1

Piotr Śniady:

"Karlin-McGregor formula for coalescing random walks"

Plan:

- Example $n=3$
  [what we want to calculate?]
- Coalescing Brownian motions
  [motivations]
- Sketch of applications
- ~~Sketch of proof~~
  - $n=2$: reflection principle
  - $n>2$: reflection principle on steroids

NOT TODAY

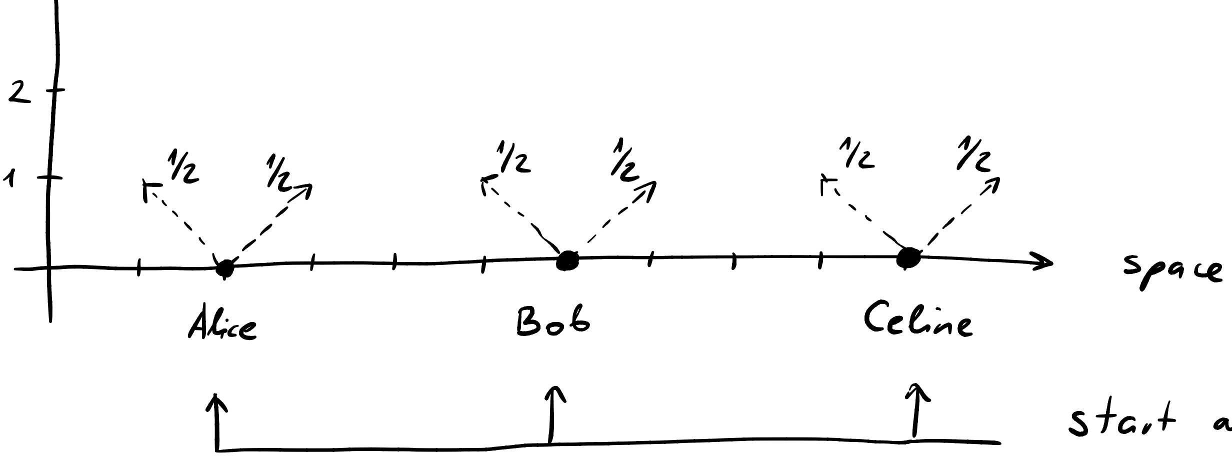
Further reading:

slides available at
→ psniady.impan.pl

- arXiv:2602.10782 🔗 | this talk
- Ákos Urbán
  arXiv:2601.12172 🔗 | parallel work

$\boxed{2}$ Very simple random walks on $\mathbb{Z}$

$\boxed{n=3 \text{ independent random walks}}$



start at EVEN integers

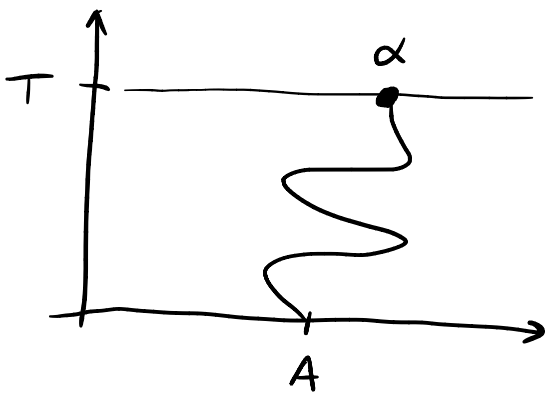
⚠ one-dimensional world, where a continuous / discrete $\boxed{\text{Darboux property}}$ holds

results hold in much wider generality:

- Brownian motions,
- Poisson processes
- birth-and-death processes
- exact formulas, no approximations

---

we will express everything in terms of

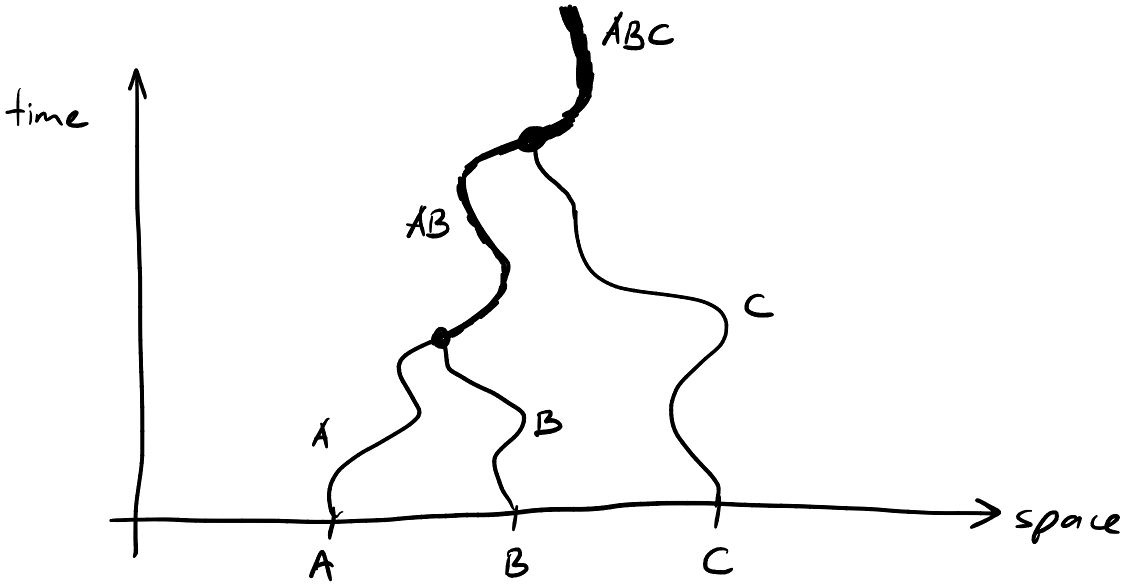


$$\mathbb{P}(A \longrightarrow \alpha) =$$

= transition probability for a single particle

3

Coalescing random walks:



when random walk X collides with Y

they are both replaced with a new random walk XY

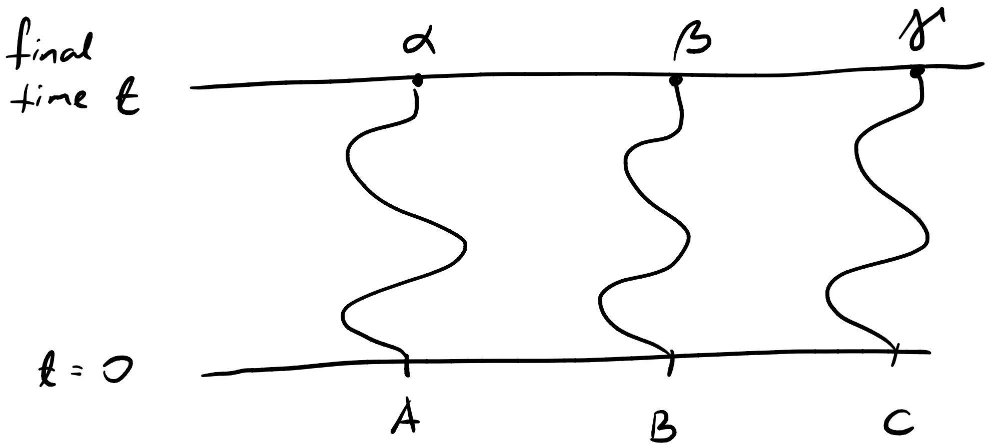
Problem

For
- specified starting positions,
- specified which particles glue together...
- ... and the final positions

Find probability of this final state

4 Karlin-McGregor formula 1959

$n = 3$



if nothing has coalesced

$$\mathbb{P}(A \to \alpha,\ B \to \beta,\ C \to \gamma \text{ AND THEY DID NOT COLLIDE}) =$$

$$= \det \begin{bmatrix} \mathbb{P}(A \to \alpha) & \mathbb{P}(A \to \beta) & \mathbb{P}(A \to \gamma) \\ \mathbb{P}(B \to \alpha) & \mathbb{P}(B \to \beta) & \mathbb{P}(B \to \gamma) \\ \mathbb{P}(C \to \alpha) & \mathbb{P}(C \to \beta) & \mathbb{P}(C \to \gamma) \end{bmatrix}$$

each row = = source

each column = = target

also known as
Lindström-Gessel-Viennot lemma

$\to$ random matrix theory, algebraic combinatorics

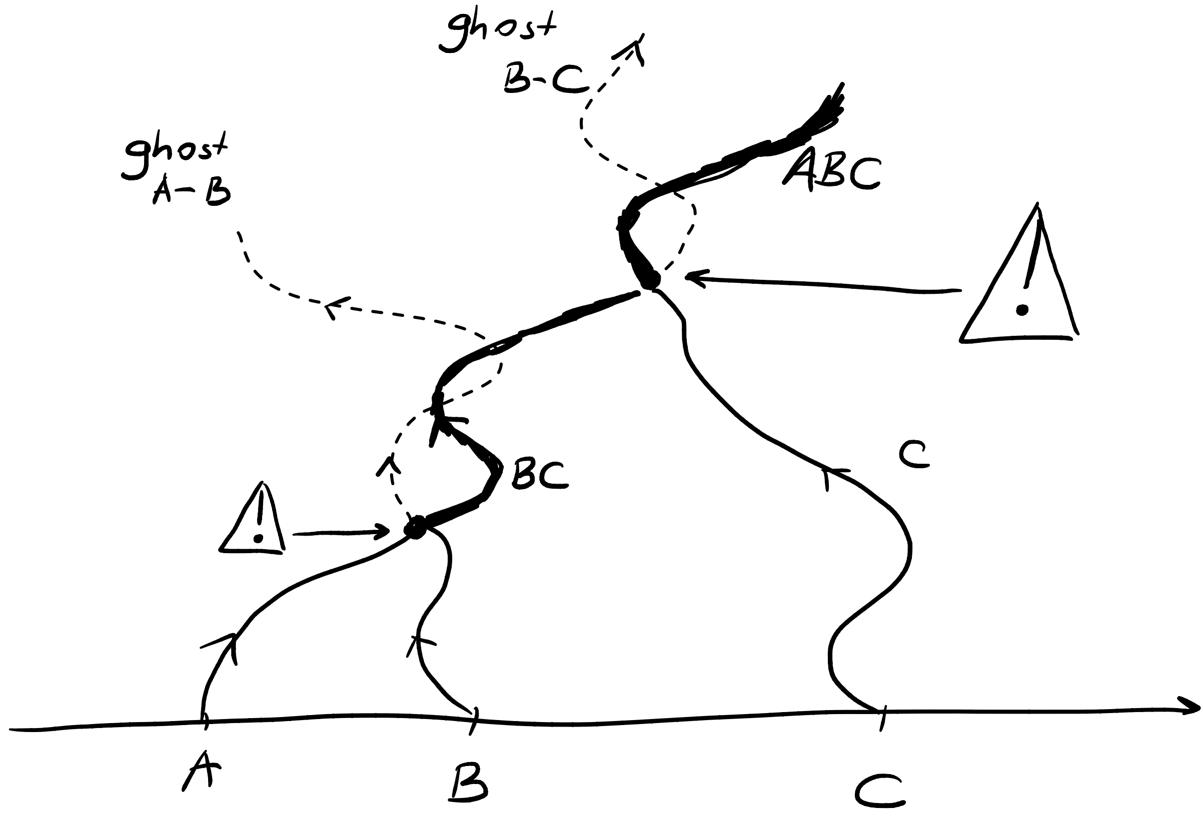
Extensions?

$$\mathbb{P}(A, B \xrightarrow[\text{to}]{\text{coalesce}} \delta,\ C \to \gamma) = ?$$

number of particles changes $\Rightarrow$ no square matrix
$\Rightarrow$ no determinant ☹

Coalescence, with ghosts

NEW IDEA



when two particles coalesce,
at collision we start an independent random walk
which does not interact with anything
= GHOST

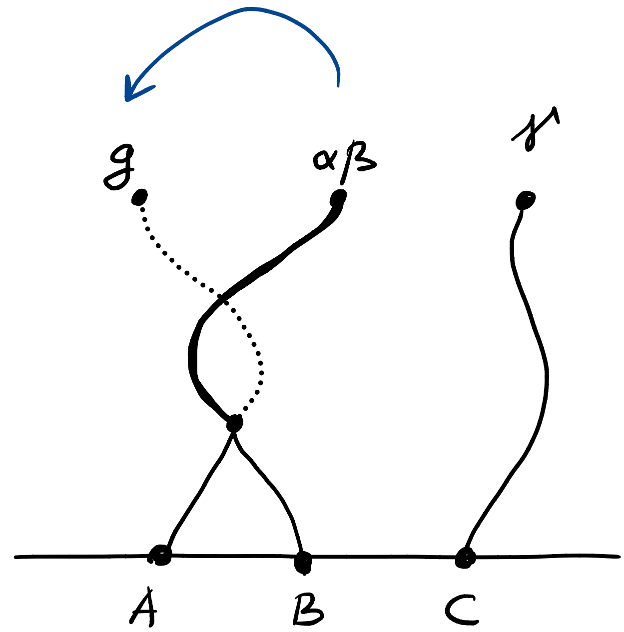
total number of particles remains constant

system becomes larger
but its description is simpler

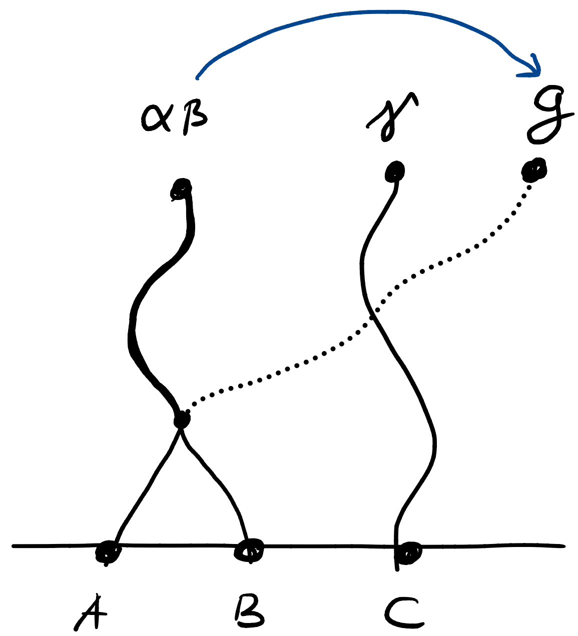
today:
simple case of only 1 ghost

6 Example : coalescence AB + C

Case $g \leq \alpha\beta$ $\boxed{\varepsilon = +}$



Case $g \geq \alpha\beta$ $\boxed{\varepsilon = -}$



Two formulas depending on relative final position of ghost

$$\mathbb{P}\left(A, B \xrightarrow{\text{coalesce}} \alpha\beta \,,\; C \xrightarrow[\text{collide}]{\text{does not}} \gamma \,,\; \text{ghost} \longrightarrow g\right) =$$

MINUSES

diagonal

PLUSES

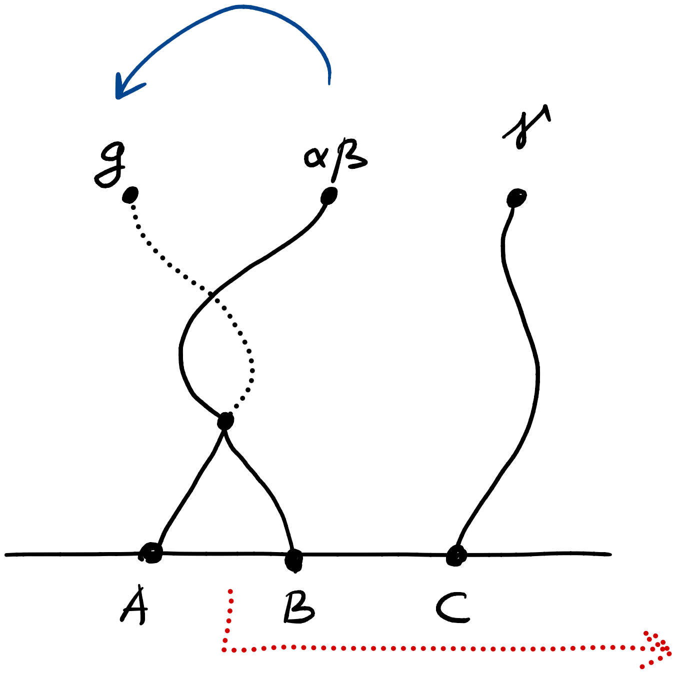
$$= \underbrace{[t^{\varepsilon}]}_{\substack{\text{extract coefficient} \\ \text{depending on} \\ \text{the case}}} \det \begin{array}{c} \left[\begin{array}{ccc} \mathbb{P}(A \to \alpha\beta) & -t^{-}\,\mathbb{P}(A \to g) & \mathbb{P}(A \to \gamma) \\ \mathbb{P}(B \to \alpha\beta) & +t^{+}\,\mathbb{P}(B \to g) & \mathbb{P}(B \to \gamma) \\ \mathbb{P}(C \to \alpha\beta) & +t^{+}\,\mathbb{P}(C \to g) & \mathbb{P}(C \to \gamma) \end{array}\right] \\ \begin{array}{ccc} \underbrace{\hphantom{\mathbb{P}(A \to \alpha\beta)}}_{\text{particle column}} & \underbrace{\hphantom{-t^{-}\mathbb{P}(A \to g)}}_{\text{ghost column}} & \underbrace{\hphantom{\mathbb{P}(A \to \gamma)}}_{\text{particle column}} \end{array} \end{array}$$

t^{-}, t^{+} are FORMAL VARIABLES

7 **Coalescence $AB + C$, continued**

Case $g \leq \alpha\beta$

$\boxed{\varepsilon = +}$



concrete formula

$$\mathbb{P}(\dots) = [t^+] \det \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & t^+\dots & \cdot \\ \cdot & t^+\dots & \cdot \end{bmatrix} =$$

$\varepsilon = +$
$\Rightarrow$ only terms below the diagonal

$$+ \begin{vmatrix} \mathbb{P}(A \to \alpha\beta) & \mathbb{P}(A \to \gamma) \\ \mathbb{P}(C \to \alpha\beta) & \mathbb{P}(C \to \gamma) \end{vmatrix} \cdot \mathbb{P}(B \to g)$$

$$- \begin{vmatrix} \mathbb{P}(A \to \alpha\beta) & \mathbb{P}(A \to \gamma) \\ \mathbb{P}(B \to \alpha\beta) & \mathbb{P}(B \to \gamma) \end{vmatrix} \cdot \mathbb{P}(C \to g)$$

"restricted Leibniz expansion of Karlin-McGregor determinant"

$\boxed{\text{4 summands}}$

8 If you do not like ghosts – single marginalized formula

$$\mathbb{P}\left(A, B \xrightarrow{\text{coalesce}} \alpha\beta,\ C \xrightarrow[\text{collide}]{\text{does not}} \gamma\right) =$$

tail of cumulative distribution function

$$\det\begin{bmatrix} \mathbb{P}(A\to\alpha\beta) & -\mathbb{P}(A\to(\alpha\beta,\infty)) & \mathbb{P}(A\to\gamma)\\ \mathbb{P}(B\to\alpha\beta) & +\mathbb{P}(B\to(-\infty,\alpha\beta]) & \mathbb{P}(B\to\gamma)\\ \mathbb{P}(C\to\alpha\beta) & +\mathbb{P}(C\to(-\infty,\alpha\beta]) & \mathbb{P}(C\to\gamma) \end{bmatrix}$$

cumulative distribution function

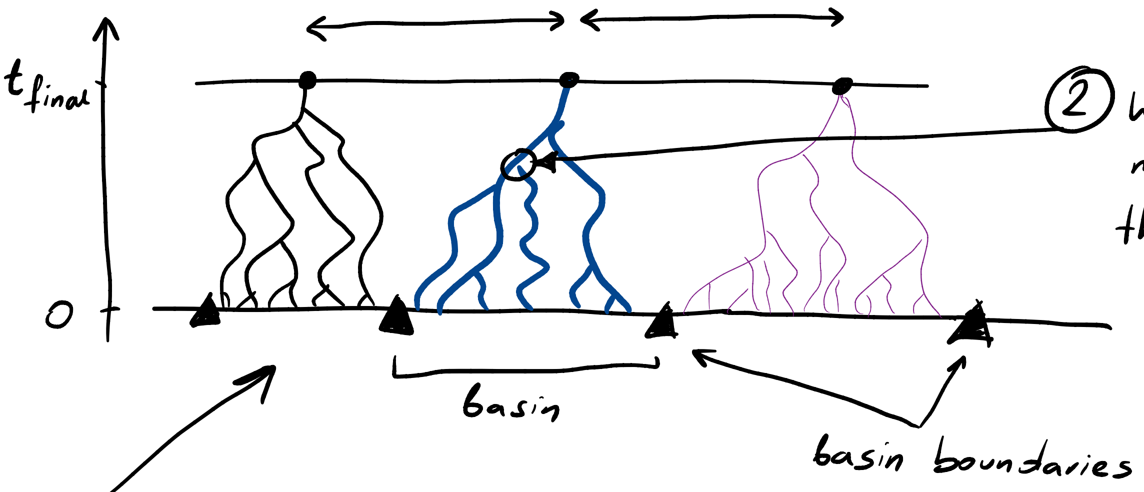
$\longrightarrow$ Ákos Urbán 2025 for a special model

Nice because:

- weak assumptions
- exact formula
- determinant
  - $\Rightarrow$ row / column operations
  - $\Rightarrow$ good asymptotics if initial/final positions are almost equal

Motivation: coalescing Brownian motions

3 gaps between particles



2 when Brownian motions collide, they coalesce

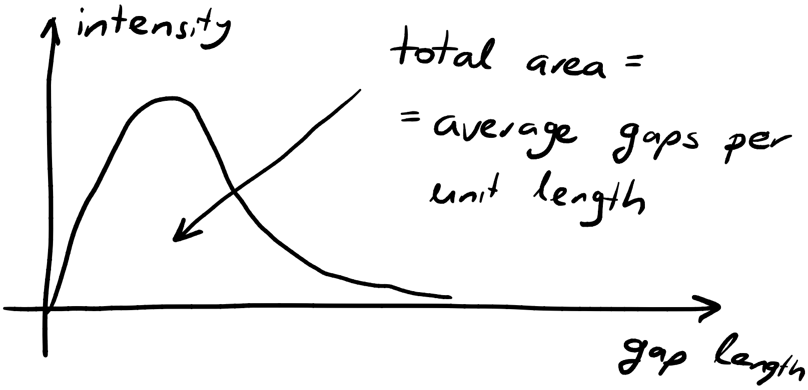
1 at time $t=0$ at each point of $\mathbb{R}$ we start an independent Brownian motion

3 at final time $t>0$ locally finite set of particles.

?

What are the statistics of spacings $\ell_1, \ell_2, \dots$?

"histogram"

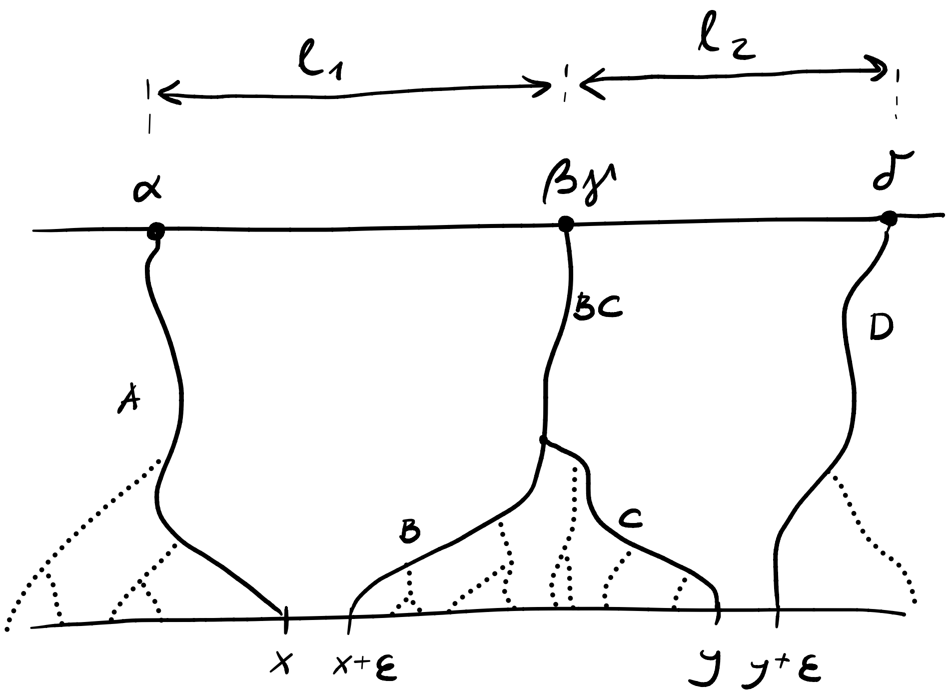


total area = = average gaps per unit length

10 Toy application:

coalescing Brownian motions

distribution of a PAIR of consecutive gaps



4 particles

schema $A + BC + D$

$$\mathbb{P}(\text{consecutive gaps } \ell_1, \ell_2) =$$

$$\iint_{x<y} dx\, dy \;\; \underbrace{\mathbb{P}\left(\begin{array}{c} x \\ \downarrow \\ \alpha \end{array} \quad \begin{array}{c} \underbrace{x+\varepsilon \quad y} \\ \downarrow \\ \beta\gamma \end{array} \quad \begin{array}{c} y+\varepsilon \\ \downarrow \\ \delta \end{array} \right)}_{\substack{4\times 4 \text{ determinant with} \\ \text{Gaussian kernel and} \\ \text{Gaussian CDF}}}$$

alternative derivation $\longrightarrow$ ben Avraham, Brunet 2005