

DETERMINANT AND PFAFFIAN FORMULAS FOR PARTICLE ANNIHILATION

PIOTR ŚNIADY

ABSTRACT. When particles on a line collide, they may annihilate—both are destroyed. Computing exact annihilation probabilities has been difficult because collisions reduce the particle count, while determinantal methods require a fixed count throughout. The ghost particle method, introduced in a companion paper for coalescence, keeps destroyed particles walking as invisible ghosts that restore the missing dimension. We apply this method to annihilation: when two particles annihilate, both trajectories continue as invisible walkers, yielding an exact determinantal formula that specifies the number of annihilations, where survivors end up, and where ghosts end up. For complete annihilation (no survivors), the formula simplifies to a Pfaffian—an algebraic relative of the determinant built from pairwise quantities—connecting to Pfaffian point process theory. The annihilation formula also yields results about coalescence: pairwise coalescence can be reinterpreted as complete annihilation, producing a Pfaffian coalescence formula. These formulas are exact for any finite initial configuration and apply to discrete lattice paths, birth-death chains, and continuous diffusions including Brownian motion.

*To Marek Bożejko,
who taught me that in creation and annihilation,
order is everything.*

1. INTRODUCTION

1.1. The problem. Consider n particles performing independent random walks on a line. When two particles meet, they annihilate: both are destroyed. This $A + A \rightarrow \emptyset$ reaction-diffusion model appears throughout statistical physics, modeling recombination kinetics, domain wall dynamics [GPTZ18], and pair annihilation [AH00]. *What is the probability that exactly k annihilations occur, with survivors reaching specified positions?*

For *non-colliding* particles, exact probabilities are classical. The Karlin–McGregor theorem [KM59] and its combinatorial cousin, the Lindström–Gessel–Viennot (LGV) lemma [Lin73; GV85], express the probability that n particles starting at positions $x_1 < \dots < x_n$ reach positions $y_1 < \dots < y_n$ without colliding as a determinant:

$$\Pr(\text{particles reach } y_1, \dots, y_n \text{ without colliding}) = \det(p(x_i \rightarrow y_j))_{1 \leq i, j \leq n},$$

Date: March 19, 2026.

2020 Mathematics Subject Classification. Primary 05A15; Secondary 05A19, 15A15, 60C05, 60J65, 82C22.

Key words and phrases. annihilating random walks, ghost particles, Lindström–Gessel–Viennot lemma, Pfaffian, determinant, lattice paths, interacting particle systems.

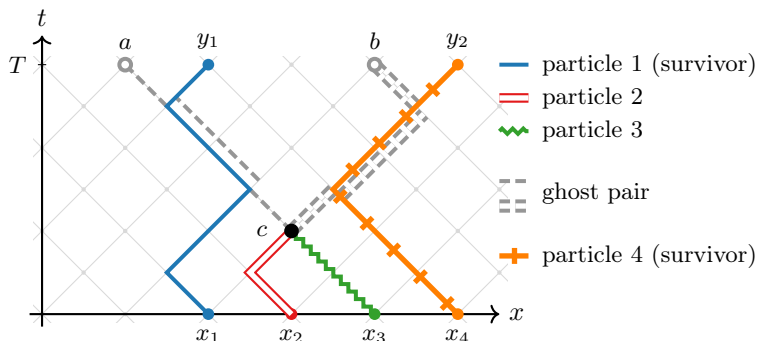


Figure 1. **Annihilation on the checkerboard lattice.** Four particles start at $x_1 < x_2 < x_3 < x_4$. Particles 2 (double) and 3 (zigzag) annihilate at c ; both are destroyed and an ordered pair of ghosts emerges (dashed paths). Particles 1 (solid) and 4 (tick marks) survive. Ghost paths freely cross survivor paths (shown offset)—ghosts do not interact. Final positions: $a < y_1 < b < y_2$.

where $p(x_i \rightarrow y_j)$ is the transition probability from x_i to y_j .

When particles annihilate, the count decreases. After k collisions, only $n - 2k$ particles remain. The number of rows (initial particles) exceeds the number of columns (final particles), and there is no square matrix to write down. This dimensional mismatch places annihilating systems outside the scope of the classical LGV framework.

1.2. The ghost method. The companion paper [SU26] introduced *ghost particles* for coalescence: when two particles merge, one heir and one ghost emerge, keeping the entity count at n and enabling determinantal formulas.

For annihilation, the same idea applies with a twist: when two particles annihilate, an ordered pair of ghosts emerges from the collision point—invisible walkers that perform independent random walks onward, without interacting with anything. The ghosts are anonymous: only the twin pairing is preserved (Section 2.3; see also Section 1.5). The entity count—survivors plus ghosts—remains exactly n , restoring the dimensional structure needed for determinantal methods. We call this variant the *ghost pair method*. Figure 1 illustrates: particles 2 and 3 annihilate at c , and a ghost pair emerges, while particles 1 and 4 survive.

1.3. Main results.

1.3.1. The annihilation formula. Consider n particles starting at $x_1 \leq \dots \leq x_n$. Suppose k collisions occur, producing $s = n - 2k$ survivors at positions $y_1 < \dots < y_s$ and k ghost pairs. The ghost pairs are physically indistinguishable—they do not remember which particles created them—so we label them $1, \dots, k$ by sampling a uniform random numbering. Ghost pair j then has positions (a_j, b_j) .

Theorem 1.1 (Annihilation formula with ghosts). *For a fixed final state—survivor positions y_1, \dots, y_s and ghost pair positions $(a_1, b_1), \dots, (a_k, b_k)$ —the probability is:*

$$\Pr = \frac{1}{k!} \left[\prod_{j=1}^k t_j^{\varepsilon_j} \right] \det(M),$$

where $\varepsilon_j = +1$ if $a_j \leq b_j$ (ghost pair ordered left-right), -1 if $a_j > b_j$ (right-left).

The notation $\left[\prod_j t_j^{\varepsilon_j} \right]$ means: expand the determinant in formal variables t_j^+ and t_j^- , then extract the coefficient where each ghost pair j contributes t_j^+ if $\varepsilon_j = +1$, or t_j^- if $\varepsilon_j = -1$. The factor $1/k!$ is the probability of the specific numbering $1, \dots, k$.

The $n \times n$ matrix M has rows indexed by initial particles and columns indexed by final entities (survivors and ghosts). Survivor columns contain transition probabilities $p(x_i \rightarrow y_\ell)$. Ghost columns additionally carry formal variables.

1.3.2. Example: four particles, one collision. Four particles start at $x_1 < x_2 < x_3 < x_4$, with one collision ($k = 1$). Two survivors end at $y_1 < y_2$ and one ghost pair at positions a and b (as in Figure 1).

The 4×4 matrix M has two survivor columns and two ghost columns (one pair). Since $k = 1$, we drop the pair index, writing τ_I^\pm for $\tau_I^{(1)\pm}$ and t^\pm for t_1^\pm :

$$M = \begin{pmatrix} p(x_1 \rightarrow y_1) & p(x_1 \rightarrow y_2) & \tau_1^+ p(x_1 \rightarrow a) & \tau_1^- p(x_1 \rightarrow b) \\ p(x_2 \rightarrow y_1) & p(x_2 \rightarrow y_2) & \tau_2^+ p(x_2 \rightarrow a) & \tau_2^- p(x_2 \rightarrow b) \\ p(x_3 \rightarrow y_1) & p(x_3 \rightarrow y_2) & \tau_3^+ p(x_3 \rightarrow a) & \tau_3^- p(x_3 \rightarrow b) \\ p(x_4 \rightarrow y_1) & p(x_4 \rightarrow y_2) & \tau_4^+ p(x_4 \rightarrow a) & \tau_4^- p(x_4 \rightarrow b) \end{pmatrix}.$$

The first two columns are plain transition probabilities (survivor columns). The last two columns carry additional formal variables τ_I^\pm indexed by the row. These variables obey the multiplication rule:

$$\tau_I^+ \tau_J^- = \begin{cases} -t^+ & \text{if } I > J, \\ t^- & \text{if } I < J. \end{cases}$$

To extract the annihilation probability, expand $\det(M)$ by Laplace expansion along the two ghost columns. Each term assigns a pair of particles $\{I, J\}$ with $I < J$ to the ghost pair and the remaining two to the survivors. Working out the product rules and extracting $[t^\varepsilon]$ for either sign gives a single formula:

$$(1.1) \quad \Pr = \frac{1}{1!} \sum_{\substack{I < J \\ \{I, J\} \subset \{1, 2, 3, 4\}}} (-1)^{I+J+1} p(x_I \rightarrow \max(a, b)) p(x_J \rightarrow \min(a, b)) \\ \cdot \det(p(x_i \rightarrow y_\ell))_{\substack{i \in \{1, \dots, 4\} \setminus \{I, J\}, \\ \ell \in \{1, 2\}}},$$

where the sign $(-1)^{I+J+1}$ comes from the Laplace expansion along columns 3 and 4, and each 2×2 determinant is a Karlin–McGregor probability for the two surviving particles. The higher-indexed particle J ends at $\min(a, b)$ and the lower-indexed particle I at $\max(a, b)$: the index ordering is reversed relative to the ghost pair's spatial ordering.

1.3.3. *The Pfaffian connection.* For complete annihilation (all particles destroyed), the formula simplifies dramatically. When $n = 2k$ particles annihilate completely, the probability is a Pfaffian of pairwise quantities:

$$P = \text{Pf}(A),$$

where A_{IJ} measures the total weight of crossing paths for particles I and J —equivalently, the probability that I and J would annihilate if they were the only two particles. The Pfaffian arises because pairwise coalescence can be reinterpreted as complete annihilation via a deterministic relabeling (Section 5). This connects to the Pfaffian point process theory for annihilating and coalescing particles [TZ11; GPTZ18]; Section 1.5 explains why matchings lead to Pfaffians, and Section 5 derives the Pfaffian structure via a purely combinatorial route.

1.3.4. *Relation to the LGV lemma.* The annihilation formula generalizes the Karlin–McGregor / Lindström–Gessel–Viennot (LGV) determinant, just as the coalescence formula does [SU26]. In both cases, the proof uses a sign-reversing involution that swaps path segments at the first wrong crossing—but in annihilation, “wrong” means “crossing that violates the prescribed collision pattern” rather than any crossing at all. When no collisions occur (all particles survive, no ghosts), the formula reduces to the standard LGV determinant. See [SU26] for the full discussion of the LGV connection, including the relationship to Stembridge’s D -compatibility [Ste90] and Fisher’s vicious walkers [Fis84].

1.4. Prior work.

1.4.1. *Pfaffian point processes.* Tribe and Zaboronski [TZ11] proved Pfaffian point process structure for annihilating Brownian motions under the maximal entrance law: their Pfaffian structure (Theorem 2 in [TZ11]) is exact, while their n -point density formula (Theorem 1) is asymptotic for large t . Garrod, Poplavskiy, Tribe, and Zaboronski [GPTZ18] extended this to continuous-time random walks on \mathbb{Z} with spatially inhomogeneous rates and all deterministic initial conditions, covering mixed coalescence-annihilation ($\theta \in [0, 1]$). Tribe and Zaboronski [TZ26] further extended this to all entrance laws. These proofs use a time-homogeneous Markov generator and ODE uniqueness. Conversely, they reach results that the ghost method does not: mixed coalescence-annihilation with arbitrary θ , and the classification of all entrance laws.

In an earlier, independent direction, Mattera [Mat03] established a bijection between annihilating random walk configurations and perfect matchings of a planar spacetime graph, and observed that the remaining particles form a Pfaffian point process. Mattera’s setting is a single specific lattice (simple random walk on $\mathbb{Z}/n\mathbb{Z}$ with synchronous updates), and his Pfaffian is the Kasteleyn Pfaffian of the spacetime graph.

1.4.2. *The IPDF method.* The interparticle distribution function (IPDF) method computes density and correlation functions by tracking the probability distribution of interparticle gaps. Doering and ben-Avraham [DA88] introduced it for diffusion-limited coalescence; ben-Avraham [Avr98] extended it to the full hierarchy of n -point correlation functions; ben-Avraham and Havlin [AH00] applied it to annihilation via the method of intervals. Unlike the generator/ODE approach, the IPDF method does not require a Markov generator, but relies on Brownian-motion-specific integrals.

1.4.3. *Cancellative duality.* The connection between coalescence and annihilation via parity labeling goes back to Griffeath’s cancellative duality [Gri79]. ben-Avraham and Brunet [AB05] made this connection explicit at the particle level. Section 5 uses the same idea—the *cancellative labeling*—in the ghost framework, where it is deterministic and yields the Pfaffian formula.

1.5. **Scope and structure of the method.** Beyond the specific formulas, the ghost pair method has several structural features—wide applicability, fine resolution, and a natural connection to Pfaffians—that distinguish it from the analytic approaches and give it a specific role in the four-paper series.

1.5.1. *Wide scope.* The ghost pair method is combinatorial, as in the companion coalescence paper [SU26]. It requires only the Karlin–McGregor assumptions: order preservation, the strong Markov property, and the requirement that meeting times are stopping times [KM59]. These hold for any *skip-free* process—one whose transitions go only to neighboring states, so that particles cannot change order without first meeting. No generator, no PDE, and no spin-pair identity is needed. The formula therefore applies uniformly to lattice paths, birth-death chains, and Brownian motion, including discrete lattices with arbitrary inhomogeneous transition probabilities (varying in both space and time) where no generator is available. The present paper works in the discrete spacetime-graph setting (Section 2); for continuous processes (Brownian motion, birth-death chains), the extension is carried out in the companion paper [SU26]. By contrast, the generator/ODE approach (Section 1.4) requires a time-homogeneous Markov generator and the spin-pair identity, while the IPDF method relies on Brownian-motion-specific integrals. Neither framework subsumes the ghost method, nor vice versa.

1.5.2. *Exact finite-time formulas.* The ghost formula gives exact probabilities for specific outcomes of finite configurations: how many annihilations occur, where the survivors end up, and where the ghost pairs end up. This is finer resolution than the density and correlation functions provided by the analytic approach, which describes the statistical structure of infinitely many particles but does not resolve individual collision histories. For Ising–Glauber domain walls (Section 1.6), this means exact transition probabilities for the full wall configuration, not just asymptotic densities.

1.5.3. *Why a Pfaffian.* Complete annihilation of $2k$ particles requires them to pair up: each pair meets and destroys both members. The different ways to pair $2k$ particles are perfect matchings of $\{1, \dots, 2k\}$ —exactly the combinatorial objects that define a Pfaffian. The ghost pair structure makes this correspondence precise: the Leibniz expansion of the determinant naturally groups its terms by matching, with each matching assigning particles to ghost pairs. This is why annihilation, not coalescence, is the natural home of the Pfaffian structure: coalescence involves compositions (how many particles merge into each heir), while annihilation involves matchings (which pairs meet and destroy each other).

1.5.4. *Ghost anonymity.* Ghost pairs do not remember which initial particles produced them. Can this be refined to prescribe *which* particles annihilate? Section A provides computational evidence that no such refinement exists: the prescribed annihilation probability cannot be expressed as any linear combination of Karlin–McGregor products, even allowing arbitrary rational coefficients. Ghost anonymity

is not a bookkeeping convenience but a structural necessity—the formula exists precisely because ghosts are anonymous.

1.5.5. *Role in the series.* The annihilation formula is the keystone of the four-paper series. Section 5 of the present paper introduces the *cancellative labeling*, which converts pairwise coalescence into complete annihilation, and derives the Pfaffian formula from the annihilation formula. The companion paper [Śni26] then applies this Pfaffian formula to the wall system (basin boundaries of the coalescing process), obtaining an empty-interval formula, a cumulant coloring formula, and a central limit theorem for the wall count. Without the Pfaffian reduction proved here, these results would have no combinatorial foundation.

At a structural level, the ghost pair method is a minimal extension of the Karlin–McGregor theorem to annihilation: the same sign-reversing involution applied to the same determinantal language, with one new ingredient—ghost pairs that restore the matrix to square form.

1.6. Applications.

1.6.1. *Domain wall dynamics.* In the one-dimensional Ising–Glauber model [Gla63], a spin chain evolves by single-spin flips at rate proportional to the energy change. Domain walls—boundaries between spin-up and spin-down regions—perform random walks on the dual lattice [GPTZ18]. When two domain walls meet, the intervening region disappears and both walls are destroyed: this is precisely the $A + A \rightarrow \emptyset$ annihilation dynamics.

The annihilation formula gives exact probabilities for the evolution of any finite domain wall configuration: given n domain walls, the probability that k pairs annihilate with survivors and ghost pairs reaching specified positions.

1.6.2. *Reaction-diffusion systems.* The $A + A \rightarrow \emptyset$ reaction models diffusion-limited pair annihilation: particles diffuse independently until two meet and react, destroying both. This appears in recombination kinetics (electron-hole pairs in semiconductors), chemical reactions at low concentrations, and population dynamics with fatal encounters [AH00]. Previous exact results for these systems describe asymptotic densities and correlation functions (Section 1.4). The ghost formula complements these by resolving individual collision histories: given a specific initial configuration, it yields the probability that a prescribed number of annihilations occur with survivors and ghost pairs at specified positions.

1.7. **Companion papers and organization.** The ghost method applies equally to coalescence ($A + A \rightarrow A$). The companion paper [ŚU26] develops the coalescence formula, a ghost-free coalescence determinant (integrating out ghost positions), and the full continuous-time treatment. The two papers share the same proof architecture—sign-reversing involution via segment swap—but differ in the ghost structure: annihilation produces ghost pairs with paired formal variables, while coalescence produces single ghosts with a staircase pattern. The Pfaffian pairwise coalescence formula (Section 5) bridges the two models, as described in Section 1.5. A further companion paper [Śni26] builds on this Pfaffian formula to derive an empty-interval formula for the walls (basin boundaries) of any skip-free coalescing system, together with explicit cumulants and a central limit theorem for the wall count.

Sections 2 and 3 set up the annihilation model and state the formula. Section 4 proves it via a sign-reversing involution. Section 5 derives the Pfaffian pairwise coalescence formula by converting coalescence to complete annihilation via the cancellative labeling.

2. SETUP

The annihilation formula holds for random walks on \mathbb{Z} , Brownian motion on \mathbb{R} , and birth-death chains on arbitrary state spaces. Following the companion paper [SU26], we work with *spacetime graphs*—a combinatorial abstraction that captures two structural properties common to all these settings:

- (i) **Planarity**: paths with swapped endpoints must cross, and non-adjacent particles cannot meet without an intermediate particle involved;
- (ii) **Weight-preserving segment swap**: exchanging path segments at a shared vertex preserves the product of weights.

For discrete models, the spacetime graph is literal; for continuous processes, the combinatorial structure is identical but the proof uses measure-theoretic arguments [SU26].

2.1. Spacetime graphs.

Definition 2.1 (Spacetime graph). A *spacetime graph* is a directed, acyclic graph $D = (V, E)$ with edge weights $w: E \rightarrow R$, where R is a commutative ring. The acyclicity induces a *time ordering*: $u \prec v$ if there is a directed path from u to v . This is a partial order; we fix a linear extension of \prec . The phrase “first crossing” (used in the proof, Section 4.6) means first in this linear order; the proof works for any such extension.

Definition 2.2 (Paths and weights). A *path* from x to y is a sequence of vertices (v_0, \dots, v_ℓ) with $v_0 = x$, $v_\ell = y$, and each $(v_i, v_{i+1}) \in E$. The *weight* of a path is $w(P) = \prod_i w(v_i \rightarrow v_{i+1})$. The *path generating function* is

$$W(x \rightarrow y) = \sum_{P:x \rightarrow y} w(P).$$

2.2. Planarity.

Definition 2.3 (Source and target sets). The *source set* $\mathcal{X} \subseteq V$ and *target set* $\mathcal{Y} \subseteq V$ are each equipped with a linear order \prec .

Definition 2.4 (Planar configuration). The pair $(\mathcal{X}, \mathcal{Y})$ is *planar* if:

- (P1) **Crossing property**. For $x \prec x'$ in \mathcal{X} and $y' \prec y$ in \mathcal{Y} (targets swapped), every path from x to y intersects every path from x' to y' .
- (P2) **Consecutive collision property**. For $x \prec x' \prec x''$ in \mathcal{X} , if paths from x and x'' meet at vertex v , then every path from x' must pass through v or intersect one of those paths before v .

The crossing property (P1) is Stembridge’s D -compatibility [Ste90]: paths with swapped endpoints must meet. The classical LGV lemma needs only this condition, because it forbids all crossings. The consecutive collision property (P2), introduced in the companion paper [SU26], is needed because we allow collisions but require them to respect the spatial ordering: non-adjacent particles cannot collide without involving intermediate ones. Both properties hold for lattice paths, random walks

on \mathbb{Z} , birth-death chains, and Brownian motion; see [ŚU26] for verification. The consecutive collision property is essential for the sign-reversing involution: it ensures that when two paths cross, the corresponding particles are adjacent in the active set (Section 4.6).

These planarity conditions constrain particle paths (trajectories from initial to final positions). Ghost paths—introduced below as invisible walkers emerging from collision points—are not subject to these constraints: they may freely share vertices with any other path.

2.3. The annihilation model. Fix n source vertices $x_1 \preceq x_2 \preceq \cdots \preceq x_n$ in \mathcal{X} . The target set \mathcal{Y} consists of all vertices reachable from \mathcal{X} that serve as potential endpoints (survivor positions and ghost positions). Each source vertex represents an initial particle; a *path* from x_I to a target vertex y represents the particle's trajectory, with weight $W(x_I \rightarrow y)$ equal to the path generating function on the spacetime graph D .

When two particles occupy the same vertex, they *annihilate*: both are destroyed and a *twin pair* of ghosts emerges from the collision point. The ghosts do not remember which particles produced them; only the twin pairing is preserved as the ghosts drift apart. In particular, particles that share a starting position annihilate instantly: the collision occurs at time zero and only ghost paths emerge.

With k collisions, we have $s = n - 2k$ survivors and $2k$ ghosts (forming k pairs), preserving the total count of n entities.

2.4. Actors and roles.

Definition 2.5 (Actors). The *actor set* $\mathcal{A} = \{1, \dots, n\}$ indexes the initial particles. Actor $I \in \mathcal{A}$ starts at position x_I .

Definition 2.6 (Roles). The *role set* (final entities) is:

$$\mathcal{R} = \mathcal{S} \cup \mathcal{G},$$

where:

- $\mathcal{S} = \{1, \dots, s\}$: survivor slots (with $s = n - 2k$);
- $\mathcal{G} = \{1, \dots, k\} \times \{1, 2\}$: ghost slots.

The cardinality is $s + 2k = n$, matching $|\mathcal{A}|$.

Each role $f \in \mathcal{R}$ has a final position y_f :

- Survivor slot $\ell \in \{1, \dots, s\}$: position y_ℓ (with $y_1 \prec \cdots \prec y_s$);
- Ghost slot $(j, m) \in \{1, \dots, k\} \times \{1, 2\}$: position $y_{(j,m)}$.

We write $a_j = y_{(j,1)}$ and $b_j = y_{(j,2)}$ as shorthand for the positions of the two ghosts in pair j .

2.5. Collision diagrams.

Definition 2.7 (Collision diagram). A *collision diagram* is a graph embedded in the spacetime graph D with three types of vertices:

- *Initial vertices* x_1, \dots, x_n : each has in-degree 0 and out-degree 1;
- *Collision vertices*: each has $m \geq 2$ incoming edges, out-degree $d \in \{0, 1\}$, and even total degree $m + d$ (so that all incoming particles can be paired). The collision produces $(m - d)/2$ ghost pairs (and one outgoing edge if $d = 1$);

- *Survivor vertices* y_1, \dots, y_s : each has in-degree 1 and out-degree 0.

Each initial vertex has a directed path to either a collision vertex or a survivor vertex. Distinct paths meet only at collision vertices (no crossings between collisions). The collision diagram records which collisions occur and where, but not what happens after each collision; the ghost paths are specified separately as part of the performance (Definition 2.8). When two initial particles share a vertex, that vertex serves simultaneously as initial vertex and collision vertex; the incoming paths have length zero.

2.6. Performances.

Definition 2.8 (Performance). An *annihilation performance* \mathcal{P} specifies:

- A collision diagram: which particles annihilated and where;
- Final positions for each survivor;
- A global numbering $j = 1, \dots, k$ of all ghost pairs across all collision vertices (a vertex with m incoming edges and out-degree d produces $(m - d)/2$ ghost pairs);
- For each ghost pair j : an ordered pair of ghost paths from the collision vertex to positions a_j and b_j .

The *weight* of a performance is

$$w(\mathcal{P}) = \frac{1}{k!} \prod_{\text{edges } e \text{ of diagram}} w(e) \cdot \prod_{j=1}^k w(\Gamma_{j,1}) w(\Gamma_{j,2}),$$

where $\Gamma_{j,1}$ is the ghost path to position $a_j = y_{(j,1)}$ and $\Gamma_{j,2}$ is the ghost path to position $b_j = y_{(j,2)}$.

Performances are *role-based*: they specify what happened (collisions, final positions) without tracking which initial particle ended where. Ghost pairs are physically indistinguishable—they do not remember which particles created them. We convert the unordered collection of ghost pairs into an ordered sequence by sampling a uniform random numbering. The factor $1/k!$ is the probability of any specific numbering; summing over all $k!$ numberings of a fixed physical outcome recovers its probability.

The global numbering determines the final state: ghost pair j occupies slots $(j, 1)$ and $(j, 2)$ with positions a_j and b_j . The proof (Section 4) introduces the complementary *actor-based* perspective via castings.

Figure 2 shows a larger example with five particles and two collisions on the \mathbb{Z}^2 lattice.

2.7. Final state.

Definition 2.9 (Ghost sign). For ghost pair j , the *ghost sign* is:

$$\varepsilon_j = \begin{cases} +1 & \text{if } a_j \preceq b_j, \\ -1 & \text{if } b_j \prec a_j. \end{cases}$$

When $a_j = b_j$ (ghosts at the same final position), we have $\varepsilon_j = +1$ by convention. For continuous state spaces, this case has probability zero.

Definition 2.10 (Final state). The *final state* \mathcal{F} specifies:

- The number k of collisions (with $2k \leq n$);

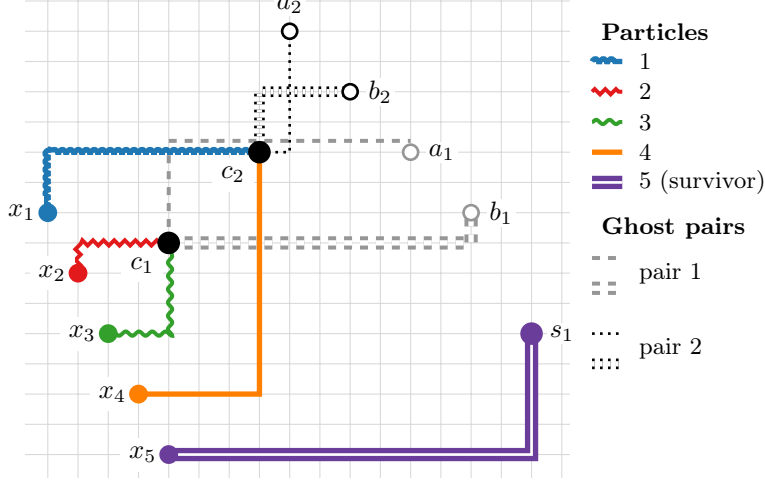


Figure 2. An annihilation performance on the lattice \mathbb{Z}^2 with North/East steps. Five particles start at x_1, \dots, x_5 . Particles 2 and 3 meet at c_1 ; both are destroyed and ghost pair 1 emerges (dashed paths). Particles 1 and 4 meet at c_2 ; both are destroyed and ghost pair 2 emerges (dotted paths). Particle 5 survives, reaching s_1 . Within each ghost pair, one ghost follows a single line, the other a double line. Ghost pairs are distinguished by line pattern (dashed vs. dotted)—they carry no memory of which particles were destroyed.

- Final positions y_f for each $f \in \mathcal{R}$.

The *sign* of the final state is the product of ghost signs: $\text{sgn } \mathcal{F} = \prod_{j=1}^k \varepsilon_j$.

3. THE ANNIHILATION FORMULA

The matrix M defined below has one row per actor and one column per role. Its Leibniz expansion sums over all bijections $\pi: \mathcal{A} \rightarrow \mathcal{R}$, but only those consistent with the ghost configuration should contribute. We introduce formal variables that track the relative ordering of the two actors assigned to each ghost pair, so that coefficient extraction selects the correct terms.

3.1. Formal variables. For each ghost pair $j \in \{1, \dots, k\}$, introduce a formal parameter t_j and formal variables $\tau_I^{(j)+}, \tau_I^{(j)-}$ for each particle I , subject to the relations (for $I \neq J$):

$$(3.1) \quad \tau_I^{(j)+} \tau_J^{(j)-} = \begin{cases} -t_j^+ & \text{if } I > J, \\ t_j^- & \text{if } I < J. \end{cases}$$

(The case $I = J$ does not arise, since bijections assign distinct actors to distinct slots.)

Concretely, the product rule (3.1) assigns $-t_j^+$ when the higher-indexed particle goes to slot $(j, 1)$ and t_j^- when the lower-indexed one does. The coefficient extraction $[t_j^{\varepsilon_j}]$ in the theorem then selects the terms consistent with the ghost sign ε_j .

3.2. **The matrix.** Define the $n \times n$ matrix M with:

- Rows indexed by particles $I \in \mathcal{A}$;
- Columns indexed by roles $f \in \mathcal{R}$: survivor slots $1, \dots, s$ followed by ghost slots $(1, 1), (1, 2), \dots, (k, 1), (k, 2)$.

Entries:

$$M_{I,f} = \begin{cases} W(x_I \rightarrow y_\ell) & \text{survivor column } \ell, \\ \tau_I^{(j)+} W(x_I \rightarrow a_j) & \text{column for ghost slot } (j, 1), \\ \tau_I^{(j)-} W(x_I \rightarrow b_j) & \text{column for ghost slot } (j, 2). \end{cases}$$

3.3. **The theorem.**

Theorem 3.1 (Annihilation formula with ghosts). *For a fixed final state \mathcal{F} , the total weight of performances is:*

$$Z = \frac{1}{k!} \left[\prod_{j=1}^k t_j^{\varepsilon_j} \right] \det(M).$$

When $k = 0$ (no collisions), the formula reduces to the Karlin–McGregor determinant $Z = \det(M)$.

Here $[\prod_j t_j^{\varepsilon_j}]$ denotes coefficient extraction: for each ghost pair j , extract the coefficient of t_j^+ if $\varepsilon_j = +1$ (slot $(j, 1)$ has smaller or equal position), or t_j^- if $\varepsilon_j = -1$ (slot $(j, 2)$ has smaller position). The proof occupies Section 4.

4. PROOF OF THE ANNIHILATION FORMULA

This proof parallels the coalescence proof in [SU26]: identical four-part structure (castings, attribution, rehearsal, sign-reversing involution) and identical involution mechanism (segment swap at the first wrong crossing). The annihilation proof is simpler because annihilation destroys both particles at a collision—no heir continues—reducing the case analysis in the involution. Nevertheless, ghost pairs differ enough from single ghosts that the success criterion, attribution map, and involution case analysis all require reworking. The present proof is self-contained; familiarity with the companion paper is not required.

4.1. **Performances vs. castings.** Throughout the proof, we fix a final state \mathcal{F} (Section 2.7): the number k of collisions, survivor positions y_1, \dots, y_s , and ghost pair positions (a_j, b_j) for $j = 1, \dots, k$. The generating function $Z = Z_{\mathcal{F}}$ counts *performances* for \mathcal{F} : complete descriptions of what collisions occur and where the final entities end up.

4.1.1. *Performances.* A performance is **role-based**—it specifies outcomes without tracking which particle “becomes” which final entity. In probabilistic terms, each performance is an elementary event in the sample space.

Consider Figure 1: four particles start at $x_1 < x_2 < x_3 < x_4$, particles 2 and 3 collide at c , and the system ends with two survivors at $y_1 < y_2$ and a ghost pair at (a, b) . The role-based description records:

- A collision occurred at spacetime point c .
- The ghost pair emerged from that collision, reaching positions a and b .
- Two particles survived, reaching y_1 and y_2 .

- The paths taken by each entity.

This description says nothing about *which* of particles 2 and 3 ended at which ghost position—only that the collision produced a ghost pair. In theater terms: the script specifies where each character should stand at the final scene, but does not specify which actor plays which character.

4.1.2. *The determinant perspective.* The determinant $\det(M)$ gives something different: a signed sum over *castings*—assignments of particles to final positions via non-interacting paths. A casting is **actor-based**: it tracks each particle’s complete trajectory. Figure 4a shows a successful casting where line styles persist through the collision, revealing who went where.

The proof establishes a bijection between performances and a subset of castings (the “successful” ones), while the remaining “failed” castings cancel in pairs (Figure 4b).

4.1.3. *Outline of the argument.* The argument has four parts:

- (1) **Castings from the determinant.** The Leibniz expansion produces castings; coefficient extraction selects *candidate* castings consistent with the ghost configuration.
- (2) **Attribution.** Every performance determines a casting by tracking which particle ends where. Attribution always produces a candidate casting.
- (3) **Rehearsal.** We attempt to interpret each casting as a performance by scanning crossings in temporal order. This may succeed (“successful casting”) or fail (“failed casting”).
- (4) **The sign-reversing involution.** Failed castings pair up via segment swap and cancel. Only successful castings remain.

4.2. **Castings.** The Leibniz formula expands the determinant as a sum over bijections $\pi: \mathcal{A} \rightarrow \mathcal{R}$:

$$\det(M) = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{I \in \mathcal{A}} M_{I, \pi(I)}.$$

Each matrix entry $M_{I,f}$ is either a path generating function $W(x_I \rightarrow y_f)$ (for survivor columns) or $\tau_I^{(j)\pm} W(x_I \rightarrow y_{(j,m)})$ (for ghost columns).

Definition 4.1 (Casting). A *casting* (π, \mathbf{P}) consists of:

- A bijection $\pi: \mathcal{A} \rightarrow \mathcal{R}$;
- A path family $\mathbf{P} = \{P_I\}_{I \in \mathcal{A}}$ where P_I goes from x_I to $y_{\pi(I)}$.

The *weight* of a casting is

$$w(\mathcal{C}) = \frac{1}{k!} \prod_{I \in \mathcal{A}} w(P_I).$$

Crucially, castings are pure geometry: the paths are non-interacting. They may cross freely—a crossing is simply a shared vertex with no physical consequence.

Definition 4.2 (Candidate bijection). A bijection $\pi: \mathcal{A} \rightarrow \mathcal{R}$ is a *candidate* for \mathcal{F} if, for each ghost pair $j \in \{1, \dots, k\}$:

- when $\varepsilon_j = +1$: $\pi^{-1}((j, 1)) > \pi^{-1}((j, 2))$;
- when $\varepsilon_j = -1$: $\pi^{-1}((j, 1)) < \pi^{-1}((j, 2))$.

In both cases, the higher-indexed particle maps to the leftward ghost position.

Candidacy arises from the coefficient extraction. Each ghost pair j contributes two columns to $\det(M)$: column $(j, 1)$ with entries $\tau_I^{(j)+}$ and column $(j, 2)$ with entries $\tau_I^{(j)-}$. When a bijection π sends particles I to $(j, 1)$ and J to $(j, 2)$, the product rule (3.1) yields $\tau_I^{(j)+}\tau_J^{(j)-} = -t_j^+$ if $I > J$ and t_j^- if $I < J$. Extracting $t_j^{\varepsilon_j}$ selects those bijections satisfying the candidacy condition.

Write $\Pi_{\mathcal{F}}$ for the set of candidate bijections. After coefficient extraction, only candidates survive. Each candidate π contributes a sign from the formal variable product (determined by the ordering of actors within each ghost pair) in addition to $\text{sgn}(\pi)$:

$$\left[\prod_j t_j^{\varepsilon_j} \right] \det(M) = \sum_{\pi \in \Pi_{\mathcal{F}}} (\text{formal sign}) \cdot \text{sgn}(\pi) \prod_I W(x_I \rightarrow y_{\pi(I)}).$$

The sign analysis in Section 4.8 shows that these signs combine to give +1 for each successful casting.

4.3. Attribution. A performance specifies collision locations and final positions but not identities. *Attribution* constructs the underlying casting by tracking which particle ends where. We first describe what happens at a single collision, then explain how to combine these local operations.

4.3.1. *The two-particle case.* At a pairwise annihilation, two path segments arrive and two ghost paths depart. The *swap principle* determines the gluing:

Principle 4.3 (Swap principle). At a collision between particles $I < J$ filling ghost pair j : the higher-indexed particle J is glued to the ghost at the leftward position; the lower-indexed particle I is glued to the ghost at the rightward position.

Figure 3 illustrates the case $\varepsilon_j = +1$ (that is, $a_j \preceq b_j$). Two ghost paths depart the collision vertex toward positions a_j and b_j , see Figure 3a. The swap principle glues each incoming path to an outgoing ghost path, see Figure 3b: particle J (higher index) is glued to the leftward ghost at a_j , and particle I (lower index) is glued to the rightward ghost at b_j . The case $\varepsilon_j = -1$ (meaning $b_j \prec a_j$) is symmetric: the labels a_j and b_j swap sides, so particle J again goes to the leftward position and particle I to the rightward one. In both cases, the index ordering is reversed relative to the ghost pair's spatial ordering.

4.3.2. *From local to global.* Attribution takes a performance as input and produces a casting (π, \mathbf{P}) . For each initial particle I , follow its path to the collision vertex (if any) and apply the swap principle to determine which ghost path it joins. The bijection π combines local and global structure: the second coordinate of $\pi(I) = (j, m)$ comes from the swap principle (which assigns I to either the first or second path in the ordered pair, giving $m \in \{1, 2\}$), while the first coordinate j comes from the performance's global numbering of ghost pairs.

Definition 4.4 (Attribution). *Attribution* constructs a casting (π, \mathbf{P}) from a performance by applying the swap principle at each collision vertex. For each initial particle I :

- If I collides, the swap principle glues its incoming path to an outgoing ghost path, determining the endpoint $\pi(I) \in \mathcal{G}$ and the glued path P_I .
- If I survives, $\pi(I) \in \mathcal{S}$ and P_I is the path from x_I to the survivor position (this path avoids all collision vertices).



Figure 3. **Two-particle annihilation, case $\varepsilon_j = +1$: $a_j \preceq b_j$.**
 (a) Schema: particles I (thick) and J (wavy) collide; both are destroyed and two ghost paths emerge (dashed). Four distinct styles emphasize that no identity persists through the collision.
 (b) Attribution via the swap principle: particle I (left, thick) is glued to the rightward ghost at b_j (thick dashed); particle J (right, wavy) is glued to the leftward ghost at a_j (wavy dashed). The spatial ordering is reversed.

The output is a casting $(\pi, (P_I)_{I \in \mathcal{A}})$.

Since both colliding particles are destroyed, each particle participates in at most one collision. Attribution therefore applies the swap principle independently at each collision vertex—no chaining arises (unlike coalescence, where the heir continues and may collide again).

At a high-indegree vertex where $m > 2$ particles $I_1 < I_2 < \dots < I_m$ meet simultaneously, we glue the first pair (I_1, I_2) to the first ordered pair of ghosts in the vertex's list, the second pair (I_3, I_4) to the second ordered pair, and so on. If m is odd, the last incoming particle I_m is glued to the outgoing edge. Each pair gluing applies the swap principle. (Section 4.6, page 20, gives the detailed treatment.)

4.3.3. Example.

Example 4.5 ($n = 4, k = 1$). In Figure 1, particles 2 and 3 ($I = 2 < 3 = J$) collide at c . The ghost pair has $a \preceq b$, so $\varepsilon = +1$. The swap principle glues particle 3 (J , higher index) to the leftward ghost at a , and particle 2 (I , lower index) to the rightward ghost at b . Survivors 1 and 4 pass through the collision diagram to their final positions. The resulting casting has $\pi(1) = 1$ (survivor slot at y_1), $\pi(2) = (1, 2)$, $\pi(3) = (1, 1)$, $\pi(4) = 2$ (survivor slot at y_2), with glued paths:

- P_1 : from x_1 to y_1 (survivor);
- P_2 : from x_2 to c , then to b (ghost);
- P_3 : from x_3 to c , then to a (ghost);
- P_4 : from x_4 to y_2 (survivor).

This is the successful casting shown in Figure 4a.

4.3.4. Candidacy emerges.

Proposition 4.6. *Let (π, \mathbf{P}) be the casting from attributing a performance. Then π is a candidate.*

Proof. At the collision creating ghost pair j , let particles $I < J$ collide. When $a_j \preceq b_j$ (so $\varepsilon_j = +1$), the swap principle assigns $\pi(J) = (j, 1)$ (position a_j , the leftward ghost) and $\pi(I) = (j, 2)$ (position b_j , rightward). Then $\pi^{-1}((j, 1)) = J > I = \pi^{-1}((j, 2))$, matching the candidacy condition for $\varepsilon_j = +1$. When $b_j \prec a_j$ (so $\varepsilon_j = -1$), the swap principle assigns $\pi(I) = (j, 1)$ (position a_j , now rightward) and $\pi(J) = (j, 2)$ (position b_j , leftward). Then $\pi^{-1}((j, 1)) = I < J = \pi^{-1}((j, 2))$, matching the candidacy condition for $\varepsilon_j = -1$. \square

4.4. Rehearsal. *Rehearsal* reverses attribution: given a candidate casting (π, \mathbf{P}) , we attempt to interpret it as a performance by scanning crossings in temporal order.

The rehearsal algorithm:

- (1) Initialize: all particles are active.
- (2) Find the earliest crossing among active paths. If none exist, success.
- (3) Test whether this crossing is a valid annihilation (see below).
 - If yes: record the collision, deactivate both particles, continue from step 2.
 - If no: failure—the casting is “failed.”

Since both participants are deactivated at each valid collision, each particle is processed at most once.

4.4.1. Valid vs. spurious crossings. A crossing between particles $I < J$ is *valid* if both particles are destined for the same ghost pair: $\{\pi(I), \pi(J)\} = \{(j, 1), (j, 2)\}$ for some ghost pair j —that is, the casting assigns I and J to the two slots of a single twin pair. If the constraint fails (one or both particles are destined for survivor slots, or they map to different ghost pairs), the crossing is *spurious*.

Definition 4.7 (Successful casting). A candidate casting is *successful* if rehearsal processes all crossings as valid annihilations without encountering a spurious one.

Figure 4 illustrates both outcomes. In (a), particles 2 and 3 are assigned to the ghost pair: their crossing at c is valid, and rehearsal succeeds. In (b), particles 1 and 3 are assigned to the ghost pair instead, but the first crossing involves particles 1 and 2—a spurious crossing, since particle 2 is destined for a survivor position.

Remark 4.8 (Dual interpretation). The rehearsal algorithm has two equivalent readings:

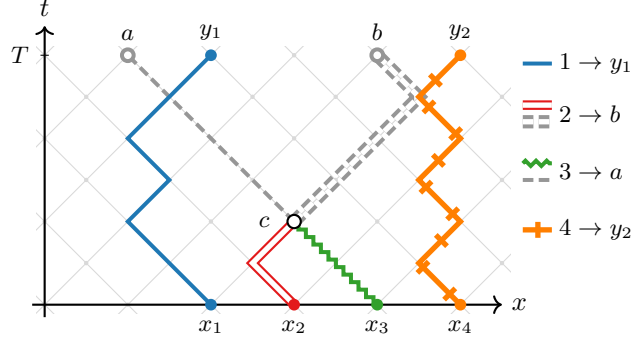
- **As a bijection:** does this casting correspond to a performance? Success produces a performance; failure means it does not.
- **As an involution:** is this casting a fixed point of ι ? Success means fixed point; failure triggers segment swap at the first spurious crossing, pairing the casting with another failed casting (Section 4.6).

The two viewpoints are unified: a casting is successful if and only if it is a fixed point of the global involution ι .

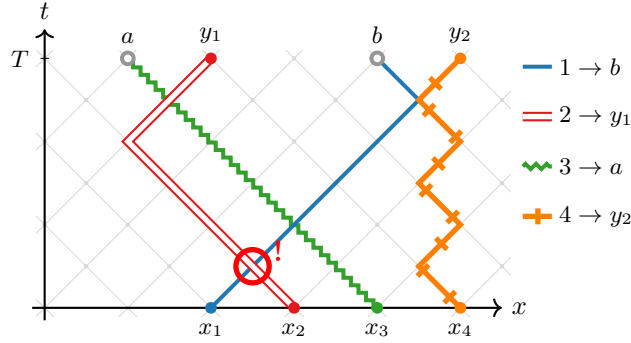
Proposition 4.9. *Attribution and rehearsal are mutual inverses between performances and successful castings.*

Proof.

$\text{Rehearsal} \circ \text{Attribution} = \text{id}$. Let \mathcal{P} be a performance. The casting $(\pi, \mathbf{P}) = \mathfrak{A}(\mathcal{P})$ is a candidate (Proposition 4.6). The crossings in \mathbf{P} are exactly the collisions in \mathcal{P} , so rehearsal encounters them in the same temporal order and processes each as valid



(a) Successful casting



(b) Failed casting

Figure 4. Successful and failed castings (annihilation, $n = 4$, $k = 1$). Same final state as Figure 1: survivors at y_1, y_2 and ghost pair at a, b . (a) The bijection π assigns particles 2 and 3 to the ghost pair, particles 1 and 4 to survivors. At the collision point c , particles 2 and 3 are destined for the same ghost pair—a valid annihilation. Both are destroyed; ghost paths emerge from the collision point (dashed). (b) The bijection assigns particles 1 and 3 to the ghost pair, particles 2 and 4 to survivors. At their first crossing, particles 1 and 2 meet, but particle 2 is destined for a survivor position—the crossing is spurious (red marker). Rehearsal fails.

(the constraint $\{\pi(I), \pi(J)\} = \{(j, 1), (j, 2)\}$ holds by construction of attribution). Rehearsal therefore succeeds and reconstructs \mathcal{P} .

Attribution \circ *Rehearsal* = id. Let (π, \mathbf{P}) be a successful casting and \mathcal{P} the performance produced by rehearsal. We verify that $\mathfrak{A}(\mathcal{P}) = (\pi, \mathbf{P})$ by checking each collision in temporal order.

At a collision between particles $I < J$ filling ghost pair j , candidacy ensures the swap principle assigns the same roles as π : if $\varepsilon_j = +1$, then $\pi^{-1}((j, 1)) > \pi^{-1}((j, 2))$, so the higher-indexed particle maps to $(j, 1)$ and the lower-indexed to $(j, 2)$ —matching attribution. The case $\varepsilon_j = -1$ is symmetric. Since both particles

deactivate, later collisions are unaffected. After all collisions, surviving particles retain their original paths and endpoints. Thus attribution recovers (π, \mathbf{P}) . \square

4.4.2. *Crossings must exist.* The rehearsal algorithm terminates either when it encounters a spurious crossing (failure) or when no crossings remain among active paths (success). We must verify that success can only occur when all ghost roles have been filled—otherwise the algorithm could “stall” with unfilled ghost roles.

Lemma 4.10 (Order preservation excludes ghosts). *If π is a candidate bijection that preserves the spatial ordering ($I < J$ implies $y_{\pi(I)} \preceq y_{\pi(J)}$), then π assigns every actor to a survivor slot.*

Proof. Suppose π assigns actors to ghost pair j with $\varepsilon_j = +1$ (meaning $a_j \preceq b_j$). Write $I = \pi^{-1}((j, 1))$ and $J = \pi^{-1}((j, 2))$. Candidacy requires $I > J$, but order preservation and $a_j \preceq b_j$ require $I < J$. Contradiction. The case $\varepsilon_j = -1$ is symmetric. \square

Corollary 4.11 (Crossings exist). *If some active actor is assigned to a ghost slot in a candidate bijection, then at least two active paths must cross.*

Proof. If no active paths cross, the restriction of π to active actors preserves spatial order. By Lemma 4.10, no active actor maps to a ghost slot—contradiction. \square

This ensures rehearsal cannot terminate with unfilled ghost roles: if any particle is still destined for a ghost position, crossings must exist among active paths.

Proposition 4.12. *Let (π, \mathbf{P}) be a successful candidate casting. The performance produced by rehearsal has the prescribed final state \mathcal{F} .*

Proof. By Corollary 4.11, rehearsal continues as long as any active particle maps to a ghost slot. Each valid collision fills one ghost pair and removes both participants. After k collisions, no active particle maps to a ghost slot; the remaining paths reach survivor positions without crossing. \square

4.5. **Segment swap.** Failed castings cancel via *segment swap*.

Definition 4.13 (Segment swap). *Segment swap* is a transformation on castings. Given a casting (π, \mathbf{P}) where paths P_I and P_J cross at vertex v , segment swap at v produces a new casting (π', \mathbf{P}') :

- P'_I : prefix of P_I from x_I to v , then suffix of P_J from v to $y_{\pi(J)}$;
- P'_J : prefix of P_J from x_J to v , then suffix of P_I from v to $y_{\pi(I)}$.

The bijection updates to $\pi' = (I J) \circ \pi$, swapping the destinations.

Lemma 4.14 (Swap properties). *Segment swap is:*

- (i) *an involution: swapping twice recovers the original;*
- (ii) *weight-preserving: $w(P_I) \cdot w(P_J) = w(P'_I) \cdot w(P'_J)$;*
- (iii) *sign-reversing: $\text{sgn}(\pi') = -\text{sgn}(\pi)$.*

Proof. (i) After swapping at v , the paths P'_I and P'_J still cross at v . Swapping again restores the original suffixes.

(ii) Each edge appears in exactly one path before and after the swap, so the product of weights is unchanged.

(iii) The bijection $\pi' = (I J) \circ \pi$ differs from π by a transposition, so $\text{sgn}(\pi') = -\text{sgn}(\pi)$. \square

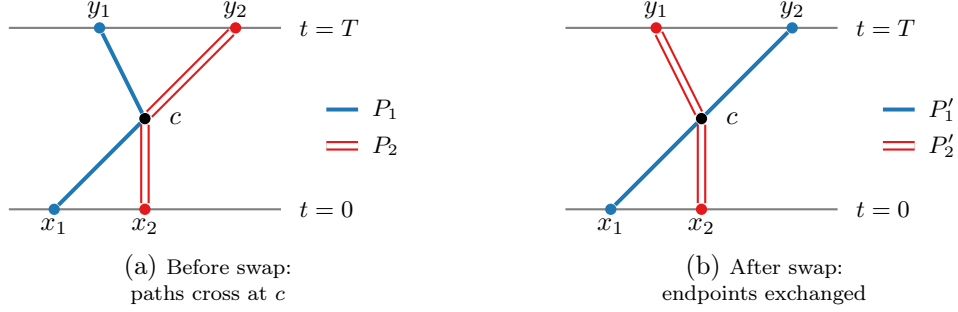


Figure 5. The segment swap operation. (a) Paths P_1 (solid) and P_2 (double) cross at vertex c . (b) After the swap, final segments are exchanged: P'_1 follows P_1 to c , then P_2 's tail to y_2 ; P'_2 follows P_2 to c , then P_1 's tail to y_1 . The paths still cross at c , but now go to swapped endpoints.

Figure 6 shows the involution in action on the failed casting from Figure 4b. Segment swap at the spurious crossing exchanges the suffixes of particles 1 and 2, producing a paired failed casting with reversed permutation sign but identical weight. The crossing persists after the swap and remains spurious, confirming $\iota^2 = \text{id}$.

4.6. The sign-reversing involution. For each pair (I, J) with $I < J$, the *local involution* $\iota_{I,J}$ acts on candidate castings. Given a candidate casting $\mathcal{C} = (\pi, \mathbf{P})$:

$$(4.1) \quad \iota_{I,J}(\mathcal{C}) = \begin{cases} \text{swap at first crossing} & \text{if } P_I \text{ and } P_J \text{ cross and swap preserves} \\ \mathcal{C} & \text{candidacy,} \\ & \text{otherwise (fixed point).} \end{cases}$$

Lemma 4.15 (Fixed points are valid annihilations). *Let (π, \mathbf{P}) be a candidate casting whose paths P_I and P_J cross (with $I < J$ adjacent in the active set). The following are equivalent:*

- (π, \mathbf{P}) is a fixed point of $\iota_{I,J}$ (swapping at the first crossing would break candidacy);
- the first crossing is a valid annihilation: $\{\pi(I), \pi(J)\} = \{(j, 1), (j, 2)\}$ for some ghost pair j .

Proof. The swap exchanges the roles of I and J : the new bijection π' satisfies $\pi'(I) = \pi(J)$ and $\pi'(J) = \pi(I)$.

Valid annihilation ($\{\pi(I), \pi(J)\} = \{(j, 1), (j, 2)\}$). Before swap, candidacy requires $\pi^{-1}((j, 1))$ and $\pi^{-1}((j, 2))$ to be ordered consistently with ε_j . After swap, their ordering is reversed. Candidacy is violated, so the casting is a fixed point.

Spurious crossing ($\{\pi(I), \pi(J)\} \neq \{(j, 1), (j, 2)\}$ for all j). We verify that π' is still a candidate. For each ghost pair j : if both slots $(j, 1), (j, 2)$ are outside $\{\pi(I), \pi(J)\}$, the swap does not affect pair j . Otherwise, exactly one slot belongs to I or J , and the other belongs to some third particle K . This K must be active (if K were inactive, its deactivation partner in pair j would be I or J , contradicting their being active). Since I and J are adjacent in the active set, K lies outside the interval $[I, J]$: either $K < I < J$ or $I < J < K$. In both cases, swapping I and J

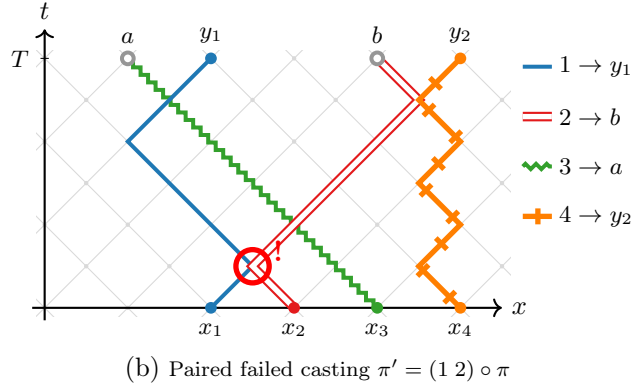
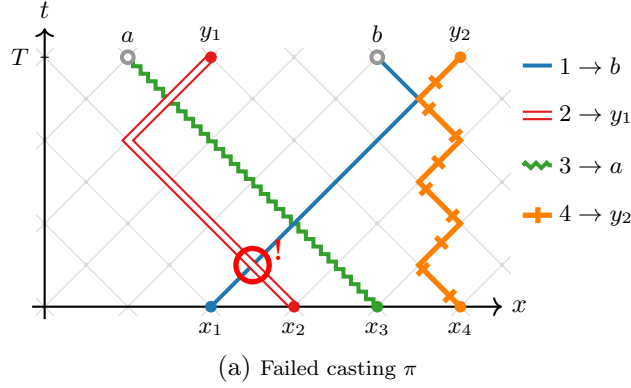


Figure 6. The sign-reversing involution in action: a matched pair of failed castings related by segment swap at the spurious crossing. (a) Particles 1 and 2 meet at the circled vertex v , but particle 2 is destined for a survivor position—spurious crossing. (b) After the swap, suffixes are exchanged: particle 1 now reaches y_1 , particle 2 now reaches b . The crossing persists and is still spurious, confirming $\iota^2 = \text{id}$. The permutation sign reverses ($\text{sgn}(\pi') = -\text{sgn}(\pi)$) while the weight is preserved, so the two castings cancel.

preserves the ordering of K relative to the participant of pair j , so candidacy for pair j is maintained. \square

The consecutive collision property (P2) guarantees that the first crossing among active paths always involves adjacent particles, so the adjacency hypothesis is satisfied whenever the algorithm invokes this lemma.

Local involutions do not commute in general, but they do commute when applied to disjoint pairs of particles. Since annihilation deactivates both participants, any two pairs checked by the algorithm are necessarily disjoint.

Lemma 4.16 (Restricted commutativity). *If the algorithm checks both (I, J) and (K, L) , then $\iota_{I,J} \circ \iota_{K,L} = \iota_{K,L} \circ \iota_{I,J}$.*

Proof. Any two checked pairs satisfy $\{K, L\} \cap \{I, J\} = \emptyset$. Disjoint local involutions act on different paths and commute. \square

The *global involution* ι composes local involutions, processing crossings in temporal order. Successful castings are fixed points; failed castings pair up via segment swap.

Input: A candidate casting $\mathcal{C} = (\pi, \mathbf{P})$.

Output: A candidate casting—either \mathcal{C} itself (fixed point) or a different casting $\mathcal{C}' = \iota_{I,J}(\mathcal{C})$ with $\text{sgn}(\pi') = -\text{sgn}(\pi)$.

Algorithm.

- (1) Initialize: all particles are active.
- (2) While active paths cross:
 - (a) Find the pair (I, J) with the earliest first crossing among active paths. (Tie-breaking: lexicographically minimal pair, ensuring the algorithm is deterministic.)
 - (b) **Test:** Is this a valid annihilation ($\{\pi(I), \pi(J)\} = \{(j, 1), (j, 2)\}$ for some j)?
 - (c) If **no** (spurious crossing): return $\iota_{I,J}(\mathcal{C})$.
 - (d) If **yes** (valid collision): deactivate both particles and continue.
- (3) **Success**—return \mathcal{C} (a fixed point).

High-indegree vertices. When $m > 2$ active particles meet at a single vertex, sort them by initial index: $I_1 < I_2 < \dots < I_m$. Process consecutive pairs (I_1, I_2) , (I_3, I_4) , etc.; each pair produces an annihilation. If m is odd, the last incoming particle I_m is not paired at this vertex and remains active. This pairing rule is a convention: the only requirement is that attribution and rehearsal use the same rule, so that the bijection between successful castings and performances is preserved. Ghost anonymity ensures that the specific convention does not affect the annihilation weight.

Theorem 4.17 (Involution theorem). *The map ι is a weight-preserving, sign-reversing involution on candidate castings. Its fixed points are exactly the successful castings.*

Proof. We verify each claimed property.

Involution. Suppose $\iota(\mathcal{C}) \neq \mathcal{C}$, meaning the algorithm swapped at pair (I, J) . Write $\mathcal{C}' = \iota(\mathcal{C})$. Since all valid collisions before the swap point deactivated both participants, the active sets in \mathcal{C} and \mathcal{C}' are identical up to pair (I, J) . The swapped paths P'_I, P'_J still cross at the same first vertex, and the constraint status (valid or spurious) depends on $\{\pi(I), \pi(J)\}$, which equals $\{\pi'(J), \pi'(I)\}$ —the same unordered set. The crossing is still spurious, so ι swaps again: $\iota^2 = \text{id}$.

Weight preservation and sign reversal. These follow from Lemma 4.14: the swap preserves total path weight and reverses permutation sign.

Fixed points. The algorithm is fixed precisely when every crossing encountered is a valid annihilation. By Corollary 4.11, if any active particle maps to a ghost slot, crossings exist among active paths. By Lemma 4.16, the order in which valid collisions are processed does not matter (any two checked pairs are disjoint, since annihilation deactivates both participants). The algorithm terminates when all ghost roles are filled and remaining paths do not cross: this is exactly a successful casting. \square

4.7. The sign identity.

Proposition 4.18 (Sign identity). *For any successful casting (π, \mathbf{P}) :*

$$\text{sgn}(\pi) = (-1)^{\#\{j:\varepsilon_j=+1\}}.$$

Proof. Each collision fills one ghost pair. Consider the collision filling ghost pair j : particles $I < J$ annihilate. By the consecutive collision property (P2), they are adjacent in the active set at the time of collision (any active particle K with $I < K < J$ would cross one of their paths earlier, forcing a prior collision).

The roles $(j, 1)$ and $(j, 2)$ have canonical order $(j, 1) < (j, 2)$, and the actors have $I < J$. By the swap principle:

- When $\varepsilon_j = +1$: $\pi(J) = (j, 1)$ and $\pi(I) = (j, 2)$. The smaller actor maps to the larger role and vice versa: one inversion, contributing -1 .
- When $\varepsilon_j = -1$: $\pi(I) = (j, 1)$ and $\pi(J) = (j, 2)$. Order is preserved: no inversion, contributing $+1$.

Survivors contribute no inversions: after all ghost pairs are filled, surviving actors map to survivor slots in spatial order (their paths do not cross). Since collisions remove disjoint adjacent pairs (with high-indegree vertices decomposed into consecutive binary annihilations as in Section 4.6), contributions multiply independently. The total sign is $\text{sgn}(\pi) = (-1)^{\#\{j:\varepsilon_j=+1\}}$. \square

4.8. Completing the proof.

Proof of Theorem 3.1. The coefficient extraction $[\prod_j t_j^{\varepsilon_j}]$ selects candidate bijections. By Theorem 4.17, the involution ι partitions candidate castings into:

- *Fixed points*: successful castings;
- *Matched pairs*: failed castings paired by segment swap.

Each matched pair consists of castings (π, \mathbf{P}) and (π', \mathbf{P}') with equal path products $\prod_I w(P_I) = \prod_I w(P'_I)$ (Lemma 4.14(ii)) and $\text{sgn}(\pi') = -\text{sgn}(\pi)$ (Lemma 4.14(iii)). Since the $1/k!$ factor is the same for both, the casting weights are equal and their contributions to the determinant cancel.

Only successful castings survive. By Proposition 4.9, these biject with performances. We verify that each contributes $+1 \cdot w(\mathcal{C})$ to Z :

Weight: For a successful casting (π, \mathbf{P}) , the path product $\prod_I w(P_I)$ decomposes into the same edges as in the corresponding performance: collision diagram edges (both incoming segments to collision vertices and survivor paths), plus the ghost paths $\Gamma_{j,1}, \Gamma_{j,2}$ for each pair j . Since both casting and performance weights include the factor $1/k!$, the bijection is weight-preserving.

Sign cancellation: Each ghost pair j contributes two sign factors. First, the Leibniz expansion contributes a factor to $\text{sgn}(\pi)$; by Proposition 4.18, this factor is -1 when $\varepsilon_j = +1$ and $+1$ when $\varepsilon_j = -1$. Second, the formal variable relations (Equation (3.1)) contribute a sign: candidacy with $\varepsilon_j = +1$ requires $\pi^{-1}((j, 1)) > \pi^{-1}((j, 2))$, producing $-t_j^+$ (a factor of -1); candidacy with $\varepsilon_j = -1$ requires $\pi^{-1}((j, 1)) < \pi^{-1}((j, 2))$, producing t_j^- (no extra sign). In both cases, the two factors are identical and their product is $(-1)^2 = +1$ or $(+1)^2 = +1$. Thus the net sign contribution from each ghost pair is $+1$, and the total contribution of each successful casting is $+1 \cdot w(\mathcal{C})$.

Conclusion: By Proposition 4.9, successful castings and performances are in one-to-one correspondence. Since both have weight $\frac{1}{k!} \prod_I w(P_I)$, the bijection preserves weight:

$$\frac{1}{k!} \left[\prod_j t_j^{\varepsilon_j} \right] \det(M) = \sum_{\text{successful}} w(\mathcal{C}) = \sum_{\mathcal{P}} w(\mathcal{P}) = Z.$$

□

5. PAIRWISE COALESCENCE AND PFAFFIANS

This section demonstrates a surprising interplay between annihilation and coalescence: we prove that the total weight of *pairwise coalescence* is a Pfaffian of pairwise quantities, by converting the coalescence problem to a complete annihilation problem. The annihilation formula (Theorem 3.1)—developed for $A + A \rightarrow \emptyset$ —turns out to be the key tool for proving a result about $A + A \rightarrow A$. This connection arises from the cancellative labeling, which reinterprets coalescence events as annihilations.

5.1. The coalescence model. Before deriving the Pfaffian formula, we introduce the coalescence model from the companion paper [ŠU26]. We provide only the definitions necessary for the cancellative labeling.

5.1.1. Coalescence dynamics. In the coalescence model ($A + A \rightarrow A$), when two particles collide one *heir* and one *ghost* emerge: the heir continues as a visible particle while the ghost drifts as an invisible walker that no longer interacts with anything. This contrasts with annihilation ($A + A \rightarrow \emptyset$), where both are destroyed and a ghost pair emerges.

Definition 5.1 (Multiplicity). Each entity carries a *multiplicity*: the number of original particles it represents. An initial particle has multiplicity 1. At each coalescence, the heir has the sum of incoming multiplicities while the ghost has multiplicity 0.

Example 5.2. If a particle with multiplicity 3 (arising from earlier mergers) coalesces with a particle with multiplicity 2, the heir has multiplicity 5 and the ghost has multiplicity 0.

5.1.2. Coalescence performance. A *coalescence performance* specifies:

- A genealogy forest: which particles merged and where;
- Ghost paths: where each ghost traveled after emerging.

The weight of a performance is the product of all path weights. (Unlike annihilation, coalescence ghosts are individually identifiable—each emerges from a specific junction—so no random labeling factor is needed.) See [ŠU26] for the full treatment, including the coalescence formula and the coalescence determinant.

5.2. The pairwise coalescence problem. Given $n = 2k$ particles at positions $x_1 \preceq \dots \preceq x_n$, we ask: what is the total weight of the *pairwise coalescence* event—the event that by time T , each consecutive pair $\{x_1, x_2\}$, $\{x_3, x_4\}$, \dots , $\{x_{n-1}, x_n\}$ has coalesced, meaning that the particles within each pair have merged (directly or through a chain of intermediate collisions) into a single heir? (Different pairs may merge separately or may share collisions.)

Computing this from a coalescence formula would require summing over all coalescence scenarios consistent with the required pairings—different coalescence trees and ghost configurations. We avoid this difficulty by converting the problem to annihilation.

5.3. The coalescence-to-annihilation map. The classical connection between coalescence and annihilation via parity—assigning independent random labels and propagating by symmetric difference—goes back to Griffeath’s cancellative duality [Gri79] and was made explicit at the particle level by ben-Avraham and Brunet [AB05] (see also Athreya and Swart [AS12] for a rigorous treatment). In the ghost framework, the conversion is deterministic: each entity carries the parity of its multiplicity, and under this *cancellative labeling*, pairwise coalescence becomes complete annihilation.

5.3.1. *Parity of an entity.* Define the *parity* of an entity as the parity of its multiplicity (Definition 5.1): *odd* (representing an odd number of original particles) or *even* (representing an even number, including zero for ghosts).

5.3.2. *Characterizing pairwise coalescence.* The event “all k consecutive pairs have coalesced” is equivalent to “every final heir has even multiplicity.” For the forward direction: each required pair contributes two original particles to the heir that absorbed it; if all pairs are absorbed, every heir’s multiplicity is even. For the converse: if every final heir has even multiplicity, then every original particle has merged with at least one other. By the consecutive collision property (Definition 2.4), particles can only merge with neighbors, so the consecutive pairing accounts for all $n = 2k$ particles and each pair must have coalesced.

5.3.3. *The cancellative labeling.* Given a coalescence performance where every final heir has even multiplicity, reclassify each entity as follows:

- every entity with even multiplicity becomes a *ghost*;
- every entity with odd multiplicity remains a *particle*.

At each binary coalescence step, the three cases are:

Incoming parities	Outgoing parities	Reclassified as
odd + odd	even + zero	ghost + ghost (annihilation)
odd + even	odd + zero	particle + ghost (non-event)
even + even	even + zero	ghost + ghost (non-event)

In the first case, two odd-parity entities (particles in the annihilation picture) produce two ghosts: this is an annihilation event. In the other cases, at least one incoming entity has even parity (already a ghost); the meeting is a non-event. In the third case, both incoming entities already have even parity (both are ghosts in the reclassified picture), so their meeting is likewise a non-event: ghosts pass through each other without interacting.

Since every final heir has even multiplicity, every final heir is reclassified as a ghost. The result is *complete annihilation*: zero survivors and $n = 2k$ ghosts.

5.4. The Pfaffian formula. We now apply the annihilation formula to derive the Pfaffian structure.

5.4.1. *The antisymmetric matrix.* For particles $I < J$ (with initial positions $x_I \preceq x_J$), define the pairwise annihilation weight

$$(5.1) \quad A_{IJ} = 2 \sum_{\substack{a,b \\ a \prec b}} W(x_I \rightarrow b) \cdot W(x_J \rightarrow a) + \sum_c W(x_I \rightarrow c) \cdot W(x_J \rightarrow c),$$

where both sums are over final vertices (replaced by integrals for continuous state spaces, in which case the second sum vanishes). The first sum is twice the total weight of strictly crossing paths: particle I (starting left) ends strictly right of particle J (starting right). The second sum is a diagonal correction for ghost pairs that end at the same vertex (for continuous state spaces such as Brownian motion, coincident endpoints have measure zero and this sum vanishes).

Extend to an $n \times n$ antisymmetric matrix by setting $A_{JI} = -A_{IJ}$ for $I < J$ and $A_{II} = 0$.

Theorem 5.3 (Pfaffian pairwise coalescence formula). *For $n = 2k$ particles at positions $x_1 \preceq \dots \preceq x_n$, the total weight of the pairwise coalescence event is*

$$(5.2) \quad P = \text{Pf}(A).$$

Proof. By the coalescence-to-annihilation map (Section 5.3), the pairwise coalescence weight equals the total weight of complete annihilation of $n = 2k$ particles. Apply the annihilation formula (Theorem 3.1) with zero survivors and k ghost pairs.

For any ghost sign pattern $(\varepsilon_1, \dots, \varepsilon_k)$, expand $\det(M)$ via the Leibniz formula. With zero survivors, the bijection $\pi: \mathcal{A} \rightarrow \mathcal{G}$ assigns each actor to one of k ghost pairs (two slots each); the Leibniz terms therefore group naturally by which actors share a ghost pair. The surviving terms are indexed by perfect matchings of $\{1, \dots, n\}$. Each perfect matching corresponds to $k!$ candidate bijections (one per numbering of ghost pairs); the $1/k!$ factor in the annihilation formula absorbs this multiplicity.

Sum the annihilation formula over all ghost positions (a_j, b_j) and all 2^k ghost sign patterns. When marginalizing over ghost positions, the ordering of ghost pairs becomes irrelevant: the physical outcome is an unordered collection of ghost pairs, and the marginalization treats all orderings equivalently. The result factorizes across pairs. For particles $I < J$ forming a ghost pair, the contribution summing both ghost signs is

$$C_{IJ} = 2 \sum_{a \prec b} W(x_I \rightarrow b) W(x_J \rightarrow a) + \sum_c W(x_I \rightarrow c) W(x_J \rightarrow c) = A_{IJ}.$$

Each perfect matching $\mu = \{\{p_1, q_1\}, \dots, \{p_k, q_k\}\}$ (unordered pairs, with $p_l < q_l$ and $p_1 < \dots < p_k$) contributes $\prod_l C_{p_l, q_l}$ with the standard Pfaffian sign: the Leibniz sign of the corresponding bijection, combined with the matching-to-numbering correspondence, yields $\text{sgn}(p_1, q_1, \dots, p_k, q_k)$.

Summing over matchings gives the Pfaffian:

$$P = \text{Pf}(A). \quad \square$$

Example 5.4 (Four particles, two pairs). Four particles at $x_1 < x_2 < x_3 < x_4$ undergo pairwise coalescence: pair $\{x_1, x_2\}$ and pair $\{x_3, x_4\}$ each merge by time T . By the cancellative labeling, this is equivalent to complete annihilation of all four particles. The 4×4 antisymmetric matrix A has entries A_{IJ} given by (5.1), and the Pfaffian expands as

$$P = \text{Pf}(A) = A_{12} A_{34} - A_{13} A_{24} + A_{14} A_{23}.$$

The three terms correspond to the three perfect matchings of $\{1, 2, 3, 4\}$. The matching $\{(1, 3), (2, 4)\}$ (sign -1) is not physically realizable: particles x_1 and x_3 cannot meet without x_2 intervening first (the consecutive collision property). The remaining matchings $\{(1, 2), (3, 4)\}$ and $\{(1, 4), (2, 3)\}$ (both sign $+1$) are realizable—the first when adjacent pairs annihilate directly, the second when $\{x_2, x_3\}$

annihilate first and then $\{x_1, x_4\}$ meet. However, each product $A_{ij} A_{kl}$ counts all path configurations where the two pairs meet, including ones with wrong collision order or unexpected crossings. The counterterm $A_{13} A_{24}$ cancels exactly these spurious contributions, by the sign-reversing involution.

5.4.2. *Random walks with i.i.d. steps.* Theorem 5.3 applies to any spacetime graph satisfying the standing assumptions. A natural special case is $n = 2k$ independent random walks on \mathbb{Z} with common step probabilities: at each discrete time step, each particle jumps $+1$ with probability p or -1 with probability $q = 1 - p$, independently of all other particles.

Corollary 5.5 (Biased random walk on \mathbb{Z}). *For $n = 2k$ particles at positions $x_1 < \dots < x_n$ on \mathbb{Z} , each performing independent ± 1 random walks with jump probabilities (p, q) for T steps, the total weight of pairwise coalescence is $\text{Pf}(A)$, where A is the $n \times n$ antisymmetric matrix with entries*

$$(5.3) \quad A_{IJ} = 2 \sum_{y_1 < y_2} w_T(x_I, y_2) w_T(x_J, y_1) + \sum_y w_T(x_I, y) w_T(x_J, y)$$

for $I < J$, and the transition weight is

$$(5.4) \quad w_T(x, y) = \binom{T}{\frac{T+y-x}{2}} p^{\frac{T+y-x}{2}} q^{\frac{T-y+x}{2}},$$

with $w_T(x, y) = 0$ unless $\frac{T+y-x}{2}$ is a non-negative integer at most T . Both sums in (5.3) range over all integers y (respectively pairs $y_1 < y_2$) of the same parity as $x_I - T$ and $x_J - T$.

Proof. The spacetime graph is the path graph $\{0, 1, \dots, T\} \times \mathbb{Z}$ with edges $(t, z) \rightarrow (t+1, z \pm 1)$ weighted by p and q . The path generating function $W(x \rightarrow y) = w_T(x, y)$ is the binomial transition probability (5.4). Substituting into (5.1) gives (5.3). \square

The first sum in (5.3) counts crossing path pairs: particle I (starting left) ends strictly right of particle J (starting right). The second sum counts pairs ending at the same position; it vanishes when $x_J - x_I$ is odd, since the two particles then occupy disjoint sublattices at time T . For $p = q = \frac{1}{2}$, the transition weight simplifies to $w_T(x, y) = \binom{T}{(T+y-x)/2} / 2^T$.

5.5. Discussion.

5.5.1. *Annihilation proves coalescence.* The proof uses the annihilation formula to establish a result about coalescence. This interplay between the two models—a payoff of the ghost framework—is made possible by the cancellative labeling. In the physics literature, Masser and ben-Avraham [MA01] showed that the n -point correlation functions of annihilation and coalescence are asymptotically identical at large times. Glinyanaya and Fomichov [GF17] proved a central limit theorem for the coalescence cluster count with Fano factor (variance-to-mean ratio) $3 - 2\sqrt{2} \approx 0.172$. Our formula makes this connection exact at the level of individual configurations.

5.5.2. *From determinant to Pfaffian.* The annihilation formula is a single determinant. For complete annihilation, all columns are ghost-pair columns, and marginalizing over ghost positions produces a Pfaffian. With survivors, the formula remains a determinant (survivor columns contribute plain transition weights). The ghost structure thus interpolates between determinants (with survivors) and Pfaffians (without).

5.5.3. *Relation to prior Pfaffian formulas.* For Brownian motion, Theorem 5.3 recovers a known result. Theorem 3 of Tribe and Zaboronski [TZ11] expresses the even product moments of annihilating Brownian motions as Pfaffians of pairwise moments; specializing to the test function $g \equiv 0$ yields the complete extinction probability as Pf of pairwise extinction probabilities. Garrod, Poplavskyi, Tribe, and Zaboronski [GPTZ18] proved the analogous identity for discrete nearest-neighbor walks (Lemma 7 in [GPTZ18]). Both proofs verify that the Pfaffian satisfies the same system of differential equations as the product moment.

In an earlier, independent direction, Mattera [Mat03] established a bijection between annihilating random walk configurations and perfect matchings (dimers) of a planar spacetime graph, and observed that the remaining particles form a Pfaffian point process. Mattera’s setting is a single specific lattice (simple random walk on $\mathbb{Z}/n\mathbb{Z}$ with synchronous updates), and his Pfaffian is the Kasteleyn Pfaffian of the spacetime graph.

Theorem 5.3 extends this identity to arbitrary planar weighted directed acyclic graphs and provides a combinatorial explanation: the Pfaffian appears because annihilating particles pair up in perfect matchings, and marginalizing over ghost positions factorizes across pairs.

5.5.4. *Connection to gap statistics.* The Pfaffian pairwise coalescence formula connects to empty-interval probabilities and checkerboard duality. In the companion paper [Śni26], checkerboard duality bridges the ghost framework to Pfaffian point process theory [TZ11; GPTZ18], and the Pfaffian formula yields empty-interval probabilities for coalescing random walks.

APPENDIX A. PRESCRIBED ANNIHILATION: COMPUTATIONAL EVIDENCE

The annihilation formula (Theorem 3.1) introduces ghost particles to restore the $n \times n$ matrix after annihilation has reduced the number of surviving particles, and labels the resulting ghost pairs by a uniform random numbering: they do not remember which initial particles produced them. Can the formula be refined to specify which particles annihilate—that is, can one obtain a Karlin–McGregor-type expression for the weight of a prescribed annihilation pattern? For $n = 2$ the question is vacuous (there is only one pair).

A.1. Prescribed annihilation.

Definition A.1 (Prescribed annihilation). Given n particles at $x_1 \prec \cdots \prec x_n$, fix adjacent particles x_m and x_{m+1} . The *prescribed annihilation weight* $Z_{\text{prescribed}}$ is the total weight of all evolutions in which x_m and x_{m+1} annihilate (producing a ghost pair at positions $a \prec b$) while the remaining $n - 2$ particles reach prescribed survivor positions, without colliding with the annihilating pair before annihilation.

Since the ghost pair’s origin is determined by the constraint, the question is whether $Z_{\text{prescribed}}$ can be expressed as a linear combination of Karlin–McGregor products:

$$(A.1) \quad Z_{\text{prescribed}} = \sum_{\pi \in S_n} c_\pi \prod_{i=1}^n W(x_i \rightarrow z_{\pi(i)}),$$

with rational coefficients c_π independent of the positions and of time. Here z_1, \dots, z_n denote the n final positions (survivors and ghosts combined).

A.2. Computation. For $n = 3$ (the first nontrivial case), on the \mathbb{Z} lattice (simple random walk with ± 1 steps), the prescribed annihilation weight can be computed exactly by enumerating all 2^{nt} path evolutions and filtering to those satisfying the constraints of Definition A.1. For a fixed ordering of the n final positions, distinct final-position tuples provide linear equations in the $n!$ unknowns c_π of (A.1). The resulting system is solved over \mathbb{Q} in exact arithmetic. (The same technique was used to discover the annihilation formula of the present paper and the coalescence formula of the companion paper [ŠU26]; it succeeds when a formula of the form (A.1) exists.)

With initial positions $x_i = 2(i - 1)$ and annihilating pair $(1, 2)$, at time $t = 4$ the ordering $a \prec y_1 \prec b$ yields 4 distinct tuples (a, y_1, b) :

$$(-2, 0, 2), \quad (-2, 0, 4), \quad (-2, 2, 4), \quad (0, 2, 4).$$

The 4×6 system over \mathbb{Q} is inconsistent: no rational coefficients c_π satisfy all equations. Any three of the four equations admit a solution (three equations in six unknowns is underdetermined), so four is the minimum needed to detect inconsistency.

A.3. Interpretation. The inconsistency is not a failure of a specific ansatz such as a determinant or permanent: every expression of the form (A.1) is a linear combination of the $n!$ products $\prod_i W(x_i \rightarrow z_{\pi(i)})$, and the linear system tests all such combinations simultaneously. No rational coefficients c_π exist. Restricting to the canonical ordering $a \prec y_1 \prec b$ allows the coefficients to depend on the ordering class, weakening the requirement; the system is still inconsistent.

Ghost anonymity is not merely a convenient bookkeeping device; the computation above suggests that it is essential for the existence of the formula.

ACKNOWLEDGMENTS

We thank Maciej Dołęga, Marcin Lis, Sho Matsumoto, and Karol Życzkowski for stimulating discussions and bibliographic suggestions.

Claude Code (Anthropic) was used as an assistant during formula discovery and manuscript preparation.

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