Eigenvalues of non-hermitian matrices and invariant subspace conjecture

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Non–hermitian random matrices and operators

Selfadjoint operators on a Hilbert space and hermitian random matrices are well understood.

Non–selfadjoint operators and non–hermitian random matrices have wild properties.

Examples of problems

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- *invariant subspace conjecture: is true that for every bounded operator X on a Hilbert space $\mathcal{H}$ there exists a nontrivial closed invariant subspace $\mathcal{K} \subset \mathcal{H}$?*
- *can we have some version of the spectral theorem?*

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Elements of $\mathcal{A}$ will be called non–commutative random variables.

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Therefore random matrices fit into the framework of finite von Neumann algebras.

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Inspired by this, for any non–commutative random variable x we define its Fuglede–Kadison determinant

$$\Delta(x) = \exp\left[\phi(\log\sqrt{x^\star x})\right].$$

Brown measure of a random matrix

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Exercise

Distribution of eigenvalues of A is given by

$$\mu_A = \frac{1}{2\pi} \textit{Laplacian of logarithm of } (1) = \frac{1}{2\pi}\left(\frac{\partial^2}{(\partial \Re z)^2} + \frac{\partial^2}{(\partial \Im z)^2}\right) \log \Delta(x - z).$$

Brown measure of operators

Inspired by this, for any element x we define its Brown measure:

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Speaking informally, Brown measure tells us "how many" eigenvalues of x belong to a given subset of $\mathbb{C}$.

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$$\lim_{N \to \infty} \operatorname{tr}_N \left[\left(A^{(N)}\right)^{s_1} \cdots \left(A^{(N)}\right)^{s_n} \right] = \phi(x^{s_1} \cdots x^{s_n})$$

holds almost surely.

Is Brown measure continuous?

Conjecture

Let $A^{(N)}$ be a sequence of matrices which converges in $\star$–moments to a non–commutative random variable x.
Is it true that the eigenvalues distribution of $A^{(N)}$ converges to the Brown measure of x?

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Unfortunately, this conjecture is false! Why? because non-hermitian matrices have wild properties!

Discontinuity of Brown measure: Counterexample

Let $\Xi^{(N)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$ be $N \times N$ nilpotent matrix.

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Therefore Brown measures $\mu_{\Xi^{(N)}}$ do not converge to the Brown measure of μ_u.

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Therefore: we do not like small eigenvalues of $A^\star A$.

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Outline

1. Brown spectral measure
2. Random regularization of spectral measure
   - Main theorem
   - Other random corrections (interesting for random matrix community)
   - Proof of the main theorem (stochastic Itô integration, sketch)
   - Proof of the main theorem (stochastic Itô integration, complete)
3. Invariant subspaces in II_1 factors
4. Final remarks

The correction: Gaussian random matrices

We say that an $N \times N$ random matrix $G^{(N)}$ is a standard Gaussian random matrix if

$$\left(\Re G_{ij}^{(N)}\right)_{1 \leq i,j \leq N}, \left(\Im G_{ij}^{(N)}\right)_{1 \leq i,j \leq N}$$

are independent Gaussian variables with mean zero and variance $\frac{1}{2N}$.

Main theorem: regularization of Brown measure

Theorem (Piotr Śniady, 2001)

Let $A^{(N)}$ *be a sequence of random matrices which converges in* $\star$*–moments to* x *almost surely.*

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There exists a sequence (t_N) of numbers which converges to 0 such that the corrected sequence

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- *empirical eigenvalues distributions of matrices $A'^{(N)}$ converge to the Brown measure of x almost surely* ▸ example ⏩ skip proof

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Disadvantage of this correction: Cauchy distribution has a very heavy tail, for example none of its moments is finite.

Is Brown measure usually continuous?

Conjecture

Let $x \in \mathcal{A}$ be some element and let $(A^{(N)})$ be the most natural sequence of random matrices such that $A^{(N)}$ converges to x in $\star$–moments. Then the sequence of empirical eigenvalues distributions $\mu_{A^{(N)}}$ converges to μ_x.

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The key element of the above conjecture are the words *the most natural* since nobody knows what exactly should it mean in our context.

Nevertheless, this conjecture seems to be very important.

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Suppose that $A^{(N)}$ converges to x in $\star$–moments almost surely, let $U^{(N)}$ be a sequence of independent Haar unitary matrices, let u be a Haar unitary such that x and u are free. Is it true that for $t > 0$ we have $\mu_{A^{(N)}+tU^{(N)}} \longrightarrow \mu_{x+tu}$? Is it true that $\mu_{A^{(N)}U^{(N)}} \longrightarrow \mu_{xu}$?

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This conjecture would solve the long–standing problem about the eigenvalues of $A^{(N)}U^{(N)}$.

Matrix Brownian motion

We say that $M^{(N)}(t)$ is a matrix Brownian motion,

$$M^{(N)}(t) = \left(M_{ij}^{(N)}(t)\right)_{1 \leq i,j \leq N},$$

if

$$\left(\Re M_{ij}^{(N)}\right)_{1 \leq i,j \leq N}, \left(\Im M_{ij}^{(N)}\right)_{1 \leq i,j \leq N}$$

are independent Brownian motions.

Idea of the proof

1. We define a matrix–valued stochastic process

$$A_t^{(N)} = A^{(N)} + M_t^{(N)}.$$

Then $A^{(N)} + \sqrt{t_N} G^{(N)}$ has the same distribution as $A_{t_N}^{(N)}$.

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2. Brown measure is defined in terms of Fuglede–Kadison determinant. It is enough to construct a sequence of (t_N) such that

$$\lim_{N \to \infty} \Delta(A_{t_N}^{(N)}) = \Delta(x)$$

holds almost surely.

Idea of the proof (continued)

3. We say that $\lambda_1 \leq \cdots \leq \lambda_N$ are the singular values of a matrix A if $\lambda_1, \ldots, \lambda_N$ are eigenvalues of $\sqrt{A^\star A}$.

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$$\log \Delta(A) = \frac{\log \lambda_1 + \cdots + \log \lambda_N}{N}.$$

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Let $\lambda_1(t) \leq \cdots \leq \lambda_N(t)$ be singular values of $A_t^{(N)}$. Our goal is to show that small singular values of $A_t^{(N)}$ cannot be too small.

Idea of the proof (continued)

4. Singular values fulfill the following stochastic differential equation (in the Itô sense):

$$d\lambda_i(t) = \Re(dM_{ii}) + \frac{dt}{2\lambda_i}\left(1 - \frac{1}{2N} + \sum_{j\neq i} \frac{\lambda_i^2 + \lambda_j^2}{N(\lambda_i^2 - \lambda_j^2)}\right),$$

where M is a standard matrix Brownian motion.
The above equation allows us to show bottom bounds for singular values.

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What can we say about the singular values of $A_t^{(N)}$ which will be denoted by $\lambda_1(t) \leq \cdots \leq \lambda_N(t)$?

Eigenvalues of a perturbed matrix

Lemma

If D is a diagonal matrix with eigenvalues $\nu_1, \dots, \nu_N$ and ΔD is any matrix then the eigenvalues $\nu'_1, \dots, \nu'_N$ of a matrix $D + \Delta D$ are given by

$$\nu'_i = \nu_i + \Delta D_{ii} + \sum_{j \neq i} \frac{\Delta D_{ij} \Delta D_{ji}}{\nu_i - \nu_j} + O\big(\|\Delta D\|^3 \big)$$

for small enough $\|\Delta D\|$.

Singular values of a perturbed matrix

Corollary

If F is a diagonal matrix with positive eigenvalues $\lambda_1, \ldots, \lambda_N$, and ΔF is any matrix then the singular values $\lambda'_1, \ldots, \lambda'_N$ of $F + \Delta F$ are given by

$$(\lambda'_i)^2 = \lambda_i^2 + 2\lambda_i \Re \Delta F_{ii} + \sum_j |\Delta F_{ji}|^2$$

$$+ \sum_{j \neq i} \frac{\lambda_i^2 |\Delta F_{ij}|^2 + 2\lambda_i \lambda_j \Re(\Delta F_{ij} \Delta F_{ji}) + \lambda_j^2 |\Delta F_{ji}|^2}{\lambda_i^2 - \lambda_j^2} + O\left(\|\Delta F\|^3\right).$$

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We can always find unitaries U, V such that $F = U A_t^{(N)} V$ is diagonal; then $\Delta F = U(A_{t+\Delta t} - A_t)V$. Thus we get a relation between the singular values of A_t and $A_{t+\Delta t}$.

Stochastic differential equation for singular values

This gives a stochastic differential equation (in the Itô sense) for $\lambda_1(t), \dots, \lambda_N(t)$, the singular values of A_t:

$$d\lambda_i(t) = \Re(dM_{ii}) + \frac{dt}{2\lambda_i}\left(1 - \frac{1}{2N} + \sum_{j\neq i} \frac{\lambda_i^2 + \lambda_j^2}{N(\lambda_i^2 - \lambda_j^2)}\right), \qquad (2)$$

where M is a standard matrix Brownian motion.

Lemma

For each $k \in \{1, 2\}$ *let* $\lambda_1^{(k)}(t), \ldots, \lambda_N^{(k)}(t)$ *be a solution to the system of equations* (2)*. If*

$$\lambda_i^{(1)}(t) < \lambda_i^{(2)}(t) \tag{3}$$

holds for all $1 \leq i \leq N$ *and* $t = 0$ *then it holds true for all* $t \geq 0$*.*

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Proof.

Let t_0 be the minimal value of t for which (3) is not true; for example $\lambda_j^{(1)}(t_0) = \lambda_j^{(2)}(t_0)$.

Lemma

For each $k \in \{1, 2\}$ let $\lambda_1^{(k)}(t), \ldots, \lambda_N^{(k)}(t)$ be a solution to the system of equations (2). If

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holds for all $1 \leq i \leq N$ and $t = 0$ then it holds true for all $t \geq 0$.

Proof.

Let t_0 be the minimal value of t for which (3) is not true; for example $\lambda_j^{(1)}(t_0) = \lambda_j^{(2)}(t_0)$. If $\lambda_i^{(1)}(t_0) = \lambda_i^{(2)}(t_0)$ for all i then (uniqueness of solutions) it follows that $\lambda_i^{(1)}(t) = \lambda_i^{(2)}(t)$ for every $t \geq 0$, contradiction.

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Let t_0 be the minimal value of t for which (3) is not true; for example $\lambda_j^{(1)}(t_0) = \lambda_j^{(2)}(t_0)$. If $\lambda_i^{(1)}(t_0) = \lambda_i^{(2)}(t_0)$ for all i then (uniqueness of solutions) it follows that $\lambda_i^{(1)}(t) = \lambda_i^{(2)}(t)$ for every $t \ge 0$, contradiction. Therefore $\lambda_i^{(1)}(t_0) \le \lambda_i^{(2)}(t_0)$ and at least one inequality is sharp. Then (use stochastic differential equation)

$$d\big[\lambda_j^{(2)}(t_0) - \lambda_j^{(1)}(t_0)\big] > 0$$

contradicts the minimality of t_0.

The lemma on comparison

Corollary

For every $t \geq 0$ and a matrix A there exist random matrices $G^{(1)}, G^{(2)}$ such that each matrix $G^{(i)}$ is a standard Gaussian random matrix (but matrices $G^{(1)}$ and $G^{(2)}$ might be dependent) and such that

$$\operatorname{tr} f\left(\left|\sqrt{t}\ G^{(1)}(\omega)\right|\right) \leq \operatorname{tr} f\left(\left|A + \sqrt{t}\ G^{(2)}(\omega)\right|\right)$$

holds for every $\omega \in \Omega$ and every nondecreasing function $f : \mathbb{R} \to \mathbb{R}$.

Random regularization of Fuglede–Kadison determinant

Theorem

Let $A^{(N)}$ be a sequence of random matrices which converges in $\star$–moments to x almost surely and let $t > 0$. Then the sequence of random matrices $A'^{(N)} := A^{(N)} + \sqrt{t} G^{(N)}$ converges in $\star$–moments to some operator x_t. Furthermore,

$$\lim_{N \to \infty} \operatorname{tr} \log |A'^{(N)}| = \log \Delta(x_t)$$

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Proof.

The existence of x_t is given by free probability theory (easy). Difficult is the convergence of determinants (next transparency). □

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$$f_\epsilon(r) = \frac{\ln(r^2+\epsilon)}{2}, \qquad g_\epsilon(r) = \ln r - \frac{\ln(r^2+\epsilon)}{2}.$$

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Upper bound is easy. Bottom bound: g_ϵ is increasing!

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The distribution of the singular values of $G^{(N)}$ is known and the right-hand side can be directly computed. □

Corollary

Let $A^{(N)}$ be a sequence of random matrices which converges in $\star$–moments to x almost surely. Then there exists a sequence t_N which converges to 0 and such that the sequence of random matrices $A'^{(N)} := A^{(N)} + \sqrt{t_N} G^{(N)}$ fulfills

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For each $t > 0$

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so we can chose a sequence t_N which converges to 0 and such that

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The upper bound is easy. □

Theorem

Let (t_N) be the sequence given by the previous corollary. Then the sequence of empirical distributions of matrices $A'^{(N)} := A^{(N)} + \sqrt{t_N} G^{(N)}$ converges to the Brown measure of x almost surely.

Proof.

$$\int_{\mathbb{C}} f(\lambda)\, d\mu_y(\lambda) = \frac{1}{2\pi} \langle f(\lambda), \nabla^2 \ln \Delta(y-\lambda) \rangle =$$

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$$\lim_{N\to\infty} \operatorname{tr} \ln \left| A'^{(N)} - \lambda \right| = \ln \Delta(x-\lambda)$$

for almost all $\lambda \in K$ holds almost surely; then majorized convergence theorem and Fubini theorem finish the proof. □

Outline

Invariant subspace conjecture: Relative version

Conjecture

Let $x \in \mathcal{A} \subset B(\mathcal{H})$ and $x \notin \mathbb{C}$, where $\mathcal{A}$ is a finite von Neumann algebra (II_1 factor) and $\mathcal{H}$ is a Hilbert space.

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Idea of attack:

1. approximate operator x by random matrices;
2. study the invariant subspaces for the random matrix models;
3. deduce existence of invariant subspaces for x;

Connes' embedding problem

Conjecture

Is it true that for every $x \in \mathcal{A}$ there exists a sequence of matrices $A^{(N)}$ which converges to x in $\star$-moments?

Connes' embedding problem

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From the following on, we will study only x for which such a sequence exists.

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Theorem (Uffe Haagerup, 2001)

An analogue of Jordan's decomposition for finite matrices holds true for $x \in \mathcal{A}$:

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$$x = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}$$

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and the Brown measure of x_{11} is supported in G_1 and x_{22} is supported in G_2.

Existence of invariant subspaces

Corollary

Suppose that $x \in \mathcal{A}$ is such that:

- *x can be approximated by matrices;*
- *Brown measure μ_x is not supported in only one point in $\mathbb{C}$;*

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Corollary

Suppose that $x \in \mathcal{A}$ is such that:

- *x can be approximated by matrices;*
- *Brown measure μ_x is not supported in only one point in $\mathbb{C}$;*

Then x has a nontrivial invariant subspace $\mathcal{K}$; furthermore $p_{\mathcal{K}} \in \mathcal{A}$.

Outline

Summary

- Brown spectral measure gives information about spectral properties of random matrices and operators;

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Summary

- Brown spectral measure gives information about spectral properties of random matrices and operators;
- Brown spectral measure does not behave in a continuous way for non–hermitian random matrices and non–hermitian operators;
- random regularization by a random correction repairs the continuity. In this way questions concerning spectral properties of random matrices and operators are related to each other;
- Haagerup's spectral theorem: a generalization of Jordan form for finite matrices is true also in finite von Neumann algebras;

Postscript: are random matrices really necessary?

Theorem (Uffe Haagerup and Hanne Schultz, 2004)

New proof Haagerup's spectral theorem which does not use Connes' embedding conjecture and does not use random matrices.

Advertisement

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  Random matrices, free probability and the invariant subspace problem relative to a von Neumann algebra.
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  Inequality for Voiculescu's free entropy in terms of Brown measure.
  Internat. Math. Res. Notices, 2003(1):51–64, 2003.

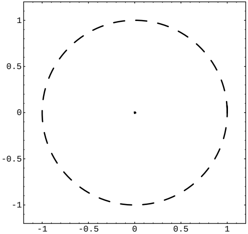
Outline I

- Non–hermitian random matrices and operators
- Examples of problems
- Solution: Brown spectral measure

1. Brown spectral measure
   - Common setup for random matrices and operators
   - Fuglede–Kadison determinant Δ
   - Brown measure
   - Discontinuity of spectral measure
2. Random regularization of spectral measure
   - Main theorem
   - Other random corrections (interesting for random matrix community
   - Proof of the main theorem (stochastic Itô integration, sketch)
   - Proof of the main theorem (stochastic Itô integration, complete)

Outline II

Random regularization of nilpotent matrix $\Xi^{(N)}$



$$\Xi^{(N)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Matrices $\Xi^{(N)}$ are nilpotent; they converge to a *Haar unitary*, the spectral measure of which is the uniform measure on the unit circle.

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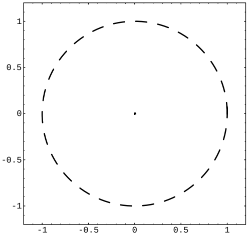

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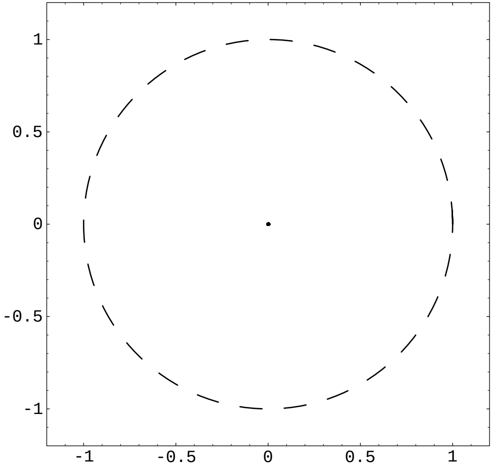
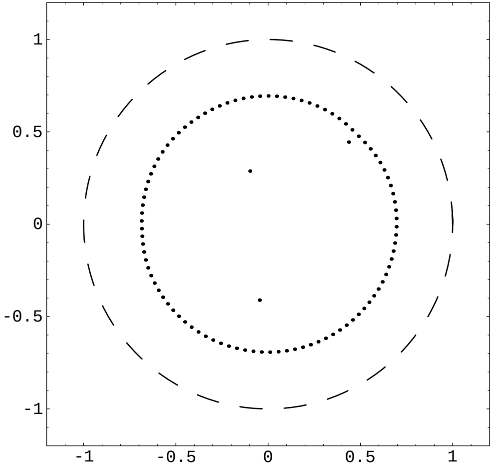
Matrices $\Xi^{(N)}$ are nilpotent; they converge to a *Haar unitary*, the spectral measure of which is the uniform measure on the unit circle.
Our goal: study eigenvalues of $\Xi^{(N)} + \sqrt{t} G^{(N)}$ for $N = 100$ and different values of t.

Example: $t = 0$



Sample eigenvalues of a random matrix $\Xi^{(N)} + \sqrt{t}G^{(N)}$ for $N = 100$ and $t = 0$.

Example: $t = 10^{-30}$



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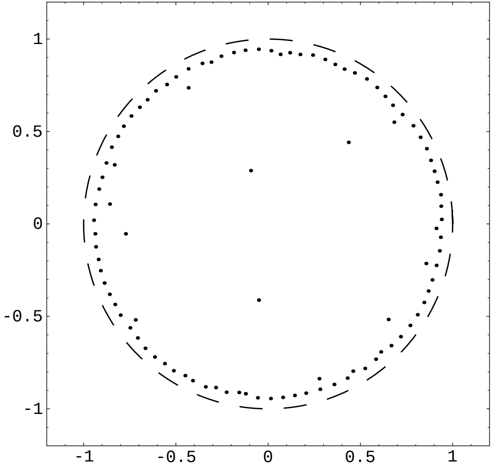

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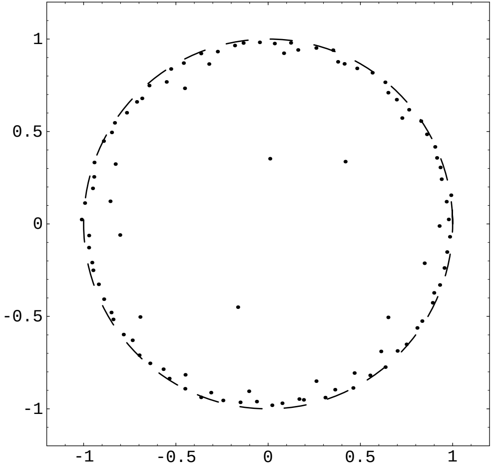
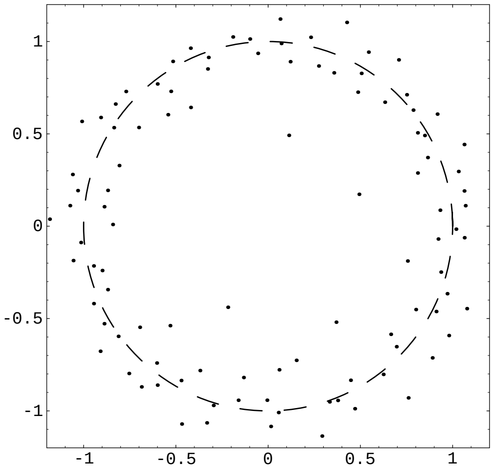
Example: $t = 10^{-2}$



Sample eigenvalues of a random matrix $\Xi^{(N)} + \sqrt{t}G^{(N)}$ for $N = 100$ and $t = 10^{-2}$.

Example: $t = 3 \cdot 10^{-1}$



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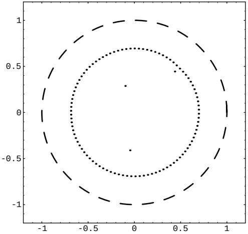
Example: summary



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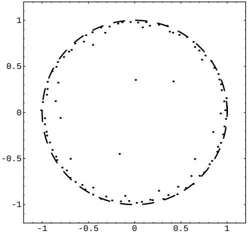
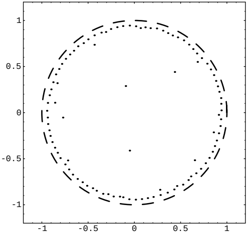
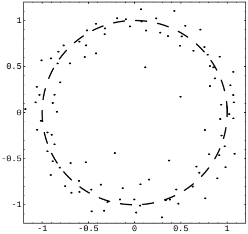


$t = 10^{-2}$



$t = 10^{-5}$



$t = 3 \cdot 10^{-1}$

go back