COALESCING RANDOM WALKS VIA THE COALESCENCE DETERMINANT

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Abstract. When identical particles on a line collide, they merge and continue as one. Exact determinantal formulas have long been available for particles conditioned never to collide, but collisions change the number of particles, and exact distributions for the survivors have been obtained only in specific settings and by ad hoc methods. Building on the coalescence determinant introduced in a companion paper, we study the wall-particle system: when every site is initially occupied, this is the joint system of survivors and the boundaries between their basins of attraction. Its finite-dimensional distributions are determinants of block matrices built from transition probabilities and their cumulative sums; a finite block matrix suffices even when the initial configuration is infinite. As applications, we recover the Rayleigh spacing density and the joint distribution of consecutive gaps—which are negatively correlated—by new methods, and give a new derivation of the determinantal formula for the joint CDF of finitely many coalescing particles starting from fixed positions. All formulas hold for arbitrary nearest-neighbor random walks and their Brownian scaling limits, with no specific transition kernels required.

1. Introduction

1.1. **Coalescing particles.** When identical particles on a line collide, they merge and continue as one (Figure 1). This coalescence rule appears in several areas of probability and mathematical physics. In the voter model [HL75], boundaries between opinion clusters perform coalescing random walks; their dynamics controls the approach to consensus. In reaction-diffusion theory ($A + A \to A$), diffusing particles that merge on contact display anomalous kinetics: the density in one dimension decays as $n(t) \sim t^{-1/2}$, slower than the mean-field prediction, because spatial correlations dominate at large times [DA88].

Arratia [Arr79] placed the infinite system on rigorous footing. Starting coalescing Brownian motions from every point of $\mathbb{R}$, he showed that the surviving population is locally finite at any positive time: the system "comes down from infinity."

1.2. **The coalescence determinant.** The Karlin–McGregor theorem [KM59] gives exact determinantal formulas for particles that *avoid* collision. By contrast, exact formulas for coalescing particles have been obtained only in specific settings and by ad hoc methods. The difficulty is that collisions change the number of particles, so the square matrices of Karlin–McGregor do not directly apply.

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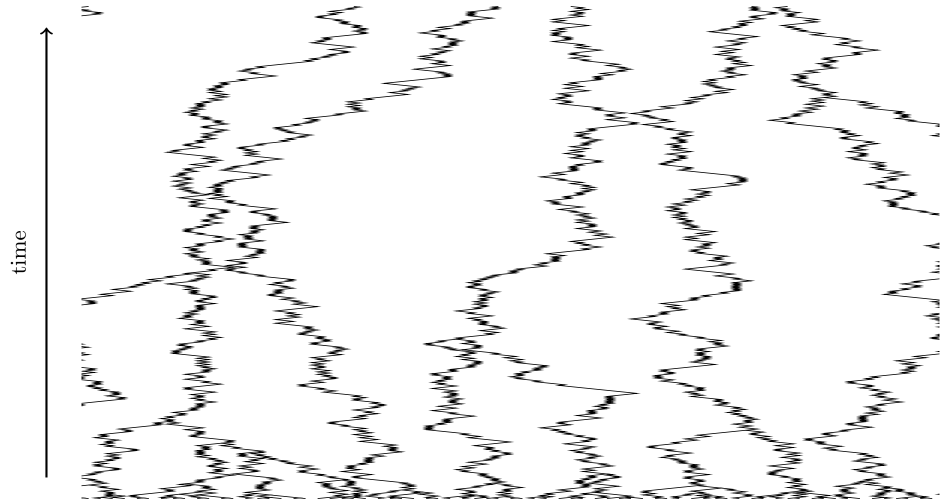

Figure 1. Coalescing random walks starting from every site of a lattice segment. Paths merge on contact; the surviving population thins over time.

1.2.1. *The formula.* The companion paper [ŚU26, Section *Integrating out the ghosts*] introduces the coalescing counterpart: the *coalescence determinant*. Given a *coalescence pattern*—which of the n initial particles merge into each survivor—the coalescence determinant gives the joint distribution of survivor positions as the determinant of an $n \times n$ matrix built from transition probabilities and their cumulative sums (Section 2 recalls the precise definition).

Much of the power of the original Karlin–McGregor theorem comes from the weakness of its assumptions: the Markov property and *skip-free* trajectories (transitions only to neighboring states, so that particles cannot change order without first meeting [KM59]). No symmetry and no specific transition kernels are needed. The coalescence determinant operates under exactly the same assumptions, and therefore applies wherever Karlin–McGregor does: we work with coalescing skip-free random walks on $\mathbb{Z}$ and their Brownian scaling limits on $\mathbb{R}$.

Urbán [Urb25] proved the same formula for binary coalescence of *Pólya walks*—a special case, since Pólya walks are birth-and-death chains. His proof reaches the determinant from a complementary direction, starting from Karlin–McGregor for non-colliding walks and handling coalescence via a total-probability decomposition; see [ŚU26] for a detailed comparison.

1.2.2. *This paper.* This paper explores what the coalescence determinant yields for general skip-free processes. A companion paper [Śni26] proves that basin boundaries carry a natural Pfaffian structure (for any skip-free process), derives explicit cumulants and a central limit theorem, and transfers these results to surviving particles via checkerboard duality. Under the maximal entrance law (every site initially occupied), we study the *wall-particle system*: the joint system of survivors and the boundaries between their basins of attraction (defined in Section 1.3). The coalescence determinant yields the finite-dimensional marginals of this system in closed form (Theorem 1.1); gap distributions follow by marginalizing over wall

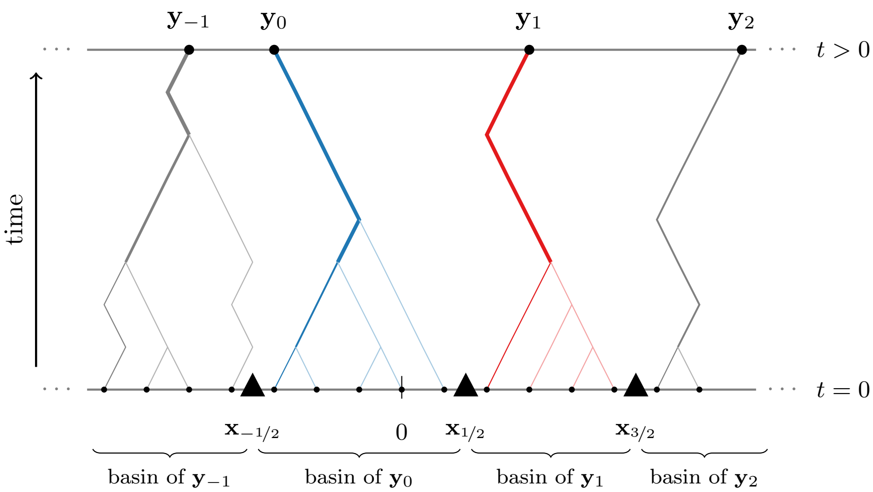

Figure 2. The $(\mathbf{X}, \mathbf{Y})$ system for coalescing random walks. Paths coalesce on meeting; line weight increases with each merger. Bottom: walls $\mathbf{X} = (\ldots, \mathbf{x}_{-1/2}, \mathbf{x}_{1/2}, \mathbf{x}_{3/2}, \ldots)$ (triangles) partition the initial line, with $\mathbf{x}_{-1/2} < 0 \leq \mathbf{x}_{1/2}$. Top: survivor positions $\mathbf{Y} = (\ldots, \mathbf{y}_{-1}, \mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots)$ (circles), one per basin.

positions. Separately, for any finite initial configuration, the coalescence determinant gives Warren's determinantal CDF formula for survivor positions (Theorem 6.1).

1.3. **The wall-particle system.**

1.3.1. *The construction.* We work under the maximal entrance law: every site of $\mathbb{Z}$ is initially occupied (for Brownian motion, every point of $\mathbb{R}$). Each surviving particle then owns a *basin*: the contiguous set of initial positions whose particles merged into it. The basins partition the initial configuration, and their boundaries are the *walls*. The joint system of wall positions $\mathbf{X}$ and survivor positions $\mathbf{Y}$—the *wall-particle system* (Section 3; see Figure 2)—pairs each survivor with its walls.

Previous work has studied the two marginals separately. Arratia [Arr79] identified the walls (his "partition points"): the points separating groups of initial positions that merge into the same survivor. For Brownian motion, he proved $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$ via a time-reversal duality—but this distributional identity requires specific symmetry of the underlying process. Fomichov [Fom16] computed the joint distribution of one wall position and two cluster values for the Arratia flow, using Karlin–McGregor determinants; he noted that this approach cannot recover the full joint distribution for three or more clusters. The wall-particle correlation function (Theorem 1.1) generalizes Fomichov's computation from $k = 2$ clusters to arbitrary k, and from the Arratia flow to any skip-free process.

1.3.2. *Correlation function.* The coalescence determinant gives the finite-dimensional marginals of $(\mathbf{X}, \mathbf{Y})$ in closed form.

Theorem 1.1 (Wall-particle correlation function)**.** *Consider a coalescing skip-free process on $\mathbb{Z}$ with every site initially occupied. The probability that $(\mathbf{X}, \mathbf{Y})$ contains* *the consecutive pattern*

$$y_0 \nwarrow x_{1/2} \nearrow y_1 \nwarrow \cdots \nwarrow x_{k-1/2} \nearrow y_k$$

(walls at $x_{1/2}, \dots, x_{k-1/2}$ *flanked by survivors at* $y_0, \dots, y_k$*) equals* $\det(\tilde{M})$*, where* $\tilde{M}$ *is the* $2k \times 2k$ *block matrix described in Section 3.2.*

The formula holds for any skip-free process—no symmetry of the transition probabilities and no specific kernels are needed. Despite the infinite initial configuration (every site occupied), k consecutive wall-particle pairs depend only on a $2k \times 2k$ block matrix; the rest of the infinite system does not enter the formula.

1.4. **Gap distributions.** Marginalizing the correlation function over wall positions gives the joint distribution of consecutive gaps between survivors. Gap distributions for coalescing Brownian motions were first studied by Doering and ben-Avraham [DA88] via the *inter-particle distribution function* (IPDF) method: the rescaled nearest-neighbor distance density converges to $x\, e^{-cx^2}$, a Rayleigh form. Ben-Avraham [Avr98] extended the method to derive the full hierarchy of empty-interval probabilities; ben-Avraham and Brunet [AB05] extracted explicit densities for two and three consecutive spacings. These formulas use the explicit transition kernel of Brownian motion. In the asymptotic regime, FitzGerald, Tribe, and Zaboronski [FTZ20; FTZ22] computed gap exponents and persistence exponents via Fredholm Pfaffian methods, rigorously confirming predictions of Derrida and Zeitak. For counting statistics, Glinyanaya and Fomichov [GF17] proved a central limit theorem for the number of surviving clusters in the Arratia flow, with Fano factor $3 - 2\sqrt{2} \approx 0.172$, reflecting the sub-Poissonian correlations induced by coalescence.

1.4.1. *Discrete gap formula.* In the discrete setting, no limiting procedure is needed.

Theorem 1.2 (Discrete gap intensity measure)**.** *Let* $P_s(n)$ *denote the transition probability of a translation-invariant skip-free process on* $\mathbb{Z}$ *(from* 0 *to* n *in time* s*), and suppose the process is symmetric:* $P_s(n) = P_s(-n)$*. Start with every site occupied.*

The gap intensity measure—the expected number of gaps of size g *per unit length—is*

$$\mu(\{g\}) = P_{2T}(g-1) - P_{2T}(g+1), \qquad g = 1, 2, 3, \dots$$

The total intensity

$$\sum_g \mu(\{g\}) = P_{2T}(0) + P_{2T}(1)$$

is the survivor density per site; dividing by it recovers the gap probability mass function (Theorem 5.1).

This formula holds for any symmetric skip-free process: simple random walk, lazy random walk, or any birth-death chain with symmetric rates. The only inputs are transition probabilities at doubled time—no PDEs, no passage to a continuous limit. For non-symmetric processes, the gap intensity is a convolution of transition probabilities at the original time (Section 5.1.1).

1.4.2. *Brownian gaps: the Rayleigh distribution.* Passing to Brownian motion, the discrete formula becomes a continuous density.

Theorem 1.3 (Single gap intensity measure)**.** *Under the maximal entrance law, the gap intensity measure in rescaled coordinates ($G = gap/\sqrt{T}$) has density*

$$\mu(dG) = \frac{G}{2\sqrt{\pi}}\, e^{-G^2/4}\, dG, \qquad G > 0.$$

The total intensity

$$\int_0^\infty \mu(dG) = \frac{1}{\sqrt{\pi}}$$

gives the rescaled survivor density; un-rescaling yields density $n(T) = 1/\sqrt{\pi T}$. Normalizing to a probability distribution gives $\mathrm{Rayleigh}(\sqrt{2})$.

This recovers the Rayleigh law of Doering and ben-Avraham [DA88] by new methods (Theorem 5.2). In Section 5.1.2 we sketch how the discrete formula from Theorem 1.2 converges to this density under diffusive scaling.

1.4.3. *Joint gap distribution.* The single-gap Rayleigh law determines the marginal distribution but says nothing about correlations between neighboring gaps.

Theorem 1.4 (Joint gap intensity)**.** *For two consecutive gaps G_1 and G_2, the joint gap intensity $h(G_1, G_2)$ is given by an explicit integral formula (Theorem 5.4). The marginal intensities are each* $\mathrm{Rayleigh}(\sqrt{2})$*, but the gaps are negatively correlated ($\rho \approx -0.163$).*

The joint density of two consecutive spacings was previously obtained by ben-Avraham and Brunet [AB05] from the IPDF hierarchy; we give an alternative derivation via a 4×4 determinant.

1.5. **Warren's formula.** Warren [War07] proved that for finitely many coalescing Brownian motions, the joint CDF of survivor positions is a determinant of Gaussian CDFs and their tails. Assiotis, O'Connell, and Warren [AOW19] extended Warren's intertwining approach to general one-dimensional diffusions, and Assiotis [Ass18; Ass23] to birth-death chains. Unlike the wall-particle correlation function—which requires the maximal entrance law (every site occupied)—Warren's formula applies to any finite configuration of particles at fixed starting positions. The coalescence determinant yields the CDF formula for any skip-free process, including discrete-time random walks on $\mathbb{Z}$ (Theorem 6.1): the proof uses only the coalescence determinant and a summation-by-parts identity.

1.6. **Scope and significance.** This paper demonstrates the coalescence determinant in action on concrete problems. Several of the results—the Rayleigh gap law [DA88], Warren's CDF determinant [War07]—were first established by other methods. The new derivations share a common mechanism (column operations on the block matrix) and reveal structural reasons why these formulas take the form they do.

1.6.1. *The wall perspective.* The wall-particle system (Section 1.3) makes walls first-class objects alongside particles. As noted in Section 1.3, previous work studied walls and survivors separately; the correlation function (Theorem 1.1) captures both in a single block matrix.

This joint perspective has consequences beyond this paper. The companion paper [Śni26] proves that the Pfaffian point process structure [TZ11; GPTZ18] belongs naturally to walls (basin boundaries), not to the surviving particles; particles inherit it only through a duality that identifies survivors with boundaries of a dual process. The wall-particle correlation function defined here is the starting point for that analysis.

1.6.2. *Algebraic manipulation in practice.* The companion paper [ŚU26] argues that the determinantal structure enables algebraic manipulation absent from non-determinantal approaches. This paper illustrates that argument concretely, in three ways.

First, the confluent limit: when adjacent starting positions in the block matrix nearly coincide, subtracting the corresponding rows and dividing by the spacing replaces a pair of transition probabilities with a density-and-derivative pair. Under the maximal entrance law, every pair of flanking sites undergoes this collapse, producing the characteristic 2×2 block structure of the Tribe–Zaboronski Wronskian kernel [TZ11] (Section 4). Second, summation-by-parts: summing the coalescence determinant over all coalescence patterns and using multilinearity of the determinant collapses columns one by one, converting transition probabilities to CDFs and recovering Warren's formula [War07] (Section 6). Third, marginalizing the correlation function over wall positions yields the gap formula at doubled time (Theorem 1.2)—a formula whose simplicity is enabled by the block matrix structure.

1.6.3. *Structural explanations.* The new derivations do more than reprove known results; they explain structural features. The CDF entries in Warren's determinant arise from integrating out ghost positions: each ghost can end anywhere to the left of the corresponding survivor, and summing transition probabilities over these positions produces cumulative distributions. The doubled-time transition probabilities in the discrete gap formula (Theorem 1.2) reflect a convolution: the gap between two survivors is mediated by the wall between them, and marginalizing over the wall position convolves two single-step transition probabilities into one at doubled time. The discrete formula is primary; the Brownian Rayleigh law (Theorem 1.3) is its scaling limit, reversing the historical order in which Brownian motion came first [DA88] and discrete processes followed.

1.6.4. *Complementary approaches.* The coalescence determinant and the analytic approaches cover complementary territory. Our formulas apply to any skip-free process with arbitrary inhomogeneous transition probabilities, but the wall-particle results require the maximal entrance law and treat only pure coalescence. The spin-pair duality of [GPTZ18] handles mixed coalescence-annihilation and all deterministic initial conditions (extended by Tribe and Zaboronski [TZ26] to all entrance laws), but requires a time-homogeneous Markov generator. For coalescing Brownian motions, FitzGerald, Tribe, and Zaboronski [FTZ20; FTZ22] derive sharp gap exponents via Fredholm Pfaffian methods, and Glinyanaya and Fomichov [GF17] prove a CLT with Berry–Esseen bound—results our approach does not yield.

1.7. **Organization.** Section 2 recalls the coalescence determinant for arbitrary patterns. Section 3 develops the wall-particle system and derives its correlation function for skip-free processes on $\mathbb{Z}$. Specializing to Brownian motion (Section 4), each pair of flanking sites collapses to a density–derivative pair as the lattice spacing vanishes, and the maximal entrance law provides translation invariance. Gap distributions (Section 5) follow by marginalizing the correlation function over wall positions for $k = 1$ and $k = 2$. Warren's formula (Section 6) is a separate application of the coalescence determinant: summing over all coalescence patterns converts density columns to CDF columns.

2. The Coalescence Determinant

We recall the coalescence determinant from [ŚU26], which applies to any finite collection of coalescing skip-free particles. We state the formula in the discrete setting; the continuous extension is recalled in Section 4.

Write $P(x, y)$ for the transition probability of the underlying skip-free process from x to y in time T (with T fixed throughout), and

$$F(x, y) = \sum_{z \le y} P(x, z)$$

for the cumulative sum.

Start n particles at positions $x_1 < x_2 < \cdots < x_n$. A *coalescence pattern* is an integer composition $n_1 + n_2 + \cdots + n_k = n$: the first n_1 initial particles merge into survivor 1, the next n_2 into survivor 2, and so on. The lth block of the composition—the initial particles merging into survivor l—has indices $n_1 + \cdots + n_{l-1} + 1$ through $n_1 + \cdots + n_l$. Write $y_1, \ldots, y_k$ for the survivor positions at time T.

Definition 2.1 (Coalescence matrix)**.** Both rows and columns of the $n \times n$ *coalescence matrix* $\tilde{M}$ are indexed by $\{1, \ldots, n\}$. The entry in row i, column j (where j lies in the lth block, with survivor position y_l) is

$$\tilde{M}_{ij} = \begin{cases} P(x_i, y_l) & \text{if } j \text{ is the first index in its block,} \\ F(x_i, y_l) - [i < j] & \text{otherwise.} \end{cases}$$

The first column of each block contains transition probabilities P; the remaining $n_l - 1$ columns contain cumulative sums F with a staircase shift.

Theorem 2.2 (Coalescence determinant [ŚU26])**.** *Under the coalescence pattern $n_1 + \cdots + n_k = n$, the joint probability of survivor positions at $y_1 < \cdots < y_k$ is $\det(\tilde{M})$.*

The formula above is stated for discrete state spaces. For continuous processes satisfying the Karlin–McGregor assumptions (such as Brownian motion), transition probabilities P become densities and cumulative sums F become CDFs; the determinant then gives a probability density rather than a probability mass. See [ŚU26] for the general statement covering both cases.

Example 2.3 (Pattern $2 + 1$). Three particles; the first two merge (survivor y_1), the third survives alone (y_2):

$$\tilde{M} = \begin{pmatrix} P(x_1, y_1) & F(x_1, y_1) - 1 & P(x_1, y_2) \\ P(x_2, y_1) & F(x_2, y_1) & P(x_2, y_2) \\ P(x_3, y_1) & F(x_3, y_1) & P(x_3, y_2) \end{pmatrix}.$$

3. The Wall-Particle System

3.1. **Two coupled sequences.** Consider a coalescing skip-free process on $\mathbb{Z}$ with every site initially occupied. When two particles meet, they coalesce and continue as one. Fix a time $T > 0$. We call each surviving particle a *survivor* and write

$\mathbf{Y} = (\mathbf{y}_j)_{j\in\mathbb{Z}}$ for the increasing sequence of survivor positions (time-T coordinates), indexed by the integers.

Each survivor owns a *basin*: the contiguous set of initial positions whose particles merged into it. The basins partition $\mathbb{Z}$. Between consecutive basins lies a *wall*: the half-integer separating the last initial position in one basin from the first in the next. Write $\mathbf{X} = (\mathbf{x}_i)_{i\in\mathbb{Z}'}$ for the increasing sequence of walls (time-0 coordinates), where $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$. The basin of $\mathbf{y}_j$ is the set of initial integers in the interval $(\mathbf{x}_{j-1/2}, \mathbf{x}_{j+1/2})$. Throughout, $\mathbf{X}$ and $\mathbf{Y}$ denote the random sequences, with components $\mathbf{x}_i$ and $\mathbf{y}_j$. When x_i and y_j appear without boldface in formulas, they denote specific (deterministic) positions. See Figure 2 for an illustration.

Because walls live at time 0 and survivors at time T, there is no interlacing constraint on their values—a survivor need not lie inside its basin. The two sequences have interleaved indices—half-integers for walls, integers for survivors:

$$\cdots \nwarrow \mathbf{x}_{1/2} \nearrow \mathbf{y}_1 \nwarrow \mathbf{x}_{3/2} \nearrow \mathbf{y}_2 \nwarrow \mathbf{x}_{5/2} \nearrow \cdots$$

We use this *zigzag notation* throughout: each arrow connects a wall to an adjacent survivor in the index order.

Remark 3.1 (Labeling convention). The labeling of walls and survivors is determined only up to a global shift of the index set. When needed, we fix a reference by requiring $\mathbf{x}_{-1/2} < 0 \leq \mathbf{x}_{1/2}$, placing the origin in the basin of $\mathbf{y}_0$. This convention plays no role in the distributional results.

Each wall x_i ($i \in \mathbb{Z}'$) is flanked by two initial integer positions:

$$a_i = x_i - \tfrac{1}{2}, \qquad\qquad b_i = x_i + \tfrac{1}{2}.$$

Position a_i (to the left of the wall) belongs to the basin of survivor $y_{i-1/2}$, and b_i (to the right) to the basin of survivor $y_{i+1/2}$.

3.2. Correlation function.

Theorem 3.2 (Wall-particle correlation function)**.** *Consider a coalescing skip-free process on $\mathbb{Z}$ with every site initially occupied. For positions $x_{1/2} < \cdots < x_{k-1/2}$ and $y_0 < \cdots < y_k$, the probability that $(\mathbf{X}, \mathbf{Y})$ contains the consecutive pattern*

$$y_0 \nwarrow x_{1/2} \nearrow y_1 \nwarrow x_{3/2} \nearrow \cdots \nwarrow x_{k-1/2} \nearrow y_k$$

equals $\det(\tilde{M})$, where $\tilde{M}$ is the coalescence matrix (Definition 2.1) for $2k$ particles started at $a_1, b_1, \ldots, a_k, b_k$ with coalescence pattern $1{+}2{+}\cdots{+}2{+}1$: particle a_1 survives alone at y_0; each pair (b_l, a_{l+1}) merges into survivor y_l; and b_k survives alone at y_k.

Proof. Consider first only the $2k$ flanking particles $a_1, b_1, \ldots, a_k, b_k$ (no other sites occupied; see Figure 3 for $k = 2$). The coalescence determinant (Theorem 2.2) gives the probability of the stated coalescence event as $\det(\tilde{M})$.

Now populate every remaining integer site. In Arratia's construction of coalescing processes [Arr79], additional particles do not alter the trajectories of existing ones: they follow the same underlying paths and merge into whatever they meet. Because the process is skip-free, every intermediate particle (at a site between b_l and a_{l+1}) is trapped between the converging paths of b_l and a_{l+1} and must coalesce into the same survivor y_l (see Figure 3). The wall-particle event in the full system therefore has the same probability as the coalescence event for the $2k$ flanking particles alone. □

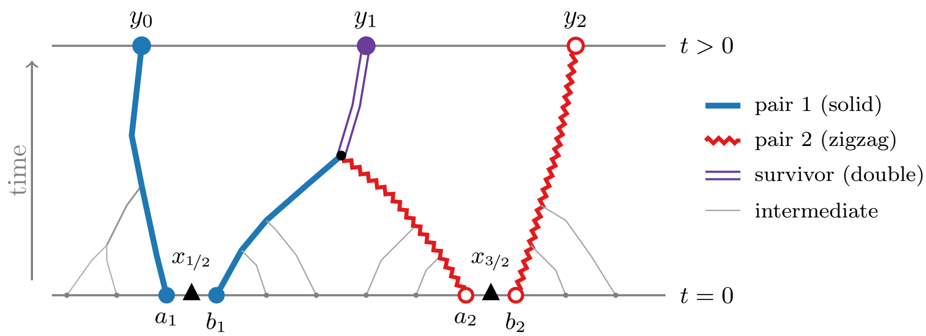

Figure 3. Proof of Theorem 3.2 for $k = 2$. The coalescence determinant applies to the four flanking particles a_1, b_1, a_2, b_2 (bold paths: solid for pair 1, zigzag for pair 2). Particles b_1 and a_2 coalesce into survivor y_1 (double line); particles a_1 and b_2 survive as y_0 and y_2. The intermediate particles (thin gray) cannot cross the flanking paths—the skip-free property traps them in the closing funnel between b_1 and a_2—so they are absorbed into the same survivors. Adding them does not change the coalescence outcome for the flanking particles.

3.3. **Block structure.** We now examine the matrix $\tilde{M}$ from Theorem 3.2 more closely. The pattern 1+2+$\cdots$+2+1 gives $\tilde{M}$ a 2×2 block structure: each wall contributes a row-pair and each interior survivor a column-pair (P and F), while the two boundary survivors contribute single P columns. Write $B_{i,j}$ ($i \in \mathbb{Z}'$, $j \in \mathbb{Z}$) for the 2×2 block at row-pair i (wall) and column-pair j (survivor):

$$B_{i,j} = \begin{pmatrix} P(a_i, y_j) & F(a_i, y_j) - [i < j] \\ P(b_i, y_j) & F(b_i, y_j) - [i < j] \end{pmatrix},$$

where $[i < j]$ is the Iverson bracket (equals 1 if $i < j$ and 0 otherwise).

Example 3.3 (Pattern 1+2+1). For $k = 2$ (two walls, three survivors), the pattern $y_0 \nwarrow x_{1/2} \nearrow y_1 \nwarrow x_{3/2} \nearrow y_2$ gives a 4×4 matrix with column structure 1+2+1. The interior survivor y_1 contributes a block $B_{i,j}$—a P column and an F column carrying a staircase step—while the boundary survivors y_0 and y_2 each contribute a single P column:

$$\tilde{M} = \begin{pmatrix} P(a_1, y_0) & P(a_1, y_1) & F(a_1, y_1) - 1 & P(a_1, y_2) \\ P(b_1, y_0) & P(b_1, y_1) & F(b_1, y_1) - 1 & P(b_1, y_2) \\ P(a_2, y_0) & P(a_2, y_1) & F(a_2, y_1) & P(a_2, y_2) \\ P(b_2, y_0) & P(b_2, y_1) & F(b_2, y_1) & P(b_2, y_2) \end{pmatrix}.$$

For the boundary case $k = 1$ (one wall, no interior survivors, no coalescence at all), there are no blocks $B_{i,j}$ and the matrix reduces to the 2×2 Karlin–McGregor determinant [KM59].

Corollary 3.4 (Block structure of $\tilde{M}$)**.** *The matrix $\tilde{M}$ from Theorem 3.2 for a pattern with k walls has a column structure* 1+2+$\cdots$+2+1*: the first and last columns are single P columns (one for each boundary survivor), and each interior*

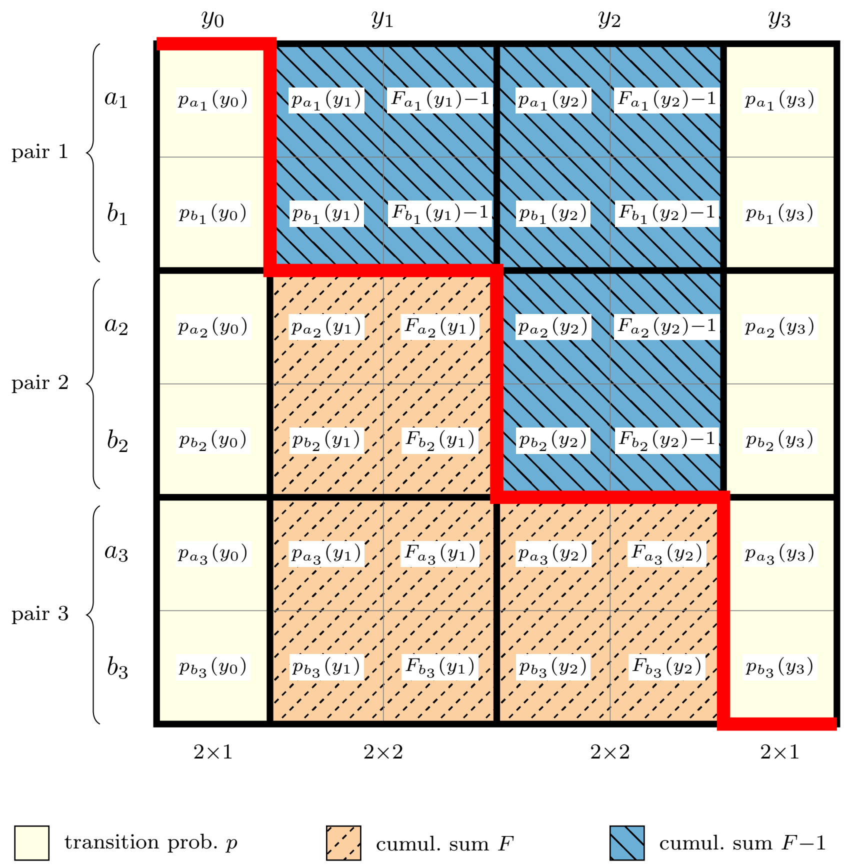

Figure 4. Block structure of $\tilde{M}$ for the 1+2+2+1 pattern ($k = 3$ walls). Rows come in pairs; columns are grouped by survivor: 2×1 boundary blocks for y_0 and y_3, and 2×2 interior blocks for y_1 and y_2, each containing a P column and an F column. The thick red staircase separates $F-1$ blocks (dark blue, solid hatching) from F blocks (orange, dashed hatching); unhatched yellow blocks contain only P entries.

survivor contributes a 2×2 *block* $B_{i,j}$*. See Figure 4 for the* $k = 3$ *case, where the block structure and the staircase pattern are fully apparent.*

Remark 3.5 (Staircase and block structure). The block formula for $B_{i,j}$ uses the Iverson bracket $[i < j]$, which steps at block boundaries. The coalescence matrix (Definition 2.1), however, defines the staircase at the level of individual rows. For the wall-particle patterns $1{+}2{+}\cdots{+}2{+}1$, each staircase step falls between the two rows of a single block, so the two conventions agree on all F columns (where the distinction matters) and may differ only on P columns—but P entries do not depend on the staircase. Warren's formula (Section 6), however, sums over all 2^{n-1} coalescence patterns and requires the original row-level staircase.

Remark 3.6 (Asymmetry between walls and survivors). The block matrix $\tilde{M}$ treats walls and survivors asymmetrically: the pattern has k walls but $k+1$ survivors, and the two boundary survivors each contribute only one column (rather than two), so walls and survivors cannot simply be interchanged. When the underlying process has a checkerboard structure—such as discrete-time ± 1 random walk—this asymmetry can be resolved: a decomposition of the lattice gives a duality between walls and survivors, connecting the wall-particle system to Pfaffian point processes. This is developed in [Śni26].

3.4. **Examples.** We describe two classes of processes satisfying the skip-free assumption of Theorem 3.2.

Example 3.7 (Simple symmetric random walk). Consider the ± 1 simple random walk: at each discrete time step, every particle moves left or right with equal probability. Space-time splits into two checkerboard sublattices: a particle at position x at time t satisfies either $x+t \equiv 0$ or $x+t \equiv 1 \pmod 2$, and each particle stays on one sublattice forever. If we start particles at *every* site of $\mathbb{Z}$, the Karlin–McGregor assumptions fail: a particle starting at an even site and a particle starting at an odd site live on complementary sublattices and can exchange order without ever sharing a site, so the non-crossing property does not hold.

The remedy is to occupy a single sublattice—say all even sites $2\mathbb{Z}$ at time 0. Particles on the same sublattice cannot cross without meeting (they share the same parity at every time step), so the skip-free assumption holds. Applying the framework to the initial sublattice $2\mathbb{Z}$ (with spacing 2 playing the role of the unit lattice), walls sit at the odd integers $2\mathbb{Z}+1$. At time T, all survivors share the same parity, so all gaps between consecutive survivors are even (see Theorem 5.1 for the gap distribution).

Example 3.8 (Birth-death chains). Continuous-time birth-death chains on the nonnegative integers are skip-free by construction (only ± 1 transitions), and every integer is a valid state at every time—no parity constraint. The wall-particle framework applies directly, with walls at $\{\frac{1}{2}, \frac{3}{2}, \dots\}$. For instance, the M/M/1 queue has transition probabilities expressible via modified Bessel functions [KM57].

3.5. **Multi-pattern correlations.** Theorem 3.2 extends to several separated consecutive patterns observed simultaneously, with an unspecified number of intermediate survivors between them. We state this for completeness; it is not used in the present paper.

Theorem 3.9 (Multi-pattern correlation function)**.** *Consider m separated consecutive patterns in the wall-particle system. Pattern α ($\alpha = 1, \dots, m$) consists of k_α walls and $k_\alpha + 1$ survivors:*

$$y_0^{(\alpha)} \nwarrow x_{1/2}^{(\alpha)} \nearrow y_1^{(\alpha)} \nwarrow \cdots \nwarrow x_{k_\alpha - 1/2}^{(\alpha)} \nearrow y_{k_\alpha}^{(\alpha)},$$

with walls and survivors each globally increasing. Between consecutive patterns, the number of intermediate survivors is unspecified.

The probability that $(\mathbf{X}, \mathbf{Y})$ contains all m patterns simultaneously equals $\det(\tilde{M})$, where $\tilde{M}$ is the coalescence matrix (Definition 2.1) for the $2K$ flanking particles ($K = \sum_\alpha k_\alpha$). Within each pattern, the coalescence is as in Theorem 3.2; between consecutive patterns, the boundary particles survive alone.

Proof. Apply the coalescence determinant (Theorem 2.2) to the $2K$ flanking positions. Within each pattern, the argument is identical to Theorem 3.2. Between patterns, intermediate particles are trapped between the last flanking particle of one pattern and the first of the next, so populating them does not change the probability. □

Example 3.10 (Two separated patterns). Consider $m = 2$ patterns: pattern 1 with $k_1 = 2$ (walls $x_{1/2}, x_{3/2}$, flanking positions a_1, b_1, a_2, b_2, survivors y_0, y_1, y_2) and pattern 2 with $k_2 = 1$ (wall $x_{5/2}$, flanking positions a_3, b_3, survivors y_3, y_4), with an unspecified number of intermediate survivors between y_2 and y_3. The 6×6 matrix is:

$$\tilde{M} = \left(\begin{array}{c||c|cc|c||c|c}
 & y_0 & & y_1 & y_2 & y_3 & y_4 \\
\hline\hline
a_1 & P & P & F-1 & P & P & P \\
b_1 & P & P & F-1 & P & P & P \\
\hline
a_2 & P & P & F & P & P & P \\
b_2 & P & P & F & P & P & P \\
\hline\hline
a_3 & P & P & F & P & P & P \\
b_3 & P & P & F & P & P & P
\end{array}\right).$$

Here P denotes a transition probability entry $P(a_l, y_j)$ or $P(b_l, y_j)$, and F, $F-1$ denote cumulative sum entries $F(\cdot, y_1)$ or $F(\cdot, y_1) - 1$.

The double lines mark the pattern boundary. Within pattern 1 (upper-left 4×4 block), the structure is the familiar 1+2+1 from Example 3.3: the interior survivor y_1 has an F column, with the staircase separating $F-1$ (wall 1, above) from F (walls 2 and 3, below). Pattern 2 (lower-right 2×2 block) is pure Karlin–McGregor.

The off-diagonal blocks couple the two patterns; their entries are all P. No F column appears between y_2 and y_3: this absence is what allows an arbitrary number of unspecified intermediate survivors between the two patterns. Compare Figure 4, where the single pattern 1+2+2+1 has F columns for *every* interior survivor.

4. Brownian Motion Setting

We specialize the block matrix $\tilde{M}$ (Corollary 3.4) to Brownian motion. In the continuous limit, each flanking pair (a_l, b_l) collapses to a single wall position; subtracting the two rows of each block and dividing by the grid spacing turns the row pair into a Gaussian density and its spatial derivative. The resulting $2k \times 2k$ matrix M_0 is explicit. We then pass to the maximal entrance law—coalescing Brownian motions starting from all of $\mathbb{R}$, a classical construction due to Arratia [Arr79]—and obtain a determinantal formula for the intensity of the wall-particle system.

4.1. Transition densities.

Transition densities. Write $p_x(y)$ for the Gaussian transition density at time T (fixed throughout):

$$p_x(y) = \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{(y-x)^2}{2T}\right),$$

and $F_x(y)$ for the Gaussian CDF:

$$F_x(y) = \int_{-\infty}^{y} p_x(z)\, dz = \Phi\left(\frac{y-x}{\sqrt{T}}\right),$$

where Φ is the standard normal CDF.

Start coalescing Brownian motions from a grid with spacing ε. Each wall is flanked by starting positions a_l and $b_l = a_l + \varepsilon$. The coalescence determinant (Theorem 2.2) gives a $2k \times 2k$ matrix as in Section 3.2, with Brownian densities $p_x(y)$ and CDFs $F_x(y)$ in place of $P(x, y)$ and $F(x, y)$.

Subtracting the a_l row from the b_l row within each pair (which preserves the determinant) and dividing by ε replaces the second row by the derivative ∂_x, up to $O(\varepsilon)$ error. Each 2×2 interior block becomes:

$$\begin{pmatrix} p_x(y) & F_x(y) - [\cdot] \\ \partial_x p_x(y) & \partial_x F_x(y) \end{pmatrix} + O(\varepsilon),$$

where $[\cdot]$ stands for the Iverson bracket as in $B_{i,j}$ (Section 3.3). The boundary columns (containing only p entries) undergo the same row operation, becoming $(p_x(y),\ \partial_x p_x(y))^T$. Each row-pair contributes one factor of ε.

Proposition 4.1 (Grid refinement)**.** *For coalescing Brownian motions starting from a grid with spacing ε, the wall-particle correlation function for k walls near $x_{1/2}, \dots, x_{k-1/2}$ and $k+1$ survivors near $y_0, \dots, y_k$ equals $\varepsilon^k \det(M_0) + O(\varepsilon^{k+1})$, where M_0 is the $2k \times 2k$ matrix with alternating density and derivative rows and the column structure of $\tilde{M}$. Each wall contributes one factor of ε.*

Proof. The row operations above give $\det(\tilde{M}) = \varepsilon^k \det(M_0) + O(\varepsilon^{k+1})$. □

Remark 4.2 (Wronskian structure)*.* For translation-invariant kernels $p_x(y) = p(y - x)$,

$$\partial_x p_x(y) = -\partial_y p_x(y), \qquad \partial_x F_x(y) = -p_x(y).$$

Thus the derivative row of each block in M_0 is $-\partial_y$ applied to the density row, giving a generalized Wronskian in the CDF F_x evaluated at y. This structure matches the Tribe–Zaboronski kernel for coalescing Brownian motions [TZ11; GPTZ18]; the connection is explored in [Śni26].

4.2. **Maximal entrance law.** Arratia [Arr79] constructs coalescing Brownian motions starting from all of $\mathbb{R}$ via dyadic approximation and proves that the set of survivors is locally finite. We use the same construction to build the joint $(\mathbf{X}, \mathbf{Y})$ system.

Fix a time $T > 0$. At step $n = 0, 1, 2, \dots$, start independent Brownian motions from every point of $2^{-n}\mathbb{Z}$ at time 0, and run them until time T with the coalescing rule: when two trajectories meet, they merge and continue as one.

The construction is incremental. Going from step n to step $n+1$, the new starting points $2^{-n-1}(2\mathbb{Z}+1)$ interleave the existing ones from $2^{-n}\mathbb{Z}$. Each newly launched Brownian motion either hits an existing trajectory before time T and is absorbed, or survives to time T without hitting any existing trajectory. The set of survivors grows monotonically with n: existing survivors are never removed. The limiting set is locally finite: two Brownian motions distance ε apart fail to coalesce by time T with probability $\varepsilon/\sqrt{\pi T} + O(\varepsilon^2)$ (this is Arratia's estimate [Arr79]; it also follows from Proposition 4.1 with $k = 1$), so the expected number of survivors per unit length remains bounded as the grid refines.

The construction produces the $(\mathbf{X}, \mathbf{Y})$ system from Section 3: the wall sequence $\mathbf{X} = (\mathbf{x}_i)$ and the survivor sequence $\mathbf{Y} = (\mathbf{y}_j)$ (positions at time T), with basins partitioning $\mathbb{R}$. The maximal entrance law inherits translation invariance from the dyadic grid: the shifts by 2^{-n} become dense as $n \to \infty$, so the joint law of $(\mathbf{X}, \mathbf{Y})$ is invariant under all translations of $\mathbb{R}$.

4.3. **Wall-particle intensity.** A consecutive pattern with k walls and $k+1$ survivors lives in $\mathbb{R}^{2k+1}$ (k wall coordinates plus $k+1$ survivor coordinates), and the wall-particle system under the maximal entrance law forms a point process on this space. Its *intensity* is the density of the point process with respect to Lebesgue measure: integrating over a region gives the expected number of patterns in that region. Combining the grid-refinement limit (Proposition 4.1) with the dyadic construction gives this intensity in closed form.

Theorem 4.3 (Wall-particle intensity). *Under the maximal entrance law for coalescing Brownian motions at time $T > 0$, the consecutive pattern*

$$y_0 \nwarrow x_{1/2} \nearrow y_1 \nwarrow \cdots \nwarrow x_{k-1/2} \nearrow y_k$$

has intensity

$$\det\big(M_0(x_{1/2}, \dots, x_{k-1/2};\, y_0, \dots, y_k)\big)$$

with respect to Lebesgue measure on $\mathbb{R}^{2k+1}$, where M_0 is the $2k \times 2k$ matrix from Proposition 4.1.

Proof. At dyadic step n, the starting grid has spacing $\varepsilon = 2^{-n}$. By Proposition 4.1, the wall-particle correlation function for k walls near $x_{1/2}, \dots, x_{k-1/2}$ and survivors near $y_0, \dots, y_k$ is $\varepsilon^k \det(M_0) + O(\varepsilon^{k+1})$. Each factor of ε matches the grid spacing per wall, so $\det(M_0)$ is the density per unit length in each wall coordinate. Since the survivor set is locally finite [Arr79] and grows monotonically with each dyadic refinement, it almost surely stabilizes on any bounded interval after finitely many steps. Passing to $n \to \infty$ gives the result. □

4.4. **Example: reflected Brownian motion.** The grid-refinement technique of Section 4.1 extends naturally to any process with continuous paths and a smooth transition density. As an illustration beyond standard Brownian motion, we treat coalescing Brownian motions on the half-line $[0, \infty)$ with reflection at 0. Translation invariance is lost, and the reflecting boundary introduces a leftmost survivor with a special role.

Reflected Brownian motion on $[0, \infty)$ has transition density

$$p_x(y) = \frac{1}{\sqrt{2\pi T}} \left[e^{-(y-x)^2/(2T)} + e^{-(y+x)^2/(2T)} \right], \quad x, y \geq 0.$$

This is skip-free (continuous paths), so the coalescence determinant applies. The maximal entrance law occupies all of $[0, \infty)$; translation invariance is broken by the boundary at 0.

The set of survivors is half-infinite: there is a leftmost survivor $\mathbf{y}_0$, with basin $[0, \mathbf{x}_{1/2})$ bounded on the left by the reflecting boundary. We write

$$\mathbf{y}_0 \nwarrow \mathbf{x}_{1/2} \nearrow \mathbf{y}_1 \nwarrow \mathbf{x}_{3/2} \nearrow \mathbf{y}_2 \nwarrow \cdots$$

for the wall-particle system, with walls $0 < \mathbf{x}_{1/2} < \mathbf{x}_{3/2} < \cdots$ and survivors $0 < \mathbf{y}_0 < \mathbf{y}_1 < \mathbf{y}_2 < \cdots$.

Theorem 4.4 (Half-line intensity). *Under the maximal entrance law for reflected Brownian motion on $[0, \infty)$ at time $T > 0$, the pattern*

$$y_0 \nwarrow x_{1/2} \nearrow y_1 \nwarrow \cdots \nwarrow x_{k-1/2} \nearrow y_k$$

with y_0 the leftmost survivor has intensity

$$\det\big(M_0(x_{1/2},\dots,x_{k-1/2};\, y_0,\dots,y_k)\big)$$

with respect to Lebesgue measure on $(0,\infty)^{2k+1}$. Here M_0 is a $(2k{+}1)\times(2k{+}1)$ matrix: its first row comes from a particle at the boundary 0, and its remaining k row-pairs are the density and ∂_x-derivative rows from the k walls. Columns are organized as in the coalescence pattern $2{+}2{+}\cdots{+}2{+}1$: a P-and-F column-pair for each group of size 2, and a single P column for the final group of size 1.

Proof. Construct the maximal entrance law on $[0,\infty)$ by dyadic approximation, as in Section 4.2: at step n, start reflected Brownian motions from every point of $\{0, 2^{-n}, 2\cdot 2^{-n},\dots\}$.

At grid spacing $\varepsilon = 2^{-n}$, the particle at 0 and the $2k$ flanking particles $a_1, b_1,\dots,a_k,b_k$ form a system of $2k+1$ particles. The coalescence determinant (Theorem 2.2) gives their joint distribution under the pattern $2{+}2{+}\cdots{+}2{+}1$: the particle at 0 and a_1 merge into y_0; each pair (b_l, a_{l+1}) merges into y_l; and b_k survives alone at y_k. Intermediate particles are trapped by the skip-free property (as in Theorem 3.2): between 0 and a_1, the reflecting boundary prevents leftward escape; between b_l and a_{l+1}, the closing funnel absorbs all intermediate particles.

The grid-refinement procedure (Proposition 4.1) carries over: each (a_l, b_l) pair collapses to a density row and a ∂_x-derivative row, contributing one factor of ε; the particle at 0 contributes a single row. Thus

$$\det(\tilde{M}) = \varepsilon^k \det(M_0) + O(\varepsilon^{k+1}).$$

Since the survivor set is locally finite [Arr79] and grows monotonically, it stabilizes on any bounded interval after finitely many steps. Passing to $n\to\infty$ gives the intensity $\det(M_0)$. □

Example 4.5 (One wall on the half-line). For $k=1$ (one wall, two survivors, leftmost survivor at y_0), the pattern $2{+}1$ gives the 3×3 matrix

$$M_0 = \begin{pmatrix} p_0(y_0) & F_0(y_0)-1 & p_0(y_1) \\ p_{x_{1/2}}(y_0) & F_{x_{1/2}}(y_0) & p_{x_{1/2}}(y_1) \\ \partial_x p_{x_{1/2}}(y_0) & \partial_x F_{x_{1/2}}(y_0) & \partial_x p_{x_{1/2}}(y_1) \end{pmatrix},$$

where $p_x(y)$ and $F_x(y)$ use the reflected Brownian motion transition density.

5. Gap Distributions

We compute gap distributions in the discrete setting (where no limiting procedure is needed) and for Brownian motion under the maximal entrance law. All results are stated in terms of the *gap intensity measure*: $\mu(\{g\})$ is the expected number of gaps of size g per unit length. Dividing by the total intensity $\sum_g \mu(\{g\})$ (the survivor density) recovers the gap probability distribution.

5.1. **Single gap.**

We derive the single-gap intensity measure, first in the discrete setting (where the formula is exact) and then for Brownian motion under the maximal entrance law. The Brownian result recovers the Rayleigh law (see Section 1.4 for prior work); we give a new proof through the wall-particle system.

5.1.1. *Discrete single-gap distribution.* Apply the wall-particle system with $k = 1$: a single wall at half-integer x separates two basins, with flanking sites $a = x - \frac{1}{2}$ and $b = x + \frac{1}{2}$ (so $b = a + 1$). The two survivors $y_0 < y_1$ satisfy the Karlin–McGregor non-intersection condition, and the coalescence determinant gives:

$$\det(\tilde{M}) = P(a, y_0)\, P(b, y_1) - P(a, y_1)\, P(b, y_0).$$

Summing the wall-particle intensity over all wall positions gives the probability that y_0 and y_1 are consecutive survivors:

$$\mathbb{P}(y_0, y_1 \text{ consecutive survivors}) = \sum_{a \in \mathbb{Z}} \big[P(a, y_0)\, P(a{+}1, y_1) - P(a, y_1)\, P(a{+}1, y_0)\big]. \tag{5.1}$$

This formula is valid for any skip-free process with every site initially occupied: no translation invariance or symmetry is needed.

Translation-invariant case. Assume $P(x, y) = P(0, y - x)$ for all $x, y \in \mathbb{Z}$; we write $P(n) = P(0, n)$ when the time T is understood. Writing $g = y_1 - y_0$ for the gap, translation invariance gives $P(a, y_0) = P(y_0 - a)$, $P(a{+}1, y_1) = P(y_1 - a - 1)$, and so on; each summand in (5.1) depends only on g and $a - y_0$, so the sum is independent of y_0. Define the *autocorrelation*

$$R(m) = \sum_{s \in \mathbb{Z}} P_T(s)\, P_T(s + m).$$

Then the two sums in (5.1) are $R(g - 1)$ and $R(g + 1)$ respectively, giving

$$\mu(\{g\}) = R(g - 1) - R(g + 1). \tag{5.2}$$

Symmetric case. When the walk is *symmetric* ($P(n) = P(-n)$ for all n), the autocorrelation reduces to a convolution: $R(m) = \sum_s P(s)\, P(m - s) = P_{2T}(m)$ by the Chapman–Kolmogorov identity, so $\mu(\{g\}) = P_{2T}(g - 1) - P_{2T}(g + 1)$. The continuous-time simple random walk (± 1 jumps, each at rate 1) is the main example: every integer is reachable from every integer, so there is no parity constraint.

Theorem 5.1 (Discrete gap intensity measure)**.** *For a symmetric translation-invariant coalescing skip-free process on $\mathbb{Z}$ with every site initially occupied, the gap intensity measure is*

$$\mu(\{g\}) = P_{2T}(g - 1) - P_{2T}(g + 1), \qquad g = 1, 2, 3, \ldots$$

The total intensity is a telescoping sum:

$$\sum_{g=1}^{\infty} \mu(\{g\}) = P_{2T}(0) + P_{2T}(1),$$

since $P_{2T}(g) \to 0$ as $g \to \infty$; this gives the survivor density per site. Dividing by the total intensity recovers the probability mass function $\mathbb{P}(G = g) = \mu(\{g\})/[P_{2T}(0) + P_{2T}(1)]$.

Proof. Equation (5.2) gives $\mu(\{g\})$. For the total intensity, the partial sum

$$\sum_{g=1}^{N} \big[P_{2T}(g - 1) - P_{2T}(g + 1)\big] = P_{2T}(0) + P_{2T}(1) - P_{2T}(N) - P_{2T}(N + 1)$$

telescopes (reindex: the first sum runs over $m = 0, \ldots, N - 1$ and the second over $m = 2, \ldots, N + 1$). Letting $N \to \infty$ gives total intensity $P_{2T}(0) + P_{2T}(1)$. ∎

5.1.2. *Scaling limit preview.* Under diffusive scaling, the discrete gap distribution from Theorem 5.1 recovers the Brownian motion Rayleigh density from Theorem 1.3. The scaling sends:

- lattice spacing $\varepsilon \to 0$;
- discrete gap $g \in \mathbb{Z}$ to continuous gap $G = g \cdot \varepsilon$ (measured in the rescaled coordinates);
- discrete time t to continuous time with $\varepsilon^2 t$ held fixed.

In this regime, the transition probability $P_{2t}(n)$ at the integer $n = (g \pm 1)$ is well approximated by the Gaussian density $p(n\varepsilon, 2\varepsilon^2 t)$ times the lattice spacing ε. The difference at unit spacing becomes a derivative:

$$P_{2t}(g-1) - P_{2t}(g+1) \approx -2\varepsilon^2 \, \partial_G \, p(G, 2T)\Big|_{G=g\varepsilon,\, T=\varepsilon^2 t},$$

and since $\partial_G \, p(G, 2T) = -\frac{G}{2T} \, p(G, 2T)$, this gives

$$\mu(dG) \propto G \, e^{-G^2/(4T)} \, dG,$$

the gap intensity measure, identifying the Rayleigh($\sqrt{2}$) family.

This argument is a scaling heuristic, not a rigorous proof: a complete derivation requires controlling the error terms in the Gaussian approximation and the convergence of the normalizing constants. The rigorous Brownian motion proof follows.

5.2. **Brownian motion single gap.** We prove Theorem 1.3 by applying the $(\mathbf{X}, \mathbf{Y})$ framework from Section 3 (with the Brownian motion specialization from Section 4.1) with $k = 1$ and passing to the maximal entrance law (Section 4.2).

We recall the *Rayleigh distribution* with scale parameter $\sigma > 0$:

$$\text{Rayleigh}(\sigma): \quad f_\sigma(G) = \frac{G}{\sigma^2} e^{-G^2/(2\sigma^2)}, \quad G > 0.$$

The mean is $\sigma\sqrt{\pi/2}$ and the variance is $(4-\pi)\sigma^2/2$.

Theorem 5.2 (Gap intensity measure under the maximal entrance law)**.** *Under the maximal entrance law, the gap intensity measure in rescaled coordinates has density*

$$\mu(dG) = \frac{G}{2\sqrt{\pi}} \, e^{-G^2/4} \, dG, \quad G > 0.$$

Normalizing to a probability distribution gives the Rayleigh($\sqrt{2}$) *density* $f(G) = \frac{G}{2} \, e^{-G^2/4}$.

Corollary 5.3 (Density of surviving particles)**.** *The total intensity is* $\int_0^\infty \mu(dG) = 1/\sqrt{\pi}$*, giving the rescaled survivor density. Since the rescaled mean gap is* $\mathbb{E}[G] = \sqrt{\pi}$*, the mean gap between consecutive survivors at time* T *is* $\sqrt{\pi T}$*, and the density of survivors per unit length is* $1/\sqrt{\pi T}$*.*

Proof of Theorem 5.2. By Theorem 4.3 with $k = 1$, the intensity of the pattern $y_0 \nwarrow x \nearrow y_1$ is $\det(M_0)$, where

$$M_0 = \begin{pmatrix} p_x(y_0) & p_x(y_1) \\ \partial_x p_x(y_0) & \partial_x p_x(y_1) \end{pmatrix}.$$

Using $\partial_x p_x(y) = p_x(y) \cdot (y-x)/T$, the determinant is:

$$\det(M_0) = p_x(y_0)\, p_x(y_1) \cdot \frac{y_1 - y_0}{T}.$$

Change coordinates: let $u = y_0 - x$ (displacement of left survivor from wall position) and $G = y_1 - y_0$ (gap). Then $p_x(y_0) = (2\pi T)^{-1/2} e^{-u^2/(2T)}$ and $p_x(y_1) = (2\pi T)^{-1/2} e^{-(u+G)^2/(2T)}$, so

$$\det(M_0) = \frac{G}{2\pi T^2} \exp\left(-\frac{u^2 + (u+G)^2}{2T} \right).$$

Completing the square: $u^2 + (u+G)^2 = 2(u + G/2)^2 + G^2/2$. Integrating over u:

$$\int_{-\infty}^{\infty} \det(M_0)\, du = \frac{G}{2\pi T^2}\, e^{-G^2/(4T)} \sqrt{\pi T} = \frac{G}{2\sqrt{\pi}\, T^{3/2}}\, e^{-G^2/(4T)}.$$

By translation invariance, this is independent of the location y_0, so it gives the gap intensity per unit (unrescaled) length. Changing to rescaled coordinates $G = (y_1 - y_0)/\sqrt{T}$, the gap intensity per unit rescaled length becomes

$$\mu(dG) = \frac{G}{2\sqrt{\pi}}\, e^{-G^2/4}\, dG,$$

which is $1/\sqrt{\pi}$ times the Rayleigh($\sqrt{2}$) density. □

5.3. **Joint distribution of consecutive gaps.** We derive the joint intensity of two consecutive gaps, first in the discrete setting and then for Brownian motion, proving Theorem 1.4. The Brownian joint density was previously obtained by ben-Avraham and Brunet [AB05] from the IPDF hierarchy; our derivation via the wall-particle system applies to any skip-free process, including discrete random walks and birth-death chains.

5.3.1. *Discrete joint gaps.* For $k = 2$ consecutive gaps, the 1+2+1 pattern from Example 3.3 gives a 4×4 determinant. As in the single-gap case (Section 5.1.1), summing the wall-particle intensity over all wall positions gives the probability that y_0, y_1, y_2 are consecutive survivors:

$$\mathbb{P}(y_0, y_1, y_2 \text{ consecutive survivors}) = \sum_{a_1, a_2 \in \mathbb{Z}} \det(\tilde{M}),$$

where $a_l, b_l = a_l + 1$ are the flanking sites of wall l, and $\tilde{M}$ is the 4×4 matrix

$$\tilde{M} = \left(\begin{array}{c|cc|c} P(a_1,y_0) & P(a_1,y_1) & F(a_1,y_1)-1 & P(a_1,y_2) \\ P(b_1,y_0) & P(b_1,y_1) & F(b_1,y_1)-1 & P(b_1,y_2) \\ \hline P(a_2,y_0) & P(a_2,y_1) & F(a_2,y_1) & P(a_2,y_2) \\ P(b_2,y_0) & P(b_2,y_1) & F(b_2,y_1) & P(b_2,y_2) \end{array}\right).$$

By translation invariance, the sum depends only on the gaps $g_1 = y_1 - y_0$ and $g_2 = y_2 - y_1$; in the Brownian limit the sums over wall positions become the integrals over u and v in Theorem 5.4 below.

5.3.2. *Brownian motion joint gaps.* We apply the $(\mathbf{X}, \mathbf{Y})$ framework from Section 3 with $k = 2$: two walls produce three survivors at positions $y_0 < y_1 < y_2$, with consecutive gaps $G_1 = y_1 - y_0$ and $G_2 = y_2 - y_1$.

Theorem 5.4 (Joint gap intensity and negative correlation)**.** *The joint gap intensity for* (G_1, G_2) *is*

$$h(G_1, G_2) = \iint_{u<v} \det M_0(u, v;\, G_1, G_2)\, du\, dv,$$

where u, v *are the rescaled positions of the two walls (the constraint* $u < v$ *is the wall ordering). The three survivors are placed at rescaled positions* 0, G_1, $G_1 + G_2$ *(translation invariance fixes the leftmost survivor at the origin). Write* $S = G_1 + G_2$ *for the total gap. The* 4×4 *matrix* M_0 *is:*

$$M_0 = \left(\begin{array}{c|cc|c} \phi(u) & \phi(G_1{-}u) & \Phi(G_1{-}u) - 1 & \phi(S{-}u) \\ -u\,\phi(u) & (G_1{-}u)\,\phi(G_1{-}u) & -\phi(G_1{-}u) & (S{-}u)\,\phi(S{-}u) \\ \hline \phi(v) & \phi(G_1{-}v) & \Phi(G_1{-}v) & \phi(S{-}v) \\ -v\,\phi(v) & (G_1{-}v)\,\phi(G_1{-}v) & -\phi(G_1{-}v) & (S{-}v)\,\phi(S{-}v) \end{array}\right),$$

where ϕ *is the standard normal density and* Φ *its CDF. The block structure is* $(2{+}2){\times}(1{+}2{+}1)$ *(Example 3.3 and Figure 4); the even rows are the source derivatives from the grid refinement of Proposition 4.1.*

The marginal gap distributions are each $\mathrm{Rayleigh}(\sqrt{2})$*, but the joint intensity does not factorize. The gaps are negatively correlated, with* $\rho \approx -0.163$*.*

General structure for k gaps. The joint intensity $h(G_1, \ldots, G_k)$ of k consecutive gaps is obtained by integrating a $2k \times 2k$ determinant with the same block structure as in Section 3.2 and Figure 4: k row-pairs (density and derivative) and $k{+}1$ density columns plus $k{-}1$ CDF columns. The case $k = 2$ above has 3 density columns and 1 CDF column. See Remark 5.5 for the general case.

Proof. By Theorem 4.3, the intensity of the consecutive pattern $y_0 \nwarrow x_1 \nearrow y_1 \nwarrow x_2 \nearrow y_2$ is $\det(M_0)$. Fix rescaled coordinates: place the three survivors at 0, G_1, $G_1 + G_2$ (translation invariance removes the location variable), and let u, v be the rescaled wall positions. The Gaussian density $\phi(y - x)$ and CDF $\Phi(y - x)$ give the matrix entries directly; the source derivative $\partial_x \phi(y - x) = (y - x)\,\phi(y - x)$ and $\partial_x \Phi(y - x) = -\phi(y - x)$ supply the even rows (Remark 4.2). Integrating $\det(M_0)$ over the wall positions with the ordering constraint $u < v$ yields the joint gap intensity. □

The determinant does not factorize as $f(G_1) \cdot g(G_2)$ because the wall variables u, v couple the two gaps: each wall position interacts with all three survivors. Numerical integration gives $\rho \approx -0.163$ (to three decimal places). An analytical proof that $\rho < 0$ remains open, as does a closed-form expression for ρ.

See Figure 5.

Remark 5.5 (Higher-order gap distributions)*.* The joint intensity of k consecutive gaps $(G_1, \ldots, G_k)$ follows from the $1{+}2{+}\cdots{+}2{+}1$ pattern (Section 3) with $2k$ particles in k pairs. By Proposition 4.1, the $2k{\times}2k$ determinant scales as $\varepsilon^k \det(M_0)$, where M_0 has alternating rows of transition densities and spatial derivatives. This determines the joint intensity $h(G_1, \ldots, G_k)$.

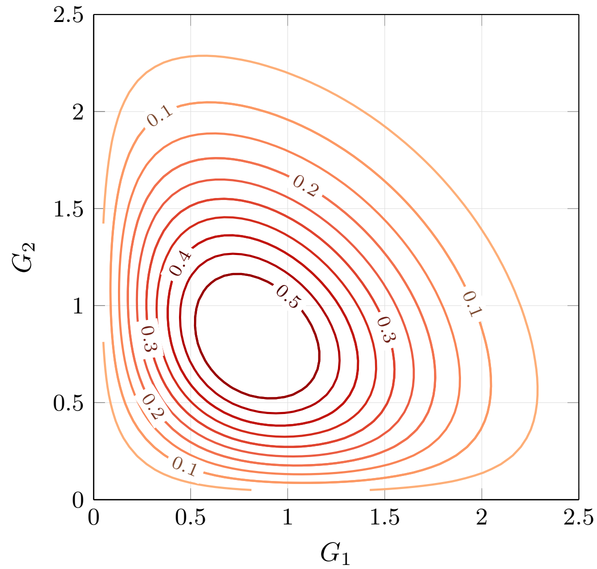

Figure 5. Joint gap intensity $h(G_1, G_2)$ from Theorem 5.4, computed by numerical integration. The tilted elliptical contours reflect negative correlation ($\rho \approx -0.163$).

Preliminary numerical computations suggest that non-adjacent gaps also have negative correlations, though weaker than for adjacent gaps.

6. Warren's Formula

We now derive the determinantal CDF formula described in Section 1.2 for all skip-free processes from the coalescence determinant.

The setting is the following. Start n particles at fixed positions $x_1 < \cdots < x_n$; each performs an independent skip-free process until it meets another particle, at which point the two coalesce and continue as one. At time T, some of the original n particles have merged, so the number of distinct positions is at most n. Warren's formula gives the joint CDF of these positions.

We use the notation $P(x, y)$ and $F(x, y)$ from Section 2. We write $\sum_z$ for summation over the state space S; in the continuous case, all sums become integrals.

Theorem 6.1 (Warren's formula for skip-free processes). *Let $X^{(1)}, \ldots, X^{(n)}$ be coalescing skip-free processes on a linearly ordered state space S, starting at positions $x_1 < x_2 < \cdots < x_n$. Let $Z_T(x_i)$ denote the position at time T of the particle that started at x_i (after possible coalescence).*

For $y_1 \leq y_2 \leq \cdots \leq y_n$ in S:

$$\mathbb{P}\big(Z_T(x_i) \leq y_i \text{ for all } i = 1, \ldots, n\big) = \det(M^W),$$

where the $n \times n$ matrix M^W has entries:

$$M^W_{ij} = F(x_i, y_j) - [i < j].$$

Proof. The plan is: sum over all coalescence patterns $n = n_1 + \cdots + n_k$, and for each pattern sum the survivor positions $z_1, \ldots, z_k$ over the region compatible with the

CDF thresholds $y_1, \dots, y_n$. We then show that the resulting sum of determinants collapses to $\det(M^W)$.

By Theorem 2.2, composition $n_1 + \cdots + n_k = n$ produces an $n \times n$ coalescence matrix $\tilde{M}$ whose determinant gives the joint density of the k survivor positions. All particles in block l share the same survivor z_l, so the constraint $z_l \le y_i$ for every i in the block reduces to $z_l \le y_{j_l}$, where $j_l = n_1 + \cdots + n_{l-1} + 1$ is the first index in the block (giving the smallest threshold). The CDF decomposes as

$$\mathbb{P}\big(Z_T(x_i) \le y_i \;\forall\, i\big) = \sum_{n_1+\cdots+n_k=n} \sum_{\substack{z_1<\cdots<z_k \\ z_l \le y_{j_l}}} \det \tilde{M}(z_1, \dots, z_k),$$

a sum over all 2^{n-1} compositions of n.

We evaluate this sum by collapsing one column at a time, from right to left.

Summing out the rightmost survivor. We sum out the survivor variable that feeds into column n, grouping patterns in pairs. For each composition $m_1+\cdots+m_r = n-1$, define two compositions of n:

- Pattern A: $m_1 + \cdots + m_r + 1$ (particle n in a new block);
- Pattern B: $m_1 + \cdots + (m_r{+}1)$ (particle n appended to the last block).

This pairs the 2^{n-1} compositions of n into 2^{n-2} pairs.

Columns $1, \dots, n-1$ of the coalescence matrix are identical for paired patterns A and B (same block sizes, same survivors): extending the last block adds column n but does not alter the existing columns.

In Pattern A, the survivor z_{r+1} of the new block $\{n\}$ appears only in column n (P-type). Summing over $z_{r+1} \in (z_r,\, y_n]$ replaces this column by the vector

$$\big(F(x_i, y_n) - F(x_i, z_r)\big)_i.$$

By multilinearity of the determinant in column n, Pattern A contributes

$$\det\big(\cdots,\ F_\cdot(y_n)\big) - \det\big(\cdots,\ F_\cdot(z_r)\big),$$

where $\cdots$ denotes columns $1, \dots, n-1$ and $F_\cdot(y)$ is the column $(F(x_i, y))_{i=1}^n$.

In Pattern B, column n is $F(x_i, z_r) - [i < n]$ (an F-column with staircase, since n is not the first index in its block). By multilinearity:

$$\det\big(\cdots,\ F_\cdot(z_r)\big) - \det\big(\cdots,\ [i < n]\big).$$

Adding: the $F_\cdot(z_r)$ terms cancel, leaving

$$\det\big(\cdots,\ F_\cdot(y_n) - [i < n]\big).$$

Column n is now $M^W_{\cdot,n} = F(x_i, y_n) - [i < n]$, independent of all survivor variables.

The rightmost survivor has been summed out: column n is in its final Warren form, and the remaining sum runs over the 2^{n-2} compositions of $n-1$ and their survivors.

Iteration. Repeating the pairing at boundary $(n-2, n-1)$ collapses column $n-1$ to $F(x_i, y_{n-1}) - [i < n-1]$, reducing to 2^{n-3} compositions of $n-2$. After $n-1$ iterations every column is in Warren form. The final step sums the sole remaining survivor z_1 over $(-\infty, y_1]$, giving column $1 = F_\cdot(y_1)$ (with $[i < 1] = 0$). The result is $\det(M^W)$. □

Theorem 6.1 applies to all the skip-free processes discussed in earlier sections, including birth-death chains (Example 3.8) and reflected Brownian motion (Section 4.4).

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