$. If $\mathbf{p}_a^T[s_0] \preceq \mathbf{p}_b^T[s_0]$ for some integer $s_0 \geq 0$, then*

$$\mathbf{p}_a^T[t] \preceq \mathbf{p}_b^T[t] \qquad \textit{for all } t \geq s_0.$$

Moreover, once the two positions coincide they remain equal at all later times.

Proof. Write $p[t] = \mathbf{p}_a^T[t]$ and $q[t] = \mathbf{p}_b^T[t]$. It suffices to prove the one-step claim that $p[t] \preceq q[t]$ implies $p[t+1] \preceq q[t+1]$; the assertion then follows by induction on $t \geq s_0$. Two facts are used repeatedly: along any trajectory each coordinate is non-decreasing in t (a step increases x or y by one, and a waiting box does not move); and $\lambda^{(t)}$ is a Young diagram, hence *down-closed* — if $(x', y') \in \lambda^{(t)}$ then every box $\trianglelefteq (x', y')$ also lies in $\lambda^{(t)}$.

Assume $p[t] \preceq q[t]$.

Case 1: at least one of the two boxes is still waiting at time t, say b, so that $q[t] = b \notin \lambda^{(t)}$. From $p[t] \preceq b$ we have $x(p[t]) \geq x_b$ and $y(p[t]) \leq y_b$. If $p[t]$ were active with $y(p[t]) = y_b$, then $b \trianglelefteq p[t]$, and down-closedness would force $b \in \lambda^{(t)}$, a contradiction; hence either $p[t]$ also waits or $y(p[t]) < y_b$. In either situation a single step keeps $x(p[t+1]) \geq x_b$ and $y(p[t+1]) \leq y_b$, while $q[t+1] = b$ (a box does not move on the step at which it is born). Thus $p[t+1] \preceq q[t+1]$. The case where a waits is symmetric, with the roles of the two coordinates exchanged.

Case 2: both boxes are active at time t. Then $p[t], q[t]$ are convex corners of $\lambda^{(t)}$ (Lemma 2.4), as are $p[t+1], q[t+1]$ of $\lambda^{(t+1)}$. If $p[t] = q[t]$, the trajectories have merged and thereafter follow identical rules, so they remain equal. Otherwise $p[t] \preceq q[t]$ yields $u(p[t]) \geq u(q[t])$ for the Russian coordinate $u = x - y$, with gap at least 2 (Lemma 2.3(ii)). Each coordinate, and hence u, changes by at most 1 along a trajectory per unit time, so

$$u(p[t+1]) - u(q[t+1]) \geq \big(u(p[t]) - u(q[t])\big) - 2 \geq 0.$$

As $p[t+1]$ and $q[t+1]$ are comparable convex corners (Lemma 2.3(i)), the inequality $u(p[t+1]) \geq u(q[t+1])$ gives $p[t+1] \preceq q[t+1]$, with equality precisely when the two positions coincide. $\square$

2.4 The cusp of a tree

Fix an IYT T and a tree τ of its jeu de taquin forest $\Phi(T)$. The first coordinates of the boxes of τ form a non-empty subset of $\mathbb{N}_0$ and so attain a minimum $x_{\min} = \min\{x : (x, y) \in \tau\}$, realised by one or more *leftmost* boxes of τ; likewise the second coordinates attain a minimum $y_{\min}$, realised by one or more *lowest* boxes. These extremal boxes need not be unique.

Definition 2.6 (Cusp of a tree)**.** The *cusp* of the tree τ is the box

$$\operatorname{cusp}(\tau) := (x_{\min}, y_{\min}) \in \mathbb{N}_0^2,$$

formed from the minimal first and second coordinates occurring in τ.

A priori the cusp is merely a point of the quarter-plane, assembled from two separate coordinate minima that need not be attained at a common box. The main result of this subsection is that it is in fact a box of τ—indeed its unique minimal box in the coordinatewise order.

Theorem 2.7 (The cusp is the minimal box of the tree)**.** *For every IYT T and every tree τ of $\Phi(T)$, the cusp $\mathrm{cusp}(\tau)$ belongs to τ. It is the unique $\trianglelefteq$-minimal box of τ; that is, $\mathrm{cusp}(\tau) \trianglelefteq (x, y)$ for every $(x, y) \in \tau$.*

Proof. Let $\ell = (x_{\min}, y_\ell)$ be a leftmost box, $m = (x_m, y_{\min})$ a lowest box, and $c = (x_{\min}, y_{\min}) = \mathrm{cusp}(\tau)$. Directly from the definition of $\preceq$,

$$m \preceq c \preceq \ell,$$

since $x_m \geq x_{\min}$ and $y_{\min} \leq y_\ell$, the remaining coordinates coinciding.

Consider the lazy trajectories $\mathbf{p}^T_m$, $\mathbf{p}^T_c$, $\mathbf{p}^T_\ell$. All tableau entries are positive, so at time $t = 0$ each rests at its starting box; hence $\mathbf{p}^T_m[0] = m \preceq c = \mathbf{p}^T_c[0] \preceq \ell = \mathbf{p}^T_\ell[0]$. By Lemma 2.5 (applied to each of the two pairs),

$$\mathbf{p}^T_m[t] \preceq \mathbf{p}^T_c[t] \preceq \mathbf{p}^T_\ell[t] \qquad \text{for all } t \geq 0. \tag{6}$$

The boxes m and ℓ both lie in τ. Two boxes of the same tree have trajectories that eventually merge (by Lemma 2.2), so $\mathbf{p}^T_m[t] = \mathbf{p}^T_\ell[t]$ for all sufficiently large t. For such t, (6) squeezes the middle term, $\mathbf{p}^T_m[t] \preceq \mathbf{p}^T_c[t] \preceq \mathbf{p}^T_m[t]$, whence $\mathbf{p}^T_c[t] = \mathbf{p}^T_m[t]$ by antisymmetry of $\preceq$. Thus the trajectory of c eventually coincides with that of m, and in particular meets τ. Since the trajectory from any box follows the directed edges of $\Phi(T)$ and therefore remains within the tree containing that box, we conclude $c \in \tau$.

Finally $c = (x_{\min}, y_{\min}) \trianglelefteq (x, y)$ for every $(x, y) \in \tau$ by minimality of $x_{\min}$ and $y_{\min}$, so c is the unique $\trianglelefteq$-minimal box of τ. □

Since the entries of T increase along rows and columns, they are monotone with respect to the coordinatewise order $\trianglelefteq$. Theorem 2.7 therefore has the following immediate consequence.

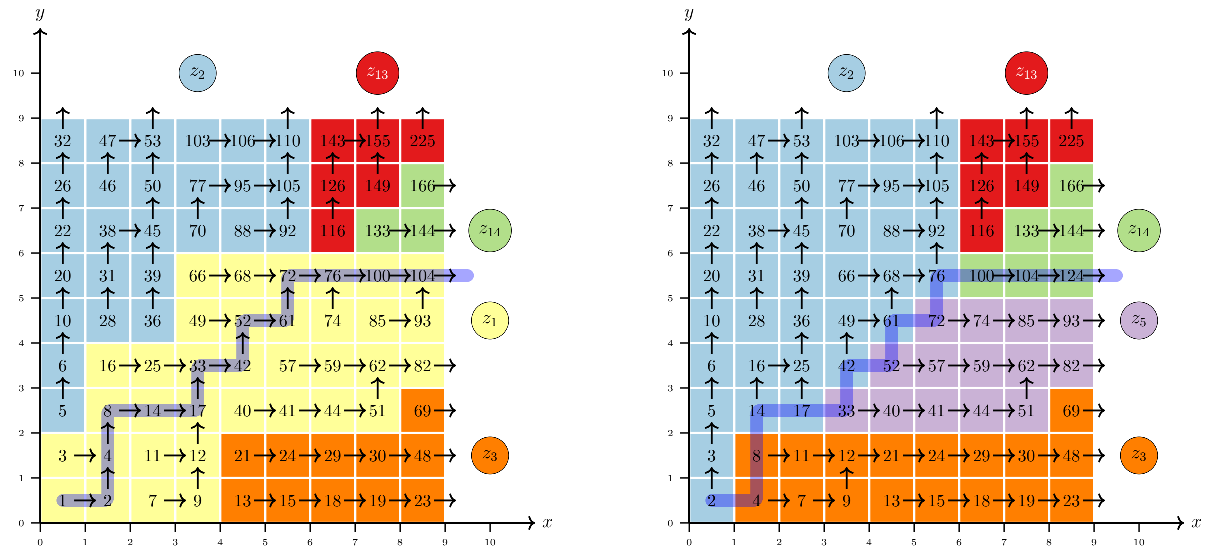
Corollary 2.8 (The cusp realizes the minimal entry)**.** *For every tree τ of $\Phi(T)$, the smallest entry occurring in τ,*

$$\min \tau := \min \{T_{x,y} : (x, y) \in \tau\},$$

is attained at the cusp: $\min \tau = T_{\mathrm{cusp}(\tau)}$.

3 Evolution of forests under jeu de taquin dynamics

In this section we bring in the jeu de taquin map J and analyse how the forest $\Phi(T)$ evolves under it. As in the previous section, T still refers to an arbitrary fixed (deterministic) IYT.



(a) (b)

Figure 3: The jeu de taquin map J acting on the tableau of Figure 2a. (a) The tableau T with its forest and jeu de taquin path. (b) The result of sliding the entries along this path, shown before the entries are decreased by 1, so that the smallest entry is 2. The tree at the origin dissolves into its neighbouring trees, while all other trees survive and only grow (Proposition 3.1).

3.1 Evolution under a single iteration of J

When the JDT map J is applied to T, the output tableau $J(T)$ has its own jeu de taquin forest structure $\Phi(J(T))$. Our first result sheds useful light on the relationship between the two forests $\Phi(T)$ and $\Phi(J(T))$. Here, we distinguish between the *tree at the origin*, which is the unique tree in $\Phi(T)$ containing the origin box $(0,0)$, and all other trees in the forest, which we refer to as *peripheral trees*. Figure 3 illustrates a single application of J and its effect on the forest.

Proposition 3.1 (Stability of peripheral trees)**.** *Let T be an IYT. There exists a canonical injection*

$$\iota\colon \big\{\textit{peripheral trees in } \Phi(T)\big\} \to \big\{\textit{all trees in } \Phi(J(T))\big\} \tag{7}$$

uniquely characterized by the following property: for each peripheral tree τ in $\Phi(T)$, τ is a subtree of its image $\iota(\tau)$ under the natural inclusion of graphs.

Proof. The map J operates by sliding boxes along the jeu de taquin path of T starting at the origin. By definition of the jeu de taquin forest, this path lies entirely within the tree at the origin. Therefore, only boxes in the tree at the origin are moved by the transformation; all other boxes remain stationary.

We now claim that for any box (x,y) in a peripheral tree of T, its outgoing edge in the JDT forest structure remains unchanged after the transformation J. Recall that the outgoing edge from (x,y) points to the smaller of its two neighbors: either to the north $(x,y+1)$ or to the east $(x+1,y)$, whichever has the smaller entry. If neither neighbor lies on the jeu de taquin path, then both entries remain unchanged (apart from a trivial global shift of values) and there is nothing to prove. We therefore consider the case where at least one of these neighbors lies on the jeu de taquin path.

Suppose the outgoing edge from (x, y) points to $(x, y+1)$, the northern neighbor. This means $T_{x,y+1} < T_{x+1,y}$. Since (x, y) belongs to a peripheral tree rather than the tree at the origin, neither (x, y) nor its northern neighbor $(x, y+1)$ can lie on the jeu de taquin path. Therefore, the eastern neighbor $(x+1, y)$ must lie on the jeu de taquin path, and its entry will be modified by sliding part of the jeu de taquin operation. However, jeu de taquin sliding replaces each entry along the path with a strictly larger value. (This refers just to the sliding step before the final step in the definition of the map J that consists of decrementing all the entries by 1, which preserves all order relations.) Consequently, after the transformation, the inequality $J(T)_{x,y+1} < J(T)_{x+1,y}$ is preserved, and the outgoing edge from (x, y) continues to point north.

The case where the outgoing edge points east is symmetric.

Since the outgoing edges from all boxes in peripheral trees of $\Phi(T)$ are preserved as edges in $\Phi(J(T))$, each peripheral tree τ in $\Phi(T)$ embeds as a subtree of some (possibly larger) tree in $\Phi(J(T))$, which we denote $\iota(\tau)$.

Moreover, we can identify the vertices of $\iota(\tau)$ as consisting of the original boxes of τ, together with those boxes (x, y) that belong to the tree at the origin in $\Phi(T)$ and whose oriented paths in the new forest $\Phi(J(T))$ eventually reach a box from τ (and remain in τ thereafter).

The map ι is an injection. Indeed, let τ, τ' be two distinct peripheral trees of T and let $u \in \tau$ and $u' \in \tau'$ be boxes. Since the outgoing edges of the boxes of peripheral trees are preserved, the trajectory of u in $\Phi(J(T))$ stays within τ, and the trajectory of u' stays within τ'; the sets τ and τ' being disjoint, the two trajectories never meet, so by Lemma 2.2 (applied to the tableau $J(T)$) the boxes u and u' lie in distinct trees of $\Phi(J(T))$. Hence $\iota(\tau) \neq \iota(\tau')$.

We constructed the map ι and verified the properties it was claimed to satisfy. It is also easy to confirm that ι is the *unique* map with these properties: since each τ is a subtree of its image $\iota(\tau)$ under the natural inclusion of graphs, and the trees of $\Phi(J(T))$ are disjoint, $\iota(\tau)$ is in fact the unique tree in $\Phi(J(T))$ that contains τ as a subtree, so this property indeed characterizes $\iota(\tau)$ uniquely. □

3.2 Finite lifetime of jeu de taquin trees

We now consider an extended scenario in which the JDT map J is applied iteratively. T will again denote a fixed deterministic IYT. For each positive integer n, we define

$$T_n = J^{n-1}(T),$$

the tableau obtained by applying the jeu de taquin transformation $n-1$ times to T. Thus $T_1 = T$, $T_2 = J(T)$, $T_3 = J^2(T)$, and so forth. Each of the tableaux T_n has its own forest structure $\Phi(T_n)$, tree at the origin, and set of peripheral trees. For each $n \geq 1$, denote the tree at the origin of $\Phi(T_n)$ by $\mathcal{O}_n$. For each $n \geq 1$, denote by γ_n the jeu de taquin path associated with T_n.

The injection ι defined in Proposition 3.1 allows us to track peripheral trees through successive applications of the jeu de taquin transformation.

Proposition 3.2 (Every tree is eventually consumed)**.** *For any tree τ in $\Phi(T)$, there exists a finite number $n = n(\tau) \in \{1, 2, \ldots\}$ such that after $n-1$ applications of the jeu de taquin transformation, the tree τ (tracked through the injections from Proposition 3.1) becomes $\mathcal{O}_n$, the tree at the* *origin in T_n. The number $n(\tau)$ satisfies the upper bound*

$$n(\tau) \leq \min_{(x,y)\in\tau} \Big(T_{x,y} - x - y - xy\Big). \tag{8}$$

We call $n(\tau)$ the *lifetime* of the tree τ. This number is the last value n for which τ can be tracked using the injection maps ι over $n-1$ iterations of the map J to be naturally associated with a tree of T_n. After one additional iteration of the jeu de taquin map applied to T_n, the tree is consumed.

Proof. Let (x, y) be a box of τ. The entry $T_{x,y}$ must satisfy

$$T_{x,y} \geq (x+1)(y+1), \tag{9}$$

since $T_{0,0} = 1$ and the entries of T are monotonically increasing along rows and columns. If there is equality in (9), then (x, y) must lie on the jeu de taquin path γ_1 of T, since otherwise, after applying the jeu de taquin map, the tableau $J(T)$ will have entry $J(T)_{x,y} = (x+1)(y+1) - 1$ at (x, y), which violates the inequality (9) with respect to $J(T)$. Since γ_1 is contained in $\mathcal{O}_1$, this means that $n(\tau) = 1$.

By similar reasoning, in the general case where (9) may be a strict inequality, since each successive application of the jeu de taquin map in which (x, y) does not lie on the jeu de taquin path has the effect of decreasing the entry at (x, y) by 1, it follows that after some number $n-1$ of successive applications of the jeu de taquin map that is at most $T_{x,y} - (x+1)(y+1)$, we will arrive at a tableau in which (x, y) lies on the jeu de taquin path γ_n. This means that the tree τ, tracked through the injections from Proposition 3.1, has become $\mathcal{O}_n$, the tree at the origin in T_n. This establishes that the tree lifetime $n(\tau)$ is well-defined, finite, and satisfies the upper bound $n(\tau) \leq T_{x,y} - (x+1)(y+1) + 1 = T_{x,y} - x - y - xy$. Since this bound holds for *any* box (x, y) in τ, taking the minimum over all boxes gives (8). □

3.3 The maximal extension of a tree

Another useful notion is the *maximal extension* of a tree τ. This object captures the full extent of τ by following it forward through the sequence of JDT map iterations until it arrives at the origin.

Definition 3.3 (Maximal extension)**.** Given an IYT T and a tree τ in $\Phi(T)$, the *maximal extension* of τ is defined by

$$\tau_{\max} = \mathcal{O}_n,$$

where as before $n = n(\tau)$ is the lifetime of τ and $\mathcal{O}_n$ is the tree at the origin in $\Phi(T_n)$.

As we track τ through the sequence of injections ι as it evolves under iterations of J, ultimately becoming the tree at the origin in $\Phi(T_n)$, each step only makes the tree grow, so in particular we have $\tau \subseteq \tau_{\max}$. The maximal extension can thus be viewed as the closure of τ with respect to jeu de taquin dynamics: it represents the maximal set of boxes that will eventually be absorbed into τ as it evolves toward the origin.

Remark 3.4 (Maximal extension in transformed tableaux)*.* If τ is a tree in the jeu de taquin forest of the transformed tableau $T_k = J^k(T)$ for some $k \geq 1$, let $n(\tau)$ denote the lifetime of τ relative to T_k, and let $\tau_{\max}$ denote its maximal extension. Then $\tau_{\max} = \mathcal{O}_m$ where $m = n(\tau) + k \geq k$.

3.4 Absorption of boxes into peripheral trees

The following lemma locates the boxes absorbed from the tree at the origin into peripheral trees during a single jeu de taquin transformation (cf. Figure 3).

Lemma 3.5 (Absorbed boxes)**.** *Let T be an IYT. For each peripheral tree τ in T, the set of boxes absorbed during the first jeu de taquin transformation*

$$\iota(\tau) \setminus \tau$$

is contained in $\mathcal{O}_1 \cap \mathcal{O}_n$ for some $n \geq 2$.

Proof. Let τ be a peripheral tree in T. The boxes in $\iota(\tau) \setminus \tau$ are precisely those absorbed by τ during the first jeu de taquin transformation. By the explicit description of the boxes of $\iota(\tau)$ in the proof of Proposition 3.1, these boxes must originate from the tree at the origin in T. Therefore,

$$\iota(\tau) \setminus \tau \subseteq \mathcal{O}_1. \tag{10}$$

On the other hand, after the first transformation, $\iota(\tau)$ is a tree in $T_2 = J(T)$. By Remark 3.4, the maximal extension of $\iota(\tau)$ is the tree at the origin in some tableau T_n where $n \geq 2$. That is,

$$[\iota(\tau)]_{\max} = \mathcal{O}_n$$

for some $n \geq 2$. Since $\iota(\tau) \subseteq [\iota(\tau)]_{\max}$, we have

$$\iota(\tau) \setminus \tau \subseteq \iota(\tau) \subseteq \mathcal{O}_n. \tag{11}$$

Combining (10) and (11) gives the claim. □

4 Geometric notions for infinite Young tableaux

We now develop some useful geometric notions that will feature in our analysis of Plancherel-random IYTs in Section 5.

4.1 The LSVK limit shape and RSK-polar coordinates

The map $F(\cdot)$ defined in (4) is closely related to the famous curve established by Logan–Shepp [LS77] and, independently, Vershik–Kerov [VK77], as the limit shape of random Young diagrams under the Plancherel measure. Consider the random Young diagram formed by the boxes labeled $1, 2, \ldots, n$ in a Plancherel-distributed random infinite tableau. (That is, if one thinks in terms of the RSK map, this is the Young diagram $\lambda^{(n)}$ described in Section 1). After rescaling by $\sqrt{n}$, the boundary of these diagrams converges almost surely to a deterministic curve as $n \to \infty$. This limiting curve is the *Logan–Shepp–Vershik–Kerov (LSVK) limit shape*.

The LSVK curve is naturally expressed in the *Russian coordinates* (u, v), [Rom15, p. 35] related to the (x, y) coordinates (sometimes called *French coordinates* in this context) by

$$u = x - y, \qquad v = x + y.$$

In those coordinates, the LSVK curve takes the explicit form

$$v = \Omega(u) = \frac{2}{\pi}\left(u \sin^{-1}\left(\frac{u}{2}\right) + \sqrt{4-u^2}\right), \quad |u| \le 2.$$

The LSVK curve can be parametrized in a way that reflects its connection to the RSK correspondence by making use of the cumulative distribution function $F(\cdot)$ from (4). For $z \in [0,1]$, define

$$q_z = F^{-1}(1-z) \in [-2,2], \tag{12}$$

the $(1-z)$-quantile of the semicircle distribution on $[-2,2]$. We introduce the *RSK trigonometric functions*:

$$\mathrm{RSKcos}\, z = \frac{\Omega(q_z)+q_z}{2},$$

$$\mathrm{RSKsin}\, z = \frac{\Omega(q_z)-q_z}{2}.$$

As is immediate to see from the definitions, the map

$$z \mapsto \big(\mathrm{RSKcos}\, z, \mathrm{RSKsin}\, z\big) \qquad (z \in [0,1]) \tag{13}$$

is a parametrization of the LSVK curve, tracing it in the French coordinates from $(2,0)$ at $z=0$ to $(0,2)$ at $z=1$.

It is helpful to generalize this parametrization to a two-dimensional curvilinear coordinate system on the positive quadrant, which we call the *RSK-polar coordinates*, and denote by (ρ,α). These coordinates relate to the (u,v) and (x,y) coordinate systems by

$$\left\{\begin{array}{lcl} x &=& \rho\, \mathrm{RSKcos}\, \alpha, \\ y &=& \rho\, \mathrm{RSKsin}\, \alpha, \end{array}\right. \quad \text{or, equivalently,} \quad \left\{\begin{array}{lcl} u &=& \rho\, q_\alpha, \\ v &=& \rho\, \Omega(q_\alpha). \end{array}\right. \tag{14}$$

We call ρ the *RSK-norm* and α the *RSK-angle* of (x,y), respectively, and denote

$$\rho = \rho(x,y), \qquad \alpha = \alpha(x,y).$$

It is easy to see that the relations (14) associate to each (x,y) in the positive quadrant, except for the origin, a unique pair (ρ,α) with $\rho \ge 0$, $\alpha \in [0,1]$, and $(\rho,\alpha) \ne (0,0)$. For the case $(x,y)=(0,0)$ we have $\rho = 0$ but the α coordinate is undefined.

The next two lemmas are not used anywhere in the paper, but have not been published before, and relate in an interesting way to existing ideas appearing in the literature — see Remarks 4.3 and 4.4 below — so we think they are worth including here.

Lemma 4.1. *The Jacobian* $\frac{\partial(x,y)}{\partial(\rho,\alpha)}$ *of the RSK-polar coordinate map* $(\rho,\alpha) \mapsto (x,y)$ *is given by*

$$\frac{\partial(x,y)}{\partial(\rho,\alpha)} = 2\rho.$$

Proof. Observe that the function $\Omega(u)$ satisfies the relation

$$\Omega(u) - u\Omega'(u) = \frac{2}{\pi}\sqrt{4-u^2}, \tag{15}$$

and that $\frac{dq_\alpha}{d\alpha} = -\left(\frac{1}{2\pi}\sqrt{4-q_\alpha^2}\right)^{-1}$, by (4) and (12). Therefore

$$\begin{aligned}\frac{\partial(u,v)}{\partial(\rho,\alpha)} &= \det\begin{pmatrix}\frac{\partial u}{\partial \rho} & \frac{\partial u}{\partial \alpha}\\ \frac{\partial v}{\partial \rho} & \frac{\partial v}{\partial \alpha}\end{pmatrix} = \det\begin{pmatrix} q_\alpha & \rho\frac{dq_\alpha}{d\alpha}\\ \Omega(q_\alpha) & \Omega'(q_\alpha)\,\rho\frac{dq_\alpha}{d\alpha}\end{pmatrix} = \rho\frac{dq_\alpha}{d\alpha}\det\begin{pmatrix} q_\alpha & 1\\ \Omega(q_\alpha) & \Omega'(q_\alpha)\end{pmatrix}\\ &= \rho\frac{-2\pi}{\sqrt{4-q_\alpha^2}}\big(q_\alpha\Omega'(q_\alpha) - \Omega(q_\alpha)\big) = 4\rho.\end{aligned}$$

It follows that $\frac{\partial(x,y)}{\partial(\rho,\alpha)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(\rho,\alpha)} = \frac{1}{2}\cdot 4\rho = 2\rho$, as claimed. □

The parametrization (13) has a natural geometric interpretation, illustrated in Figure 4a, and described in the following lemma.

Lemma 4.2. *For any $z \in [0,1]$, the area of the curvilinear triangle $E(z)$ bounded by the x-axis, the ray from the origin to $(\mathrm{RSKcos}\, z, \mathrm{RSKsin}\, z)$, and the arc of the LSVK curve with parameters ranging in $[0,z]$, is equal to z.*

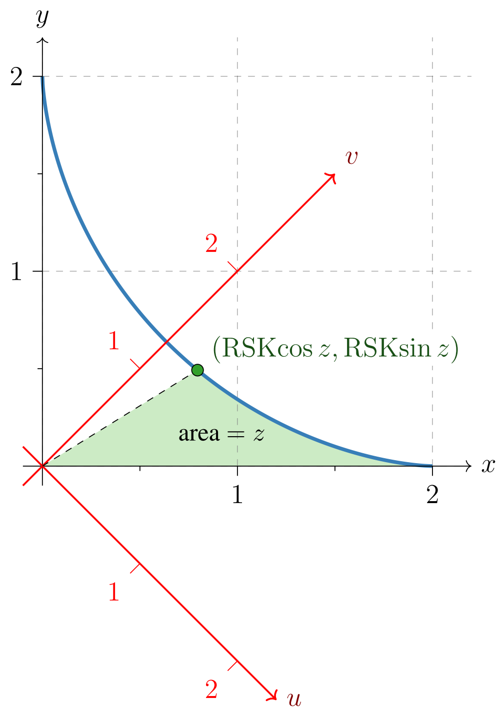
Proof. Observe that in RSK-polar coordinates the region $E(z)$ maps to the rectangle $\{(\rho,\alpha) : 0 \le \rho \le 1,\ 0 \le \alpha \le z\}$. It follows that

$$\mathrm{area}(E(z)) = \iint_{E(z)} dx\,dy = \int_0^z \int_0^1 \left|\frac{\partial(x,y)}{\partial(\rho,\alpha)}\right| d\rho\, d\alpha = \int_0^z \int_0^1 2\rho\, d\rho\, d\alpha = z. \qquad \square$$

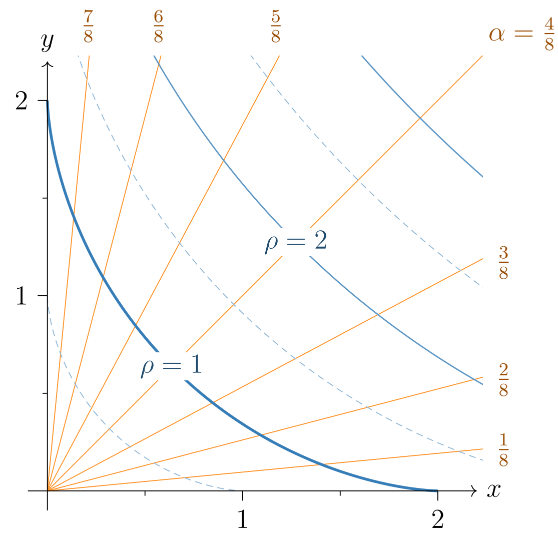
Remark 4.3. The relation (15) is equivalent to the statement that the *radial distribution* of the curve $\Omega(u)$, as defined by Kerov [Ker96, Sec. 4.4], is the semicircle distribution. This fact was used by Kerov to characterize $\Omega(\cdot)$ as the unique continual Young diagram shape whose radial distribution coincides with another natural distribution associated with the shape, known as its transition measure.

Remark 4.4. The geometric characterization in Lemma 4.2 has circulated as folklore, but we have not found a written proof in the literature. Generalizations have been discussed in the context of *"geographic coordinate systems on continual Young diagrams,"* also without proof [MŚ22, Remark 2.1].

Remark 4.5. Our parametrization (13) of the LSVK curve runs in the opposite direction to the one implicit in [RŚ15]: here the parameter increases from $z = 0$ at the point $(2,0)$ on the x-axis to $z = 1$ at the point $(0,2)$ on the y-axis, whereas in [RŚ15] the corresponding parameter traverses the curve in the reverse order. Concretely, we set $q_z = F^{-1}(1-z)$ in (12) rather than $F^{-1}(z)$; the two conventions are thus related by the substitution $z \mapsto 1-z$, the reflection of the parameter interval $[0,1]$ about its midpoint.



(a)



(b)

Figure 4: (a) The Logan–Shepp–Vershik–Kerov (LSVK) limit shape (blue curve) in dual coordinate systems. Black axes: the French coordinates (x, y). Red axes: the Russian coordinates (u, v) with $u = x - y$, $v = x + y$. The curve is parametrized by $(x, y) = (\mathrm{RSKcos}\, z, \mathrm{RSKsin}\, z)$ for $z \in [0, 1]$. The green shaded curvilinear triangle has area equal to the parameter z. (b) The RSK-polar coordinate system.

4.2 Asymptotic directions of planar sets

Definition 4.6 (Asymptotic direction)**.** Let $A \subset \mathbb{N}_0^2$ be an infinite set of boxes in the positive quarter-plane. We say that A has an *asymptotic direction* if the limit

$$\alpha_{\text{asym}}(A) := \lim_{n\to\infty} \alpha(x_n, y_n)$$

exists for any enumeration $((x_n, y_n))_{n=1}^{\infty}$ of the elements of A for which $x_n + y_n \xrightarrow[n\to\infty]{} \infty$. The limit value $\alpha_{\text{asym}}(A) \in [0, 1]$ is called the *asymptotic RSK-angle of* A, or, in a slight abuse of language, the asymptotic direction of A.

5 Plancherel-normal and Plancherel-random IYTs

5.1 Plancherel-normal tableaux

We continue using the notation T_n, $\mathcal{O}_n$, and γ_n from Section 3. Recall from Section 2 the jeu de taquin trajectories $\mathbf{p}^T_{x,y}$ and their lazy parametrization; in this notation the paths γ_n are the trajectories from the origin in the iterated tableaux T_n, that is, $\gamma_n = \mathbf{p}^{T_n}_{0,0}$.

We now extend the analysis of Section 3 to the more restricted case of Plancherel-random IYTs. It will be convenient to encapsulate the probabilistic nature of our results by stating them in the generality of a class of IYTs satisfying a certain minimal set of *deterministic* assumptions, which are known to hold for almost every IYT chosen according to Plancherel measure. We call such IYTs *Plancherel-normal*.

Definition 5.1 (Plancherel-normal tableau)**.** An IYT T is called *Plancherel-normal* if it satisfies the following conditions:

(N1) Each jeu de taquin path γ_n has an asymptotic direction

$$z_n := \alpha_{\text{asym}}(\gamma_n). \tag{16}$$

(N2) The values $z_1, z_2, z_3, \ldots$ are distinct.

(N3) The set $\{z_n : n \geq 1\}$ is dense in $[0, 1]$.

The justification for this definition is contained in the following result.

Proposition 5.2 (Plancherel-random tableaux are Plancherel-normal)**.** *Almost every IYT chosen at random according to Plancherel measure is Plancherel-normal.*

Proof. This is a consequence of the results of [RŚ15] cited in the Introduction. Specifically, let T be a random IYT chosen according to Plancherel measure. Theorem 1.1 implies that $z_1, z_2, \ldots$ are i.i.d. random variables with the uniform distribution $U(0, 1)$. This gives properties (N2) and (N3) in the definition above. Moreover, the formula (3) is equivalent to the statement that $z_1 = \alpha_{\text{asym}}(\gamma_1)$. When combined with the explicit formula (2) for the inverse map, this gives (16). $\square$

Assumption 5.3 (Standing assumption for this section)**.** *Throughout this section, T is a fixed (deterministic) Plancherel-normal IYT in the sense of Definition 5.1.*

By Proposition 5.2, any property that we prove for T under this assumption automatically holds with probability 1 for a Plancherel-random IYT.

The next lemma explains that the standing assumption is preserved under the jeu de taquin map; the proof is immediate from the definitions (removing finitely many points from a dense subset of $[0, 1]$ leaves it dense), and is omitted.

Lemma 5.4 (Plancherel-normality is preserved by jeu de taquin)**.** *For every $k \geq 1$, the iterated tableau $T_k = J^{k-1}(T)$ is again Plancherel-normal. Its jeu de taquin paths are $\gamma_k, \gamma_{k+1}, \ldots$, with respective asymptotic directions $z_k, z_{k+1}, \ldots$; in particular, the set of asymptotic directions of the jeu de taquin paths of T_k is $\{z_n : n \geq k\}$.*

5.2 Confinement of trajectories to trees

We establish that each path γ_n eventually remains within a single tree of T.

Lemma 5.5 (Eventual confinement to a single tree)**.** *For each $n \geq 1$, after removing a suitable finite initial segment from the path γ_n, the remaining tail becomes a jeu de taquin trajectory of the original tableau T. In particular, all but finitely many boxes of γ_n lie within a single tree of T.*

Proof. The path γ_n is defined in the tableau T_n, which results from applying the jeu de taquin transformation $n-1$ times to T. Our strategy is to show that γ_n eventually enters a region where T and T_n differ only by relabeling: entries differ by the constant $n-1$, but their relative order is preserved.

We start by observing a useful separation property: since T is Plancherel-normal, the asymptotic directions $z_1, \dots, z_n$ are distinct. It follows that the paths must eventually separate: there is a finite initial segment of γ_n whose removal leaves a tail R such that no box of R coincides with, or neighbors, any box visited by $\gamma_1, \dots, \gamma_{n-1}$. Indeed, choose $\epsilon > 0$ smaller than a third of the minimal gap between the numbers $z_1, \dots, z_n$. All but finitely many boxes of each γ_k (for $1 \leq k \leq n$) have RSK-angle within ϵ of z_k; moreover, far from the origin, neighboring boxes have almost equal RSK-angles. Hence the tails of the paths, together with their neighboring boxes, are eventually confined to pairwise disjoint angular sectors. The finitely many exceptional boxes are avoided by removing from γ_n a sufficiently long initial segment, since the boxes of a trajectory eventually leave any bounded region.

This separation has a crucial consequence. The $n-1$ applications of J that transform T into T_n perform sliding operations only in boxes visited by $\gamma_1, \dots, \gamma_{n-1}$. Since the boxes of R are separated from these, no sliding operation affects R or its neighbors. The only change there is a global relabeling: each entry k at a box of R in T becomes $k-(n-1)$ in T_n.

This relabeling preserves all order relations between entries, hence the local comparison rules defining the jeu de taquin forest structure remain identical in both tableaux throughout R. Therefore, the tail R of γ_n follows the same directed edges in T_n as it would in T. Being a directed path in $\Phi(T)$, the tail coincides with the trajectory $\mathbf{p}^T_{x_0,y_0}$ started at its first box (x_0, y_0), and in particular lies entirely within a single tree of the jeu de taquin forest of T. □

5.3 Asymptotic direction for the trees $\mathcal{O}_n$

The main result of this subsection is Proposition 5.7, which establishes that each tree $\mathcal{O}_n$ has a well-defined asymptotic direction equal to z_n. We first prove this for $n = 1$ in Lemma 5.6, then extend to all n by reducing to the $n = 1$ case. We freely use the partial order $\preceq$, the lazy parametrization, and the convex-corner properties of Section 2. We will also make use of the elementary observation that $\preceq$ respects RSK-angular coordinates: if $(x_1, y_1) \preceq (x_2, y_2)$ then $\alpha(x_1, y_1) \leq \alpha(x_2, y_2)$.

5.3.1 Angular confinement of the tree at the origin

We now establish that the asymptotic direction of a trajectory is in fact a property of the entire tree containing that trajectory.

Lemma 5.6 (Asymptotic direction of the tree at the origin)**.** *The tree at the origin in $\Phi(T)$ has asymptotic direction z_1:*

$$\alpha_{\text{asym}}(\mathcal{O}_1) = z_1.$$

Proof. We establish that for any $\epsilon > 0$, all but finitely many boxes (x, y) in $\mathcal{O}_1$ satisfy $|\alpha(x, y) - z_1| < \epsilon$. By symmetry, it suffices to prove the lower bound $\alpha(x, y) > z_1 - \epsilon$; the upper bound follows analogously.

Our strategy is to construct comparison trajectories with known asymptotic directions and show that they bound trajectories within $\mathcal{O}_1$ using the partial order $\preceq$ that respects RSK-angular coordinates.

Step 1: Constructing a comparison trajectory. If $z_1 = 0$ then the lower bound holds trivially, since $\alpha(x, y) \geq 0$ for every box; we may therefore assume $z_1 > 0$. Since T is Plancherel-normal, the numbers $z_1, z_2, z_3, \ldots$ are dense in $[0, 1]$. Given $\epsilon > 0$, choose an index k such that $z_1 - \epsilon < z_k < z_1$.

By Lemma 5.5, after removing a finite initial segment from γ_k, the remaining tail becomes a jeu de taquin trajectory of T, lying within a single tree of $\Phi(T)$. Moreover, by possibly removing an additional finite prefix of this tail, we further restrict to a tail segment of γ_k lying within a single tree of $\Phi(T)$, and with the additional property that all its boxes (X, Y) satisfy $\alpha(X, Y) > z_1 - \epsilon$; this is possible, since $\alpha_{\text{asym}}(\gamma_k) = z_k > z_1 - \epsilon$ implies that all but finitely many boxes of γ_k satisfy this inequality. Let $\mathbf{p}$ denote this tail segment, denote its first box (x_0, y_0), and denote $\tau_0 = T_{x_0,y_0}$. Note that $\mathbf{p}$ coincides with the jeu de taquin trajectory $\mathbf{p}^T_{x_0,y_0}$ of the tableau T.

Step 2: Comparison setup. Consider a box $(x, y) \in \mathcal{O}_1$ with entry value $t_0 = T_{x,y} > \tau_0$. We establish the lower bound for all such boxes; since only finitely many boxes in $\mathcal{O}_1$ have entries at most τ_0, this suffices.

Let $\mathbf{p}' = \mathbf{p}^T_{x,y}$ denote the jeu de taquin trajectory starting at (x, y).

Step 3: Trajectory ordering. We claim that

$$\mathbf{p}[t_0] \prec \mathbf{p}'[t_0]. \tag{17}$$

Proof. By Lemma 2.4, both $\mathbf{p}[t_0]$ and $\mathbf{p}'[t_0]$ are convex corners of $\lambda^{(t_0)} = \{(x', y') : T_{x',y'} \leq t_0\}$. Therefore by Lemma 2.3(i), they are comparable with respect to $\preceq$: one is weakly northwest or weakly southeast of the other.

Assume by contradiction that $\mathbf{p}[t_0] \succeq \mathbf{p}'[t_0]$. Applying Lemma 2.5 to the trajectories $\mathbf{p}'$ and $\mathbf{p}$ (in that order) at time t_0, this ordering persists: $\mathbf{p}[t] \succeq \mathbf{p}'[t]$ for all $t \geq t_0$. By monotonicity of angular coordinates, $\alpha(\mathbf{p}[t]) \geq \alpha(\mathbf{p}'[t])$ for all $t \geq t_0$.

Now, since $\mathbf{p}$ is obtained by removing a finite prefix from γ_k, the asymptotic direction is preserved: $\alpha_{\text{asym}}(\mathbf{p}) = \alpha_{\text{asym}}(\gamma_k) = z_k$.

On the other hand, the path $\mathbf{p}'$ lies entirely within $\mathcal{O}_1$, as does γ_1. Since both paths lie in the same tree, they must eventually merge (Lemma 2.2). Therefore, they share the same asymptotic direction:

$$\alpha_{\text{asym}}(\mathbf{p}') = \alpha_{\text{asym}}(\gamma_1) = z_1.$$

Now taking limits in $\alpha(\mathbf{p}[t]) \geq \alpha(\mathbf{p}'[t])$ yields

$$z_k = \lim_{t\to\infty} \alpha(\mathbf{p}[t]) \geq \lim_{t\to\infty} \alpha(\mathbf{p}'[t]) = z_1,$$

contradicting $z_k < z_1$. Since we arrived at a contradiction, (17) must hold.

Step 4: Conclusion. From (17), we have $\mathbf{p}[t_0] \prec \mathbf{p}'[t_0] = (x, y)$. Since $\mathbf{p}[t_0]$ is one of the boxes of $\mathbf{p}$, all of which have RSK-angle exceeding $z_1 - \epsilon$, it follows that

$$z_1 - \epsilon < \alpha(\mathbf{p}[t_0]) < \alpha(x, y).$$

This holds for all $(x, y) \in \mathcal{O}_1$ with $T_{x,y} > \tau_0$. As mentioned earlier, this suffices to finish the proof. □

5.3.2 Angular confinement for all origin-containing trees $\mathcal{O}_n$

Lemma 5.6 established angular confinement for the tree $\mathcal{O}_1$ containing the origin in T. We now extend this result to all trees $\mathcal{O}_n$.

Proposition 5.7 (Asymptotic direction for all $\mathcal{O}_n$)**.** *For each $n \geq 1$, the tree at the origin in $\Phi(T_n)$ has asymptotic direction z_n:*

$$\alpha_{\text{asym}}(\mathcal{O}_n) = z_n.$$

Proof. Set $T' := T_n = J^{n-1}(T)$. T' is Plancherel-normal by Lemma 5.4. Its jeu de taquin path from the origin is $\gamma_n = \mathbf{p}^{T'}_{0,0}$ by definition, and has asymptotic direction $\alpha_{\text{asym}}(\gamma_n) = z_n$. The tree at the origin of T' is precisely $\mathcal{O}_n$. Therefore the result follows immediately by applying Lemma 5.6 to T'. □

5.4 Applications of angular confinement

Throughout this section, when we write $\tau \subseteq \tau'$, $\tau \cap \tau'$, or $\tau \setminus \tau'$ for trees in possibly different tableaux, we mean set-theoretic operations on the underlying sets of boxes (as elements of $\mathbb{N}_0^2$), ignoring the graph structure and tableau entries. This allows us to compare spatial positions of boxes across different stages of the jeu de taquin transformation.

5.4.1 Trees at the origin have finite intersections

Corollary 5.8 (Finite intersection property for $\mathcal{O}_n$)**.** *For all positive integers $i \neq j$, the intersection $\mathcal{O}_i \cap \mathcal{O}_j$ is finite.*

Proof. By the Plancherel-normality assumption, $z_i \neq z_j$. Choose $\epsilon < |z_i - z_j|/3$. By Proposition 5.7, all but finitely many boxes in $\mathcal{O}_i$ have angular coordinate in $(z_i - \epsilon, z_i + \epsilon)$, and similarly for $\mathcal{O}_j$. Since

$$|z_i - z_j| > 3\epsilon,$$

the intervals $(z_i - \epsilon, z_i + \epsilon)$ and $(z_j - \epsilon, z_j + \epsilon)$ are disjoint.

It follows that for any box $(x, y) \in \mathcal{O}_i \cap \mathcal{O}_j$, either $\alpha(x, y) \notin (z_i - \epsilon, z_i + \epsilon)$ or $\alpha(x, y) \notin (z_j - \epsilon, z_j + \epsilon)$. The sets of boxes satisfying each of these conditions are finite. Therefore, $\mathcal{O}_i \cap \mathcal{O}_j$ is finite. □

5.4.2 Finite growth of peripheral trees

In Proposition 3.1, we established that for any peripheral tree τ in T, its image $\iota(\tau)$ under the jeu de taquin transformation contains τ as a subtree. The following result shows that for Plancherel-normal tableaux, this growth is bounded: each peripheral tree acquires only finitely many new boxes.

Corollary 5.9 (Peripheral trees grow by finitely many boxes)**.** *For each peripheral tree τ in T, the set difference $\iota(\tau) \setminus \tau$ is finite.*

Proof. By Lemma 3.5, the boxes absorbed during the first transformation satisfy $\iota(\tau)\setminus\tau \subseteq \mathcal{O}_1 \cap \mathcal{O}_n$ for some $n \geq 2$. By Corollary 5.8, each intersection $\mathcal{O}_1 \cap \mathcal{O}_n$ is finite. Therefore, $\iota(\tau) \setminus \tau$ is finite. $\square$

5.4.3 Maximal extensions differ by finitely many boxes

Corollary 5.10 (Finite difference from maximal extension)**.** *For each tree τ in the jeu de taquin forest of T, the difference $\tau_{\max} \setminus \tau$ between the tree and its maximal extension is finite.*

Proof. By repeated application of the bijection ι from (7), we trace the evolution of τ through successive tableaux until it reaches the origin after finitely many steps. At each step, by Corollary 5.9, the growth is finite. Since the total number of steps is finite, the cumulative growth $\tau_{\max} \setminus \tau$ is a finite union of finite sets, hence finite. $\square$

5.4.4 Surjectivity of tree evolution

We now show that, for Plancherel-normal tableaux, the jeu de taquin transformation produces no new trees: every tree in $J(T)$ arises as the image of a peripheral tree from T.

Proposition 5.11 (Surjectivity of tree evolution)**.** *The injective map ι from (7) is a bijection. That is, every tree in $T_2 = J(T)$ is the image under ι of some peripheral tree in T.*

Proof. By Corollary 5.8, for all $n \geq 2$, the intersection $\mathcal{O}_1 \cap \mathcal{O}_n$ is finite.

Assume by contradiction that there exists a tree τ in T_2 that is not in the image of ι. Since ι maps all peripheral trees of T injectively to trees in T_2, and τ is not in the image, the boxes of τ cannot have come from any peripheral tree. By Proposition 3.1, the only other source of boxes during the transformation is the tree at the origin $\mathcal{O}_1$ in T. Therefore,

$$\tau \subseteq \mathcal{O}_1.$$

On the other hand, by Remark 3.4, the tree τ in T_2 satisfies $\tau \subseteq \tau_{\max} = \mathcal{O}_n$ for some $n \geq 2$. Combining these inclusions,

$$\tau \subseteq \mathcal{O}_1 \cap \mathcal{O}_n.$$

Since $\mathcal{O}_1 \cap \mathcal{O}_n$ is finite, we conclude that τ contains only finitely many boxes. However, as observed in Section 1, every tree in a jeu de taquin forest is infinite. This contradiction shows that no such tree τ can exist. Therefore ι is surjective, hence a bijection. $\square$

5.5 Lifetime is a bijection

Proposition 5.12. *The lifetime map*

$$\tau \mapsto n(\tau)$$

defines a bijection between the set of all trees in $\Phi(T)$ and the positive integers $\mathbb{N}$.

Proof. We establish that the lifetime map is a bijection by proving injectivity and surjectivity separately.

Injectivity. Suppose two trees τ, τ' in T have the same lifetime n. By Proposition 3.1, the map ι induces an injection between trees at each step. After $n-1$ iterations of the jeu de taquin transformation, both trees reach the origin: $\iota^{n-1}(\tau)$ and $\iota^{n-1}(\tau')$ are both trees at the origin of tableau T_n. Since there is a unique tree at the origin of any tableau, we have $\iota^{n-1}(\tau) = \iota^{n-1}(\tau')$. Since ι^{n-1} is an injection (being the composition of injections), we conclude $\tau = \tau'$.

Surjectivity. Let n be a positive integer. Consider the tree $\mathcal{O}_n$ at the origin in T_n. By Proposition 5.11, the map ι is a bijection at each step; here we use Lemma 5.4, by which each of the tableaux $T_1, \ldots, T_{n-1}$ is again Plancherel-normal, so that Proposition 5.11 applies to it. Tracing $\mathcal{O}_n$ backward through the inverse maps, we obtain a unique tree in T with lifetime equal to n.

Therefore, the lifetime map is a bijection. □

5.6 Asymptotic directions for all jeu de taquin forest trees

Proposition 5.13 (Asymptotic directions for all jeu de taquin forest trees)**.** *The trees of $\Phi(T)$ satisfy:*

(T1) Each tree τ in $\Phi(T)$ has an asymptotic direction

$$z_\tau := \alpha_{\text{asym}}(\tau) \in [0, 1].$$

(T2) Distinct trees have distinct asymptotic directions.

(T3) The set of asymptotic directions of the trees coincides with the set of inverse-RSK values:

$$\{z_\tau : \tau \textit{ a tree in } \Phi(T)\} = \{z_n : n \geq 1\}. \tag{18}$$

(T4) The set $\{z_\tau : \tau$ a tree in $\Phi(T)\}$ is dense in $[0, 1]$.

The key to establishing these properties is a bijective correspondence between trees and trajectories, which we now develop.

5.6.1 Proof via trajectory correspondence

The following result establishes that each tree corresponds to a unique jeu de taquin trajectory.

Proposition 5.14 (Trajectories correspond to trees)**.** *For each tree τ in $\Phi(T)$, its lifetime $n(\tau)$ is the unique positive integer k such that the path γ_k, after removing a finite initial segment, lies entirely within τ.*

Moreover, the asymptotic directions exist and coincide:

$$\alpha_{\text{asym}}(\tau) = \alpha_{\text{asym}}(\gamma_{n(\tau)}) = z_{n(\tau)} \tag{19}$$

for all trees τ in $\Phi(T)$.

Proof. Let τ be any tree in $\Phi(T)$, and let $k = n(\tau)$ denote its lifetime. We establish the claims about k and its uniqueness in three steps.

Step 1: The path γ_k lies in τ after a finite prefix. The maximal extension $\tau_{\max} = \mathcal{O}_k$ is the tree at the origin in $\Phi(T_k)$, which contains the jeu de taquin path $\gamma_k = \mathbf{p}_{0,0}^{T_k}$. By Corollary 5.10, the trees τ and $\tau_{\max}$ differ by finitely many boxes. Therefore, after removing a finite initial segment, the path γ_k is contained in τ.

Step 2: The asymptotic directions coincide. By Proposition 5.7, the asymptotic direction $\alpha_{\text{asym}}(\mathcal{O}_k)$ exists and equals $z_k = \alpha_{\text{asym}}(\gamma_k)$. Since τ is a subset of $\tau_{\max} = \mathcal{O}_k$, the asymptotic direction $\alpha_{\text{asym}}(\tau)$ also exists and equals the same number $z_k = \alpha_{\text{asym}}(\gamma_k)$; this establishes (19).

Step 3: Uniqueness of $n(\tau)$. Suppose that for some index ℓ, the path γ_ℓ with a finite initial segment removed is contained in τ. Then by the same reasoning as Step 2,

$$\alpha_{\text{asym}}(\tau) = \alpha_{\text{asym}}(\gamma_\ell) = z_\ell.$$

By Step 2, we also have $\alpha_{\text{asym}}(\tau) = z_{n(\tau)}$. Therefore, $z_\ell = z_{n(\tau)}$. Since T is Plancherel-normal, the values $z_1, z_2, z_3, \ldots$ are distinct, hence $\ell = n(\tau)$, establishing uniqueness. □

5.6.2 Proof of Proposition 5.13

We can now complete the proof of Proposition 5.13.

Proof of Proposition 5.13. Property (T1): This follows immediately from Proposition 5.14, which establishes that $\alpha_{\text{asym}}(\tau) = z_{n(\tau)}$ exists for each tree τ.

Property (T2): If $\tau \neq \tau'$ are distinct trees, then by injectivity of the lifetime map (Proposition 5.12), we have $n(\tau) \neq n(\tau')$. By Proposition 5.14, $z_\tau = z_{n(\tau)}$ and $z_{\tau'} = z_{n(\tau')}$. Since T is Plancherel-normal, the values $z_1, z_2, z_3, \ldots$ are distinct, hence $z_\tau \neq z_{\tau'}$.

Property (T3): By Proposition 5.14, $z_\tau = z_{n(\tau)}$ for each tree τ, so the left-hand side of (18) is contained in the right-hand side. Conversely, by surjectivity of the lifetime map (Proposition 5.12), for each $n \geq 1$ there exists a tree τ with $n(\tau) = n$, whence $z_\tau = z_n$ by the same correspondence. This proves the equality of sets (18).

Property (T4): Since T is Plancherel-normal, the set $\{z_n : n \geq 1\}$ is dense in $[0, 1]$. By (18), the same holds for the set of asymptotic directions of the trees, completing the proof. □

5.7 Proofs of Theorems 1.2 and 1.3

By Proposition 5.2, a Plancherel-distributed random IYT T is almost surely Plancherel-normal, so we may apply Proposition 5.13. Its property (T1) yields the first statement, and its property (T3) yields the equality of sets

$$\big\{\alpha_{\text{asym}}(\tau)\ :\ \tau \text{ is a tree in } \Phi(T)\big\} = \{z_1, z_2, \ldots\},$$

which is the second statement. Finally, its property (T2) yields the third statement.

Theorem 1.3 follows in the same way: Proposition 5.12 shows that the lifetime map is a bijection onto $\mathbb{N}$, and Proposition 5.14 identifies the asymptotic direction of the tree of lifetime n as z_n.

5.8 Trajectories repeatedly approach tree boundaries

A box $(x, y) \in A$ is called a *boundary box* of a set $A \subseteq \mathbb{N}_0^2$ if $(x+1, y-1)$ or $(x-1, y+1)$ lies outside of A.

Theorem 5.15 (Trajectories repeatedly approach tree boundaries)**.** *For any starting position $(x, y) \in \mathbb{N}_0^2$, the jeu de taquin trajectory $\mathbf{p}^T_{x,y}$ passes through infinitely many boundary boxes of the tree τ of $\Phi(T)$ that contains it.*

Proof. **Step 1: Reduction to paths γ_n.** The starting position (x, y) belongs to some tree τ in the jeu de taquin forest of T. By the lifetime bijection (Proposition 5.12), we have $\tau_{\max} = \mathcal{O}_n$ for some $n \geq 1$.

By Corollary 5.10, the trees τ and $\tau_{\max}$ differ by finitely many boxes. Consequently, their sets of boundary boxes differ by at most finitely many elements. Therefore, it suffices to prove that $\tau_{\max} = \mathcal{O}_n$ has infinitely many boundary boxes visited by the trajectory $\mathbf{p}^T_{x,y}$.

By Lemma 5.5, after a finite initial segment the path γ_n follows the forest edges of T and lies within a single tree of $\Phi(T)$; by Proposition 5.14, this tree is τ, the tree of lifetime n. Two trajectories of $\Phi(T)$ lying in the same tree eventually merge, so $\mathbf{p}^T_{x,y}$ and γ_n visit the same boxes after removing finite initial segments. Therefore, it suffices to prove that for each $n \geq 1$, the path γ_n passes through infinitely many boundary boxes in $\mathcal{O}_n$.

Step 2: Reduction to the case $n = 1$. By Lemma 5.4, the tableau $T_n = J^{n-1}(T)$ is itself Plancherel-normal, with sliding paths γ_{n+k} of asymptotic directions z_{n+k}. The argument is therefore identical for all n, and it suffices to prove the result for $n = 1$. That is, we prove that $\gamma_1 = \mathbf{p}^T_{0,0}$ passes through infinitely many boundary boxes in $\mathcal{O}_1$.

Step 3: Existence of infinitely many flip boxes. We compare the outgoing connectivity of boxes in $\mathcal{O}_1$ in two different jeu de taquin forests. Call a box $(x, y) \in \mathcal{O}_1$ a *flip box* if its outgoing edge in the forest of $T_2 = J(T)$ points to a box that was peripheral in T (i.e., not in $\mathcal{O}_1$).

We claim that there exist infinitely many flip boxes. Suppose for contradiction that there are only finitely many flip boxes. Let M denote the maximum entry $(T_2)_{x,y}$ of the tableau T_2 taken over the finitely many flip boxes (x, y).

Since $\mathcal{O}_1$ is infinite, there exists a box $(x, y) \in \mathcal{O}_1$ with entry $(T_2)_{x,y} > M$. Follow the jeu de taquin trajectory $\mathbf{p}^{T_2}_{x,y}$ in the tableau T_2 from (x, y); the entries of T_2 increase along the trajectory, so every box of the trajectory has T_2-entry exceeding M and is therefore not a flip box. Starting from $(x, y) \in \mathcal{O}_1$, an induction now shows that the trajectory remains within $\mathcal{O}_1$: any box of the trajectory lying in $\mathcal{O}_1$ is a non-flip box, so its outgoing edge in the forest of T_2 points within $\mathcal{O}_1$, placing the next box of the trajectory in $\mathcal{O}_1$ as well. Therefore, this trajectory remains within $\mathcal{O}_1$ indefinitely.

This is impossible. The trajectory $\mathbf{p}^{T_2}_{x,y}$ is an infinite jeu de taquin trajectory in $T_2 = J(T)$, so all of its boxes belong to a single tree τ' of $\Phi(T_2)$. By Proposition 5.11, $\tau' = \iota(\tau)$ for some peripheral tree τ of T. Since τ is peripheral, it is disjoint from the tree $\mathcal{O}_1$ at the origin; as the trajectory lies in $\mathcal{O}_1$, it avoids τ entirely, so that

$$\mathbf{p}^{T_2}_{x,y} \subseteq \tau' \setminus \tau = \iota(\tau) \setminus \tau.$$

By Lemma 3.5, the set $\iota(\tau) \setminus \tau$ is contained in $\mathcal{O}_1 \cap \mathcal{O}_n$ for some $n \geq 2$, which is finite by Corollary 5.8. Thus the infinite trajectory $\mathbf{p}^{T_2}_{x,y}$ would be contained in a finite set, a contradiction.

Therefore, there are infinitely many flip boxes in $\mathcal{O}_1$.

Step 4: Flip boxes imply boundary box visits. We show that each flip box in $\mathcal{O}_1$ has a neighboring boundary box that γ_1 visits. Consider a flip box $(x', y') \in \mathcal{O}_1$. By definition, its outgoing edge in the forest of T_2 points to a box outside $\mathcal{O}_1$; in the forest of T, by contrast, its outgoing edge stays within its own tree $\mathcal{O}_1$.

The outgoing edge from (x', y') is determined by comparing the entries of its two neighbors: north $(x', y' + 1)$ and east $(x' + 1, y')$. The edge points to whichever neighbor has the smaller entry. For the outgoing edge to change direction, one of the two neighbors must have had its entry modified by the jeu de taquin sliding operation. This neighbor must therefore lie on the jeu de taquin path γ_1.

Without loss of generality, suppose the eastern neighbor $(x'+1, y')$ lies on γ_1. (The case where the northern neighbor lies on γ_1 is symmetric, being its mirror image under the NE–SW diagonal.) Set $(x, y) = (x' + 1, y')$; we claim that (x, y) is a boundary box of $\mathcal{O}_1$ visited by γ_1. The geometric configuration is illustrated in Figure 5, where the flip box is marked with a red square and the boundary box with $\times$.

Since (x, y) lies on γ_1, it belongs to $\mathcal{O}_1$. It remains to show that its northwestern neighbor $(x - 1, y + 1) = (x', y' + 1)$ lies outside $\mathcal{O}_1$. After the slide, the outgoing edge of the flip box (x', y') points outside $\mathcal{O}_1$, to whichever of its northern neighbor $(x', y' + 1)$ and eastern neighbor (x, y) has the smaller entry. As $(x, y) \in \mathcal{O}_1$, this edge cannot point east; hence it points north, to $(x', y'+1)$, which therefore lies outside $\mathcal{O}_1$. (This is the transition from an eastward to a northward edge depicted in Figure 5.) Consequently $(x, y) \in \mathcal{O}_1$ while $(x - 1, y + 1) \notin \mathcal{O}_1$, so (x, y) is a boundary box of $\mathcal{O}_1$.

Each flip box (x', y') thus determines an adjacent boundary box of $\mathcal{O}_1$ lying on γ_1 (namely $(x' + 1, y')$ or $(x', y' + 1)$). A given boundary box arises in this way from at most two flip boxes—its western and its southern neighbor—so this assignment is at most two-to-one. Since there are infinitely many flip boxes, we conclude that γ_1 passes through infinitely many boundary boxes of $\mathcal{O}_1$. $\square$

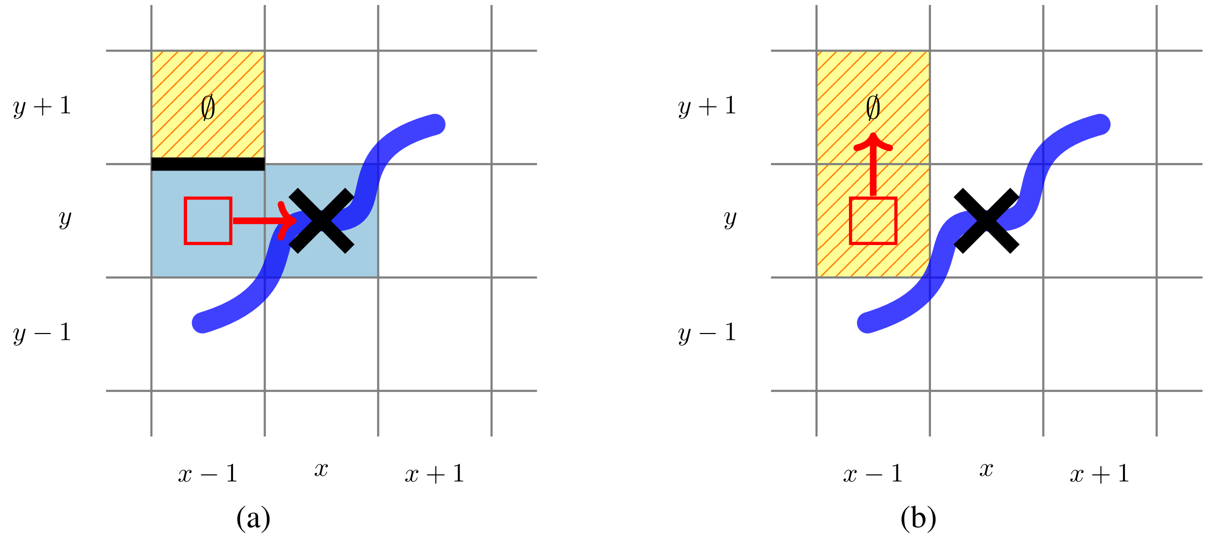


Figure 5: A jeu de taquin slide at a boundary box (x, y) (large $\times$) and its *flip box* $(x-1, y)$ (red square), illustrating Theorem 5.15. The flip box's outgoing edge is red because it changes direction across the slide: it points (a) east, into the origin tree $\mathcal{O}_1$ (light blue), before the slide, and (b) north afterwards, out of $\mathcal{O}_1$ towards $(x-1, y+1)$ (striped yellow, a neighbouring peripheral tree marked $\emptyset$): the origin tree dissolves here and (x, y) is left unpainted, its membership undetermined. The blue snake is the jeu de taquin path through (x, y), drawn vaguely as its entry and exit sides are undetermined; the complementary north $\to$ east case is the mirror image under the NE–SW diagonal.

6 Open problems

In this final section we collect a number of numerical observations and open problems suggested by the jeu de taquin forest: the point process of tree cusps and its conjectural coalescing-flow scaling limit, the level repulsion of the asymptotic directions, the hierarchical tree-of-trees structure, a random permutation of $\mathbb{N}$ comparing two orderings of the trees, and the clustering of boundary-box visits along a trajectory.

6.1 The point process of tree cusps

By Theorem 2.7, every tree τ of the jeu de taquin forest carries a well-defined *cusp* $\operatorname{cusp}(\tau)$, the $\trianglelefteq$-minimal box of τ. For a Plancherel-random IYT this produces a random point process $\{\operatorname{cusp}(\tau)\}$, indexed by the trees of the forest, whose statistics — intensity, correlations, and scaling limit — it would be interesting to describe (Figure 6). We collect here a few numerical observations, which we leave as open problems.

We use the RSK-polar coordinates (ρ, α) of Section 4.1.

First, the cusps grow sparser away from the origin. We call the smallest entry $\min \tau$ occurring in a tree τ — by Corollary 2.8, the entry at its cusp box — the *bifurcation time* of the tree. In all Monte Carlo experiments of this section, a *realization* is a Plancherel-random IYT restricted to its first $n = 10^6$ boxes, that is, to the Young diagram $\lambda^{(n)}$ of Section 1; such a realization exhibits

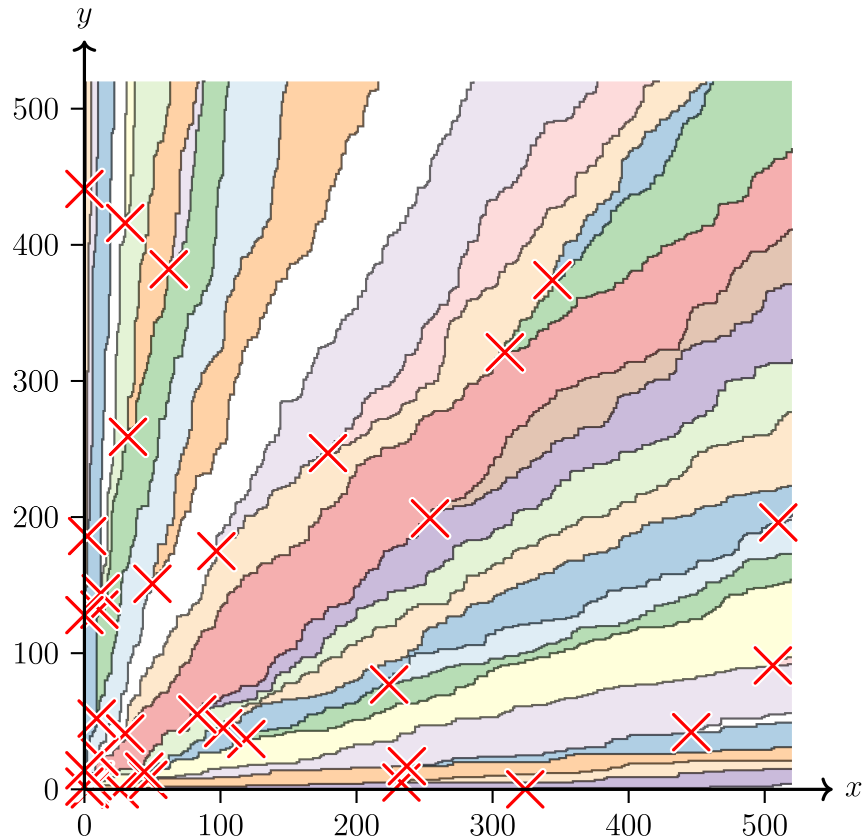


Figure 6: The point process of tree cusps for a large Plancherel-random tableau, the same realization as in Figure 2a. Each tree occupies a thin wedge-shaped region extending toward its asymptotic direction, and the cusp of a tree — its $\trianglelefteq$-minimal box, equivalently its minimum-entry box (Corollary 2.8) — is the corner where its region tapers to a point on the side of the origin. The wedge colours are subdued and a bold red cross marks each cusp, so that the cusps stand out; they grow sparser away from the origin. The wedges visibly crowd together near the two coordinate axes, reflecting the concentration of asymptotic directions discussed in the text.

precisely the cusps with bifurcation time at most n, so the truncation size n acts as a threshold on the bifurcation time. The number of cusps with bifurcation time at most t appears to grow like $K\,t^{1/4}$ as $t \to \infty$, for some absolute constant $K \approx 1.87$ whose value we do not determine (Figure 7).

Second, the asymptotic directions concentrate near the two coordinate axes $z = 0$ and $z = 1$, with a limiting law that appears singular there (Figure 8); although Proposition 5.13 shows the set of asymptotic directions to be dense in $[0, 1]$, we do not determine this law. This is not in tension with the input sequence $z_1, z_2, \ldots$ being i.i.d. uniform on $[0, 1]$: by Theorem 1.2 every tree's asymptotic direction is one of the values z_n and every z_n is realized. But Figure 8 records only the cusps inside a bounded window, ordered by bifurcation time — a geometrically selected, and hence biased, subsample rather than the full i.i.d. sequence.

The picture as a whole — the cusps as the points where the competition interfaces, drawn inward from spatial infinity, merge — is reminiscent of a system of coalescing Brownian motions,

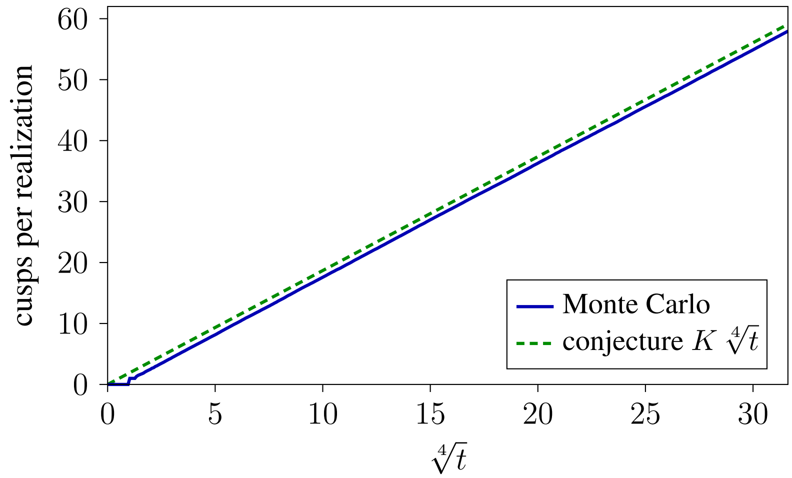


Figure 7: The number of cusps with bifurcation time at most t, per realization, against the radial coordinate $\sqrt[4]{t}$ (solid; Plancherel realizations with $n = 10^6$ boxes), compared with a straight line of slope $K \approx 1.87$ (dashed). The count grows linearly in $\sqrt[4]{t}$, the data lying a near-constant offset below the line (a finite-size correction).

in the same universality class as the exactly solvable Pólya web [BŚTU26]. Making this precise, and with it the exact growth constant K and the limiting angular law, is an open problem.

6.2 Level repulsion of the asymptotic directions

Beyond the global angular law, the asymptotic directions appear to *repel*. Fixing a threshold on the bifurcation time, forming the spacings between consecutive directions, normalizing them by the empirical mean density of the directions (estimated from all realizations, to compensate for its non-uniformity), and restricting to a bulk window of directions away from the two axes, one finds in simulations a limiting spacing law whose density vanishes — linearly — as the spacing tends to 0, conspicuously unlike the exponential law e^{-s} of independent (Poisson) points (Figure 9). Identifying this limiting spacing law is another open problem.

6.3 The tree of trees and its structure

The boundaries separating neighboring trees of the jeu de taquin forest, together with the two coordinate axes, organize the forest into a hierarchical structure. Crossing a large arc from the x-axis to the y-axis, one meets the trees in the order of their asymptotic directions; moving inward toward the origin, trees terminate at their cusps and the boundaries flanking a vanishing tree merge. Iterating this merging, the boundaries appear to form the complete infinite binary tree, rooted at the origin $(0, 0)$: its nodes are the cusps — one per tree, by Theorem 2.7 — its edges are the tree boundaries, and its ends correspond to the asymptotic directions, which are dense in $[0, 1]$ by Proposition 5.13. We have not made the notion of tree boundary precise here, and we record this binary-tree structure as a heuristic rather than a theorem (Figure 10).

In this description the combinatorial type of the tree is deterministic — it is always the infinite binary tree — while all the randomness is carried by the embedding into the quarter-plane: each

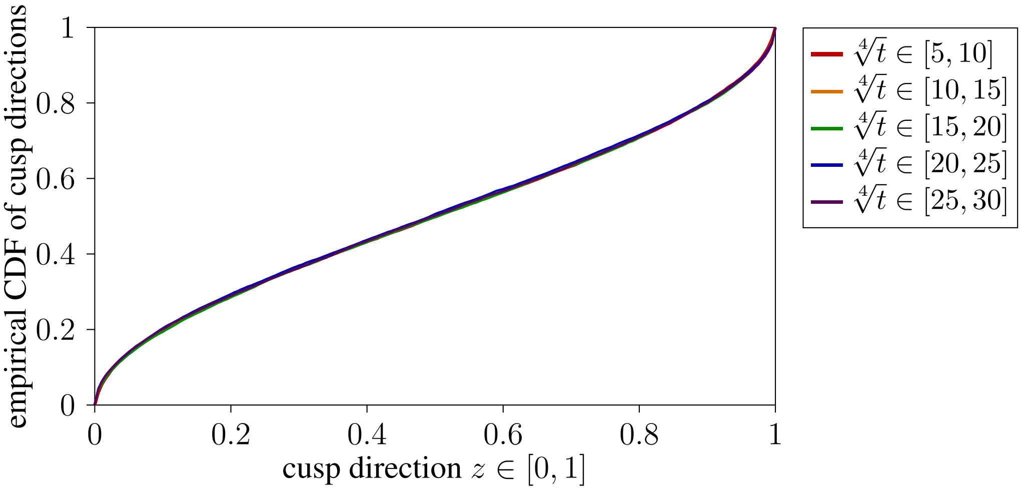


Figure 8: Empirical cumulative distribution of the asymptotic directions $z \in [0, 1]$ of the cusps, in radial shells $\sqrt[4]{t} \in [5, 30]$ (Plancherel realizations with $n = 10^6$ boxes). The curves rise steeply near the two axes $z = 0, 1$ and flatten in between — the signature of a limiting direction law that concentrates at, and is singular at, the axes. Although the input values z_n are i.i.d. uniform, this finite, geometrically selected sample of cusps is biased relative to the whole sequence.

node is placed at the cusp of its tree, so that the cusp point process is exactly the image of the node set under this embedding. The two viewpoints, geometric and combinatorial, thus describe a single object. The full "wired" tree carries more information than its nodes alone: the scaling limit of the boundaries themselves, rather than only of the cusps, should be the genealogy of a system of coalescing Brownian motions, once more in agreement with the Pólya web [BŚTU26], whose complementary boundary tree plays precisely this role.

6.4 An infinite random permutation of $\mathbb{N}$

The trees of the jeu de taquin forest can be enumerated in two natural ways, and comparing the enumerations produces a random permutation of $\mathbb{N}$.

On one hand, Proposition 5.12 assigns to each tree τ its *lifetime* $n(\tau) \in \mathbb{N}$, and by Theorem 1.2 and Proposition 5.14 the tree of lifetime n has asymptotic direction z_n; ordering the trees by lifetime is therefore the same as ordering them by the index of the input value they realize. On the other hand, the cusps furnish a purely geometric ordering: list the trees by increasing bifurcation time $\min \tau$. These values are distinct, so this is a genuine total order. Reading off the lifetime of the tree placed k-th in the cusp order defines a map

$$\sigma \colon \mathbb{N} \to \mathbb{N},$$

a bijection because the two orderings enumerate the same set of trees. The value $\sigma(1) = 1$ is forced: the tree of smallest bifurcation time is the one at the origin (it contains the box $(0, 0)$ with entry 1), and the tree at the origin has lifetime 1.

For the running example — the tableau of Figure 2a, displayed as a tree of trees in Figure 10 — reading the nodes from left to right, that is, by increasing bifurcation time $\min \tau$, and recording

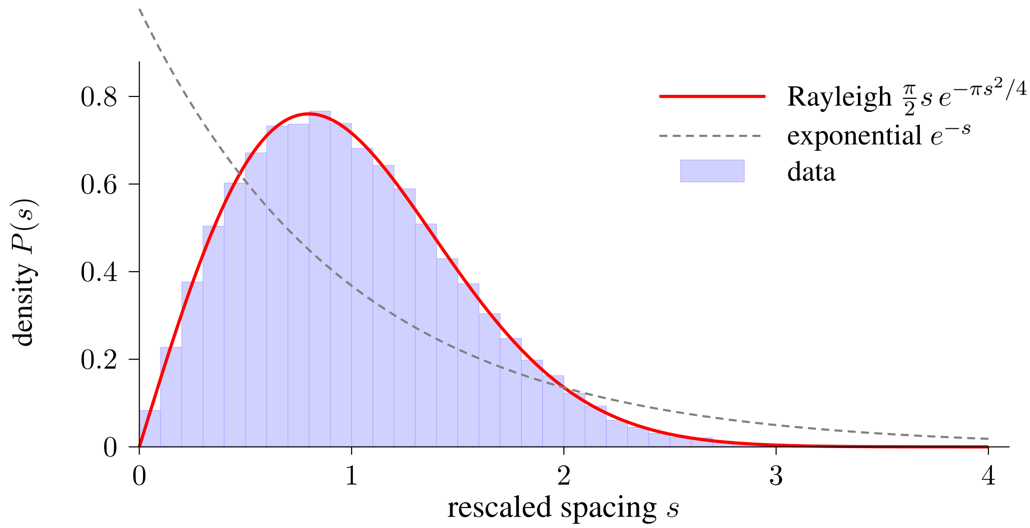


Figure 9: Histogram of the unfolded nearest-neighbor spacings between consecutive asymptotic directions: each spacing is normalized by the empirical mean density of the directions, estimated from all realizations, and only spacings in the bulk window $0.1 \leq z \leq 0.9$ are kept (204,139 spacings pooled from 5,743 Plancherel realizations with $n = 10^6$ boxes, rescaled to mean 1). The histogram is shown against the Rayleigh density $\frac{\pi}{2}\, s\, e^{-\pi s^2/4}$ (solid) and the exponential density e^{-s} of independent directions (dashed) for comparison. The linear vanishing as $s \to 0$, absent from the exponential, is the signature of level repulsion.

the lifetime n of each labelled node z_n yields the first values

$$\sigma = (1, 2, 3, 7, 13, 14, 37, 21, 5, 16, 34, 42, 9, 56, 24, \dots).$$

Already this short prefix is visibly irregular, and the lifetimes that appear do not form an initial segment of $\mathbb{N}$: a tree of small lifetime can have a large bifurcation time, having been born early but coming to rest far from the origin.

The permutation σ is a combinatorial fingerprint of the inverse RSK map, measuring how far the geometric arrangement of the cusps departs from the order in which the trees are born. In simulations it appears highly irregular, and we know of no simple description of its law; its displacement statistics $|\sigma(k) - k|$ and its cycle structure would be interesting to understand. Unlike the cusp point process and the tree structure above, σ has no counterpart in the Pólya web, which possesses cusps and a cusp order but no notion of input index: it is genuinely a feature of the RSK correspondence.

6.5 Clustering of boundary-box visits

Theorem 5.15 shows that, almost surely, every jeu de taquin trajectory passes through infinitely many boundary boxes of the tree containing it. Each boundary box is of one of two types, according to which diagonal neighbor leaves the tree: call it of type NW if $(x-1, y+1)$ lies outside the tree, and of type SE if $(x+1, y-1)$ does. It remains open whether a trajectory must visit infinitely many boxes of *each* type, or whether it might visit only finitely many of one type while visiting infinitely many of the other.

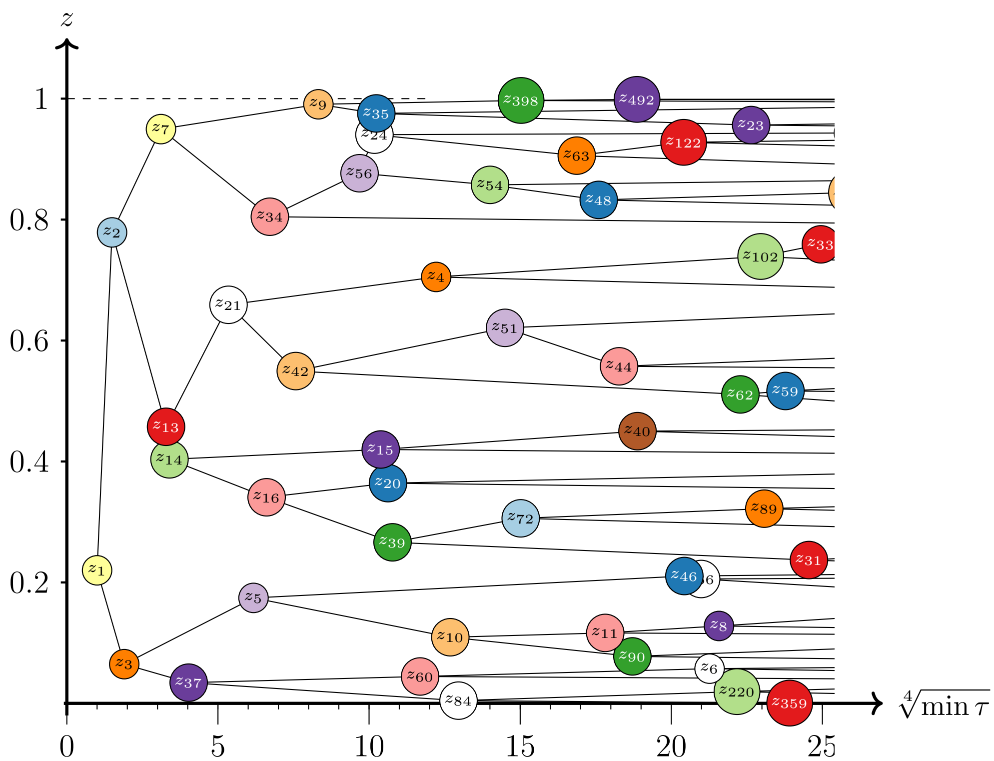


Figure 10: The tree of trees of the same Plancherel-random tableau as in Figure 2a, drawn in the coordinates $(\sqrt[4]{\min \tau},\ z)$: the horizontal coordinate of a node is the fourth root of its bifurcation time $\min \tau$ and its vertical coordinate is the asymptotic direction z of the corresponding tree. Each node is a cusp — equivalently, a bifurcation point at which a unique tree of the forest originates — and the edges are the boundaries between neighboring trees, exhibiting the binary-tree structure. A node is labelled z_n, where n is the lifetime of its tree.

Numerical experiments suggest a more detailed picture. Both types do seem to occur infinitely often, but the visits are strongly *clustered*: the trajectory runs for a long stretch alongside one side of its tree before crossing to the other. Counting the transitions between the two types along a trajectory truncated to the window $\{x + y < n\}$, the number of transitions appears to grow very slowly — roughly like the square root of the number of boundary-box visits, and hence like a small power of n. Making this growth precise, and identifying the limiting law of the alternation pattern, is another open problem. Here too the Pólya web should provide a tractable setting, in which the joint behaviour of a tree's two bounding boundaries and the trajectory running between them could be analyzed, although [BŚTU26] does not address this question.

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Data and code availability

The software that produces the figures and the Monte Carlo statistics of this paper is openly archived at doi:10.5281/zenodo.21361261 (Apache-2.0); the accompanying dataset of Plancherel-random geodesic trees is archived at doi:10.5281/zenodo.21360601 (CC-BY-4.0).

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