Representations of Lie groups and random matrices

joint work with Benoît Collins

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Outline

"big representations of the unitary groups behave like random matrices"

explanation: *this happens because representation can be viewed as a random matrix (with quantum entries)*

Representations of $U(d)$

we say that Π is a representation of the unitary group $U(d)$
if $\Pi\colon U(d) \to \operatorname{End}(V)$ for some vector space V is such that

$$\Pi(gh) = \Pi(g)\Pi(h),$$

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we say that representation Π is reducible
if $V = V_1 \oplus V_2$ and $\Pi = \Pi_1 \oplus \Pi_2$,

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irreducible representations of $U(d)$ are indexed by highest weights:
tuples $\Lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$, where $\lambda_1, \dots, \lambda_d \in \mathbb{Z}$,

notation: $\epsilon\Lambda = (\epsilon\lambda_1, \dots, \epsilon\lambda_d)$ for $\epsilon \in \mathbb{R}$

Irreducible representations of $U(d)$, examples

representation on symmetric tensors

$$\mathrm{Sym}^k \mathbb{C}^d$$

is irreducible with $\Lambda = (k, 0, 0, \ldots, 0)$

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representation on antisymmetric tensors

$$\Lambda^k \mathbb{C}^d$$

is irreducible with $\Lambda = (\underbrace{1, 1, \ldots, 1}_{k \text{ times}}, 0, \ldots, 0)$

Representations of $U(d)$

let reducible representation Π of $U(d)$ be given

Π can be written as a sum of irreducible components

we defne random highest weight associated to Π with distribution

$$P(\Lambda) = \frac{(\text{multiplicity of } \Lambda \text{ in } \Pi) \cdot (\text{dimension of } \Lambda)}{(\text{dimension of } \Pi)}$$

Part 1.

representation theory of $U(d)$
d is fixed

Problem: tensor product of representations

let $\Pi^{(1)}, \Pi^{(2)}$ be irreducible representations of $U(d)$

Kronecker tensor product is a representation $\Pi^{(1)} \otimes \Pi^{(2)}$ of $U(d)$ defined by

$$\left[\Pi^{(1)} \otimes \Pi^{(2)}\right](g) = \left[\Pi^{(1)}(g)\right] \otimes \left[\Pi^{(2)}(g)\right]$$

$$\Pi^{(1)} \otimes \Pi^{(2)} = ?$$

Problem: tensor product of representations

let $\epsilon_n \to 0$

let $(\Lambda_n^{(1)})$ and $(\Lambda_n^{(2)})$ be two sequences of highest weights such that

$$\epsilon_n \Lambda_n^{(1)} \to \Lambda^{(1)}, \qquad \epsilon_n \Lambda_n^{(2)} \to \Lambda^{(2)}$$

let $(\Pi_n^{(1)})$ and $(\Pi_n^{(2)})$ be irreducible representations of $U(d)$ corresponding to the highest weights $(\Lambda_n^{(1)})$ and $(\Lambda_n^{(2)})$

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let $\Lambda_n^{(3)}$ be the random highest weight associated to $\Pi_n^{(1)} \otimes \Pi_n^{(2)}$

$$\epsilon_n \Lambda_n^{(3)} \to ?$$

Tensor product of representations: solution

let $A^{(1)}$ and $A^{(2)}$ be independent, unitarily invariant hermitian $d \times d$ random matrices with deterministic eigenvalues $\Lambda^{(1)}$ and $\Lambda^{(2)}$

Theorem

$$\epsilon_n \Lambda_n^{(3)} \xrightarrow{\textit{in distribution}} \textit{eigenvalues of } A^{(1)} + A^{(2)}$$

Quantum random variables

matrix algebra $M_k(\mathbb{C})$ can be viewed as
algebra of quantum random variables

mean value $\mathbb{E}X = \frac{1}{k} \operatorname{Tr} X = \operatorname{tr} X$

if $X_1, X_2, \dots$ are quantum random variables, their joint distribution is a collection of their mixed moments:

$$\left(\mathbb{E} X_{i_1} \cdots X_{i_l} \right)_{i_1, \dots, i_l}$$

Spectral measure

spectral measure: for $\Lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$ we set

$$\mu_\Lambda = \frac{\delta_{\lambda_1} + \cdots + \delta_{\lambda_d}}{d}$$

if Λ is a random weight then its spectral measure is a random probability measure on $\mathbb{R}$

in a similar way, spectral measure for random matrices

Spectral measure of a quantum random matrix

if (A_{ij}) is a random matrix then its spectral measure is a random probability measure μ on $\mathbb{R}$ such that

$$\mathbb{E} M_{k_1}(\mu) \cdots M_{k_l}(\mu) = \mathbb{E}\, \mathrm{tr}\, A^{k_1} \cdots \mathrm{tr}\, A^{k_l},$$

where

$$M_k(\mu) = \int_{\mathbb{R}} x^k \, d\mu(x)$$

Spectral measure of a quantum random matrix

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Sketch of proof

a representation of the Lie group $\Pi : U(d) \to \mathrm{End}(V)$
gives a representation of the Lie algebra $\pi : \mathfrak{u}(d) \to \mathrm{End}(V)$

$$\pi = \begin{bmatrix} \pi(e_{11}) & \cdots & \pi(e_{1d}) \\ \vdots & \ddots & \vdots \\ \pi(e_{d1}) & \cdots & \pi(e_{dd}) \end{bmatrix}$$

can be viewed as a matrix with quantum entries

(spectral measure of π) $\approx$ (random highest weight Λ)

Sketch of proof

a representation of the Lie group $\Pi : U(d) \to \mathrm{End}(V)$
gives a representation of the Lie algebra $\pi : \mathfrak{u}(d) \to \mathrm{End}(V)$

$$\epsilon\pi = \begin{bmatrix} \epsilon\pi(e_{11}) & \cdots & \epsilon\pi(e_{1d}) \\ \vdots & \ddots & \vdots \\ \epsilon\pi(e_{d1}) & \cdots & \epsilon\pi(e_{dd}) \end{bmatrix}$$

can be viewed as a matrix with quantum entries

(spectral measure of $\epsilon\pi$) $\approx$ (random highest weight $\epsilon\Lambda$)

Sketch of proof: asymptotic commutativity

assume that $\epsilon \to 0$ and $\epsilon\pi$ is bounded

$$[\,\pi_{ij},\ \pi_{kl}] = \left(\delta_{jk}\ \ \pi_{il} - \delta_{li}\ \ \pi_{kj}\right)$$

so $\epsilon\pi$ converges (in distribution) to a matrix with commuting entries

this is the unitarily invariant random matrix with the distribution of eigenvalues given by the random highest weight $\epsilon\Lambda$

$$\Pi^{(3)} = \Pi^{(1)} \otimes \Pi^{(2)}$$

implies

$$\epsilon\pi^{(3)} = \underbrace{\epsilon\pi^{(1)}}_{\approx A^{(1)}} \otimes 1 + 1 \otimes \underbrace{\epsilon\pi^{(2)}}_{\approx A^{(2)}}$$

Sketch of proof: asymptotic commutativity

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$$[\epsilon\pi_{ij}, \epsilon\pi_{kl}] = \epsilon\big(\delta_{jk}\ \epsilon\pi_{il} - \delta_{li}\ \epsilon\pi_{kj}\big)$$

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Sketch of proof: asymptotic commutativity

assume that $\epsilon \to 0$ and $\epsilon\pi$ is bounded

$$[\epsilon\pi_{ij}, \epsilon\pi_{kl}] = \epsilon\left(\delta_{jk}\ \epsilon\pi_{il} - \delta_{li}\ \epsilon\pi_{kj}\right) \to 0$$

so $\epsilon\pi$ converges (in distribution) to a matrix with commuting entries

this is the unitarily invariant random matrix with the distribution of eigenvalues given by the random highest weight $\epsilon\Lambda$

$$\Pi^{(3)} = \Pi^{(1)} \otimes \Pi^{(2)}$$

implies

$$\epsilon\pi^{(3)} = \underbrace{\epsilon\pi^{(1)}}_{\approx A^{(1)}} \otimes 1 + 1 \otimes \underbrace{\epsilon\pi^{(2)}}_{\approx A^{(2)}}$$

It is trivial!

toy example:

decomposition of tensor product of two irreducible representations of $SO(3)$	$\longleftrightarrow$	addition of quantum angular momenta

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classical limit:

$$\hbar \to 0$$

commutators vanish, we recover classical addition of angular momenta

Part 2.

representation theory of $U(d)$
$d \to \infty$

Problem: tensor product of representations

let $(\Pi_n^{(1)})$ and $(\Pi_n^{(2)})$ be irreducible representations of $U(n)$ corresponding to the highest weights $(\Lambda_n^{(1)})$ and $(\Lambda_n^{(2)})$

assume that

$$\epsilon_n \Lambda_n^{(1)} \to \Lambda^{(1)}, \quad \epsilon_n \Lambda_n^{(2)} \to \Lambda^{(2)}$$

---

let $\Lambda_n^{(3)}$ be the random highest weight associated to $\Pi_n^{(1)} \otimes \Pi_n^{(2)}$

$$\epsilon_n \Lambda_n^{(3)} \xrightarrow{\text{in distribution}} ?$$

Tensor product of representations: solution

let $A_n^{(1)}$ and $A_n^{(2)}$ be independent, unitarily invariant $n \times n$ hermitian random matrices with deterministic eigenvalues $\Lambda_n^{(1)}$ and $\Lambda_n^{(2)}$

Theorem

assume that $\epsilon_n n \to 0$

then

1. *the spectral measure of $\epsilon_n \Lambda_n^{(3)}$,*
2. *the spectral measure of $A_n^{(1)} + A_n^{(2)}$*

are asymptotically Gaussian with the same mean and the same global fluctuations

extension of Biane [1995]: $\epsilon_n n^{\text{arbitrary number}} \to 0$, law of large numbers

Tensor product of representations: solution extended

I claim that if μ_n is

1. the spectral measure of $\epsilon_n \Lambda_n^{(3)}$,
2. the spectral measure of $A_n^{(1)} + A_n^{(2)}$,

and

$$M_{k,n} = \int x^k d\mu_n, \qquad \left[M_{k,n}\right]_0 = M_{k,n} - \mathbb{E} M_{k,n}$$

then

$$\lim_{n\to\infty} \mathbb{E} M_{k,n} \text{ exists for every } k \geq 1,$$

$$\left(n \left[M_{k,n}\right]_0\right)_{k\geq 1} \text{ converges to a Gaussian distribution}$$

and the limits are the same for both cases

Sketch of proof

study unitarily invariant random matrices (with quantum entries)

find relationship between

- statistical properties of the spectral measure
- joint distribution of the entries of the matrix

if the non-commutativity of the entries is small, the matrix behaves like a non-quantum random matrix

Higher-order free probability theory

Setup: algebra $\mathcal{A}$ of quantum random variables, $a, b, \dots \in \mathcal{A}$
mean value $\mathbb{E} : \mathcal{A} \to \mathbb{C}$, covariance $k_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$

$A_n, B_n, \dots$ are random matrices of size n

$(A_n, B_n, \dots) \to (a, b, \dots)$ means that

- mixed moments of $A_n, B_n, \dots$ converge, for example

$$\mathbb{E} \operatorname{tr} A_n B_n B_n \to \mathbb{E} abb,$$

- mixed covariances of $A_n, B_n, \dots$ converge, for example

$$\operatorname{Cov}(\operatorname{Tr} A_n B_n, \operatorname{Tr} B_n) \to k_2(ab, b),$$

- higher order cumulants of traces vanish quickly enough,

Higher-order free probability theory

usual freeness and freeness of higher order

higher order freeness describes fluctuations of random matrices which are sufficiently random

nice combinatorial machinery: free cumulants and free cumulants of higher order

Summary / open problems

- representation can be viewed as a random matrix with quantum entries
- (sometimes) the non-commutativity disappears
- asymptotically representation behaves like a usual random matrix

- can we use this idea to prove other connections between representations and random matrix theory?

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Greg Kuperberg.
Random words, quantum statistics, central limits, random matrices.
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