dual combinatorics of Jack polynomials and maps

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motivations for Jack characters:

- Jack polynomials,
- deformation of characters of the symmetric groups S_n,
- amazing combinatorial conjectures
  related to $\longrightarrow$ maps,
- extra deformation parameter α,
  new scaling and universality for random matrices,

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characters of the symmetric group S_n:

$$\chi_\lambda^{(1)}(\pi) := \frac{\operatorname{Tr} \rho_\lambda(\pi)}{\operatorname{Tr} \rho_\lambda(\mathrm{Id})}$$

$$\mathrm{Ch}_\pi^{(1)}(\lambda) := \begin{cases} \underbrace{|\lambda| \cdot (|\lambda|-1) \cdots (|\lambda|-|\pi|+1)}_{|\pi| \text{ factors}} \ \chi_\lambda^{(1)}(\pi) & \text{if } |\pi| \leq |\lambda| \\ 0 & \text{otherwise} \end{cases}$$

$\longrightarrow$ Kerov & Olshanski

Jack characters:

$$J_\lambda^{(\alpha)} = \sum_\pi \chi_\lambda^{(\alpha)}(\pi)\; p_\pi \frac{n!}{z_\pi}$$

$$\mathrm{Ch}_\pi^{(\alpha)}(\lambda) = \alpha^{-\frac{|\pi|-\ell(\pi)}{2}} \underbrace{|\lambda|\cdot(|\lambda|-1)\cdots(|\lambda|-|\pi|+1)}_{|\pi| \text{ factors}}\; \chi_\lambda^{(\alpha)}(\pi,1,1,\dots,1)$$

if $|\pi| \leq |\lambda|$

$\longrightarrow$ Lassalle; Dołęga & Féray

$$\mathrm{Ch}_1(\lambda) = \sum_i \lambda_i,$$

$$\mathrm{Ch}_2(\lambda) = \sqrt{\alpha} \sum_i \left(\lambda_i^2 - \lambda_i\right) - \frac{1}{\sqrt{\alpha}} \sum_{i<j} 2\lambda_j$$

polynomials in $\lambda_1, \lambda_2, \ldots$

$$\mathrm{Ch}_2(\lambda) = \sqrt{\alpha} \sum_i \left(x_i^2 - \left(\frac{-i}{\alpha} \right)^2 \right) + \left(-\sqrt{\alpha} + \frac{2}{\sqrt{\alpha}} \right) \sum_i \left(x_i - \left(\frac{-i}{\alpha} \right) \right)$$

where $x_i = \lambda_i - \frac{i}{\alpha}$

symmetric polynomials in $x_1, x_2, \dots$ proof $\longrightarrow$ LASSALLE

$\longrightarrow$ algebra of α-polynomial functions of KEROV & OLSHANSKI for each π and each $\alpha > 0$ $\longrightarrow$ Féray

$\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots)$ is the unique polynomial such that:

- $\mathrm{Ch}_\pi\left(x_1 + \frac{1}{\alpha}, x_2 + \frac{2}{\alpha}, \dots\right)$ is symmetric in $x_1, x_2, \dots$;
- polynomial $\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots)$ is of degree $|\pi|$;
  its top-degree homogeneous part is equal to

$$\alpha^{\frac{|\pi|-\ell(\pi)}{2}} \; p_\pi;$$

- for all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $|\lambda| < |\pi|$

$$\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots) = 0$$

for each π and each $\alpha > 0$ $\longrightarrow$ Féray

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- for all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $|\lambda| < |\pi|$
$$\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots) = 0$$

---

if we view α as indeterminate,

- for each Young diagram λ
$\mathrm{Ch}_\pi(\lambda) \in \mathbb{Q}\left[\sqrt{\alpha}, \frac{1}{\sqrt{\alpha}}\right]$ is a Laurent polynomial of degree (at most) $|\pi| - \ell(\pi)$

structure coefficients for Jack characters:

$$\operatorname{Ch}_2 \operatorname{Ch}_2 = 2\delta \operatorname{Ch}_2 + 2 \operatorname{Ch}_{1,1} + 4 \operatorname{Ch}_3 + \operatorname{Ch}_{2,2},$$

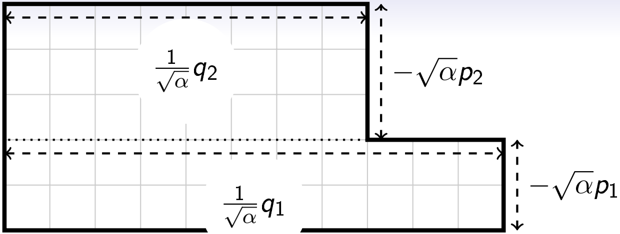
$$\operatorname{Ch}_3 \operatorname{Ch}_2 = 6\delta \operatorname{Ch}_3 + \operatorname{Ch}_{3,2} + 6 \operatorname{Ch}_{2,1} + 6 \operatorname{Ch}_4,$$

$$\begin{aligned}\operatorname{Ch}_3 \operatorname{Ch}_3 = (6\delta^2 + 3) \operatorname{Ch}_3 + 9\delta \operatorname{Ch}_{2,1} + 18\delta \operatorname{Ch}_4 + 3 \operatorname{Ch}_{1,1,1} + \\ + 9 \operatorname{Ch}_{3,1} + 9 \operatorname{Ch}_{2,2} + 9 \operatorname{Ch}_5 + \operatorname{Ch}_{3,3},\end{aligned}$$

$$\operatorname{Ch}_{2,2} \operatorname{Ch}_2 = 4\delta \operatorname{Ch}_{2,2} + 8 \operatorname{Ch}_4 + 4 \operatorname{Ch}_{2,1,1} + 8 \operatorname{Ch}_{3,2} + \operatorname{Ch}_{2,2,2}$$

$$\delta = \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}$$

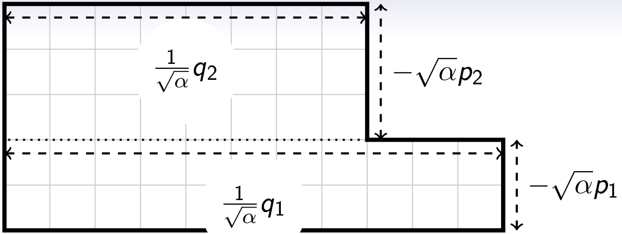
more conjectures $\longrightarrow$ Śniady `arXiv:1603.04268`;
partial results $\longrightarrow$ Burchardt



$$
\begin{aligned}
-\operatorname{Ch}_3 \quad = {} & p_1^3 q_1 + 3p_1^2 q_1^2 + p_1 q_1^3 + 3p_1^2 p_2 q_2 + 3p_1 p_2^2 q_2 \\
& + p_2^3 q_2 + 3p_1 p_2 q_1 q_2 + 3p_1 p_2 q_2^2 + 3p_2^2 q_2^2 + p_2 q_2^3 \\
& + 3p_1^2 q_1 \gamma + 3p_1 q_1^2 \gamma + 6p_1 p_2 q_2 \gamma + 3p_2^2 q_2 \gamma \\
& + 3p_2 q_2^2 \gamma + 2p_1 q_1 \gamma^2 + 2p_2 q_2 \gamma^2 + p_1 q_1 + p_2 q_2
\end{aligned}
$$

$$\gamma = -\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}$$

see also $\longrightarrow$ Kerov polynomials



$$
\begin{aligned}
-\operatorname{Ch}_3^{\text{top}} = {} & p_1^3 q_1 + 3p_1^2 q_1^2 + p_1 q_1^3 + 3p_1^2 p_2 q_2 + 3p_1 p_2^2 q_2 \\
& + p_2^3 q_2 + 3p_1 p_2 q_1 q_2 + 3p_1 p_2 q_2^2 + 3p_2^2 q_2^2 + p_2 q_2^3 \\
& + 3p_1^2 q_1 \gamma + 3p_1 q_1^2 \gamma + 6p_1 p_2 q_2 \gamma + 3p_2^2 q_2 \gamma \\
& + 3p_2 q_2^2 \gamma + 2p_1 q_1 \gamma^2 + 2p_2 q_2 \gamma^2
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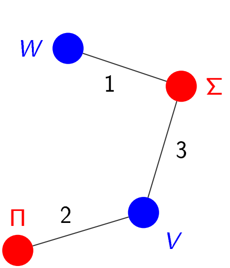
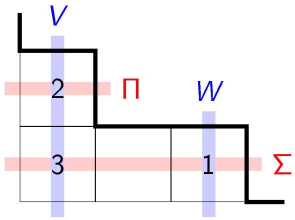
$$\mathrm{Ch}_\pi = ?$$

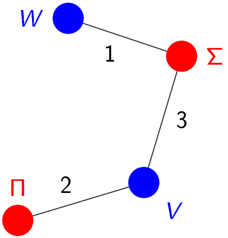
embeddings of a graph to a Young diagram

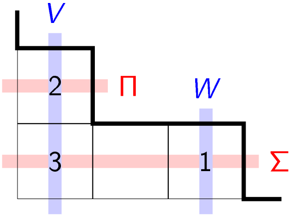




$$\mathfrak{N}_G(\lambda) = \sqrt{\alpha}^{|\text{blue vertices}|} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|\text{red vertices}|} \#\text{embeddings of } G \text{ to } \lambda$$

embeddings of a graph to a Young diagram





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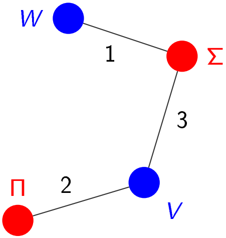
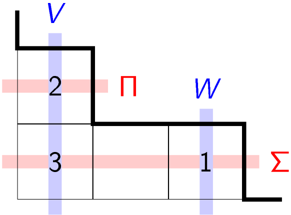
Problem

find some nice family of graphs such that

$$\mathrm{Ch}_\pi(\lambda) = \sum_G c_G \, \mathfrak{N}_G(\lambda)$$

embeddings of a graph to a Young diagram





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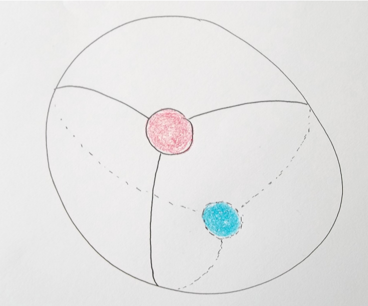
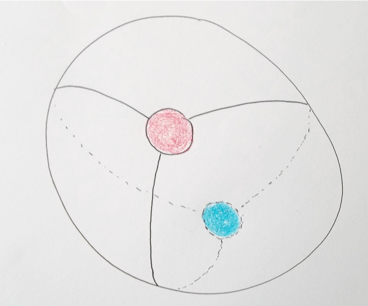
map is a graph on a surface

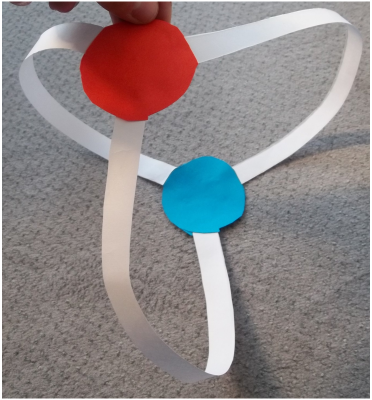


map on a sphere

map is a graph on a surface

each map can be visualized as a ribbon graph





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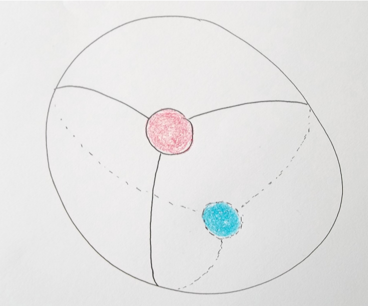
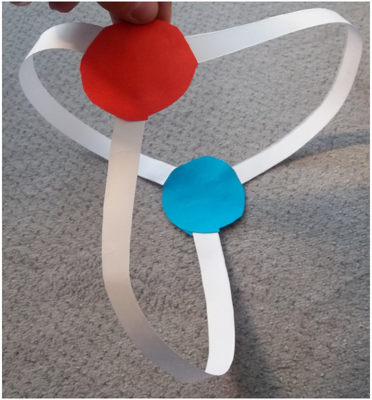
today: all maps are bicolored (red and blue vertices)





map on a sphere

we require that if we cut the surface along the edges
the surface breaks into a number of faces (=polygons)

. . . . ✂ . . . . ✂ . . . . ✂ . . . .

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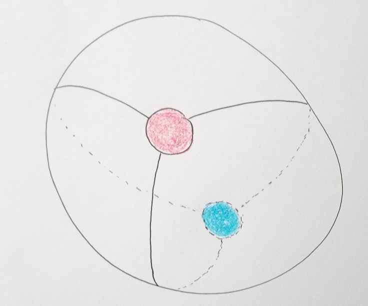

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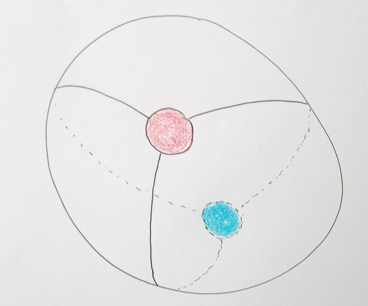

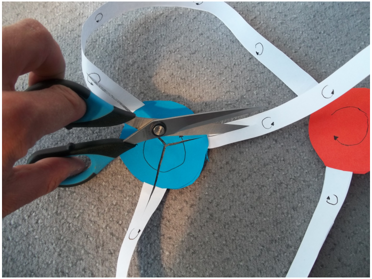

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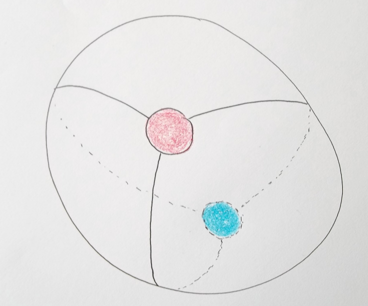

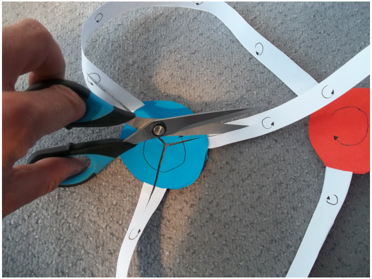

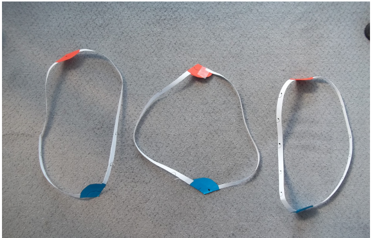

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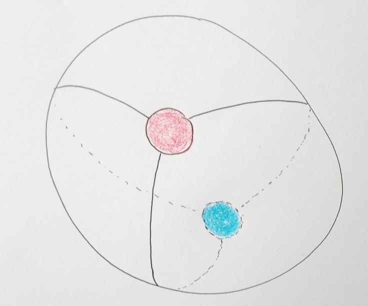
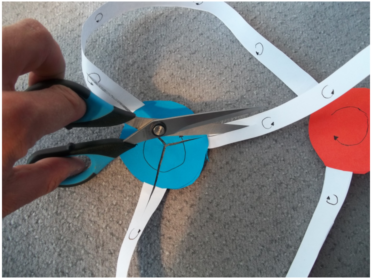
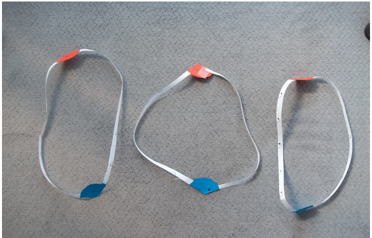
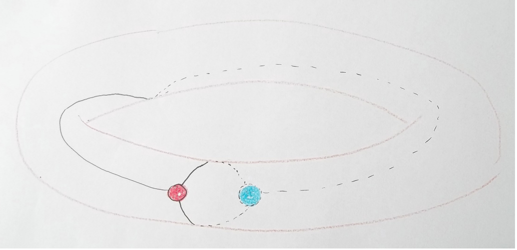


a map with three faces

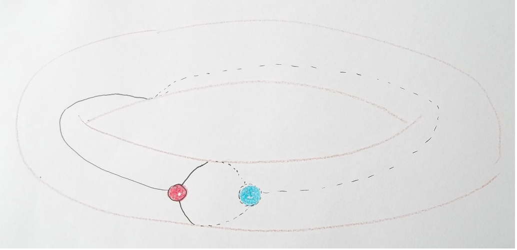
one $2 \cdot 1$-gon, one $2 \cdot 1$-gon,
one $2 \cdot 1$-gon, so
face-type $(1, 1, 1)$







map on a torus,
with one face



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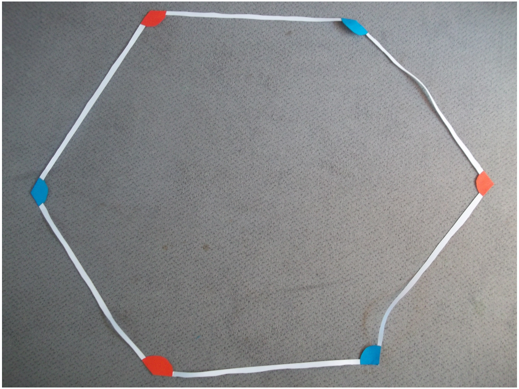

map on a torus,
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. . . ✂ . . . ✂ . . . ✂ . . .

one $2 \cdot 3$-gon, so
face-type (3)

conjecture

there exists some nice family of coefficients $\text{mon}_M \in \mathbb{Q}[\gamma]$ such that

$$\text{Ch}_\pi(\lambda) = \sum_M \text{mon}_M \, \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

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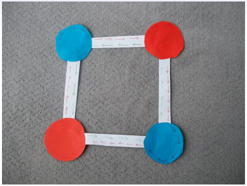
hint from Goulden & Jackson:
how (non)orientable is the map M?

for an edge E of a map M...

remove the edge E;
what happens to the number of faces of $M \setminus E$?

we say that E is:

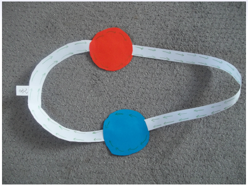
- border,
  if $\#\text{faces}(M \setminus E) = \#\text{faces}(M) - 1$;
- twisted,
  if $\#\text{faces}(M \setminus E) = \#\text{faces}(M)$;
- handle,
  if $\#\text{faces}(M \setminus E) = \#\text{faces}(M) + 1$;



border

(bad)

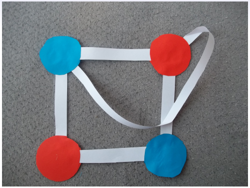
$$\text{factor} := \frac{1}{2}$$



twisted

(nice)

$$\text{factor} := \gamma$$



handle

(very nice)

$$\text{factor} := 1$$

how non-orientable is your map?

1. choose random order of the edges!

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2. take the first edge;
   is it twisted / border / handle?
   calculate the corresponding factor!

how non-orientable is your map?

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2. take the first edge;
   is it twisted / border / handle?
   calculate the corresponding factor!
3. remove this edge,

how non-orientable is your map?

1. choose random order of the edges!
2. take the next edge;
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4. take the next edge, repeat,

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this is the measure of non-orientability mon(M) of a map M

conjecture

$$\mathrm{Ch}_\pi(\lambda) = \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda),$$

where the sum runs over maps M with face-type π

bad news

$$\mathrm{Ch}_\pi(\lambda) \neq \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda),$$

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good news

$$\mathrm{Ch}^{\mathrm{top}}_\pi(\lambda) = \sum_M (\text{top-degree part in } \gamma) \, \mathrm{mon}_M \, \mathfrak{N}_M(\lambda),$$

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new scaling:

$$\text{degree of a map} = (\text{number of vertices}) + (\text{exponent of } \gamma)$$

main contribution from maps (with an order on egdes) such that
(a) during the edge removal there are no border edges $\iff$
(b) during the edge removal each connected component = one face

such maps are called top-twisted

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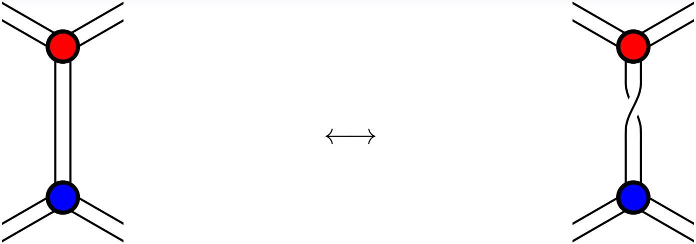
there is a bijection between:

- pairs $(M, \prec)$, where
  M is a non-oriented, rooted map with n edges, one face;
  $\prec$ is an order on the edges which makes M top-twisted;
- pairs $(M, \prec)$, where
  M is an oriented, rooted map with n edges,
  arbitrary number of faces;
  $\prec$ an arbitrary order on the edges of M;



there is a bijection between:

- pairs $(M, \prec)$, where
  M is a non-oriented, rooted map with n edges, one face;
  $\prec$ is an order on the edges which makes M top-twisted;
- pairs $(M, \prec)$, where
  M is an oriented, rooted map with n edges,
  arbitrary number of faces;
  $\prec$ an arbitrary order on the edges of M;

$$\mathrm{Ch}_n^{\mathrm{top}}(\lambda) =$$

$$\sum_{\substack{\text{non-oriented map M} \\ \text{with face-type } (n)}} (\text{top-degree part in } \gamma)\, \mathrm{mon}_M \;\; \mathfrak{N}_M(\lambda) =$$

$$\sum_{\substack{\text{oriented map M} \\ \text{with } n \text{ edges}}} \gamma^{n+1-|\text{vertices}|} \;\mathfrak{N}_M(\lambda)$$

proof: abstract characterization of Jack characters

for each π and each $\alpha > 0$
$\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots)$ is the unique polynomial such that:

- $\mathrm{Ch}_\pi\left(x_1 + \frac{1}{\alpha}, x_2 + \frac{2}{\alpha}, \dots\right)$ is symmetric in $x_1, x_2, \dots$;
- polynomial $\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots)$ is of degree $|\pi|$; its top-degree homogeneous part is equal to
$$\alpha^{\frac{|\pi|-\ell(\pi)}{2}} \; p_\pi;$$
- for all partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $|\lambda| < |\pi|$
$$\mathrm{Ch}_\pi(\lambda_1, \lambda_2, \dots) = 0$$

---

if we view α as indeterminate,

- for each Young diagram λ
$\mathrm{Ch}_\pi(\lambda) \in \mathbb{Q}\left[\sqrt{\alpha}, \frac{1}{\sqrt{\alpha}}\right]$ is a Laurent polynomial of degree (at most) $|\pi| - \ell(\pi)$

open problem

$$\mathrm{Ch}_\pi(\lambda) = \sum_M c_M \, \mathfrak{N}_m(\lambda);$$

$$c_M = ?$$

Maciej Dołęga, Valentin Féray, Piotr Śniady
Jack polynomials and orientability generating series of maps
Séminaire Lotharingien de Combinatoire 70 (2014),
Article B70j

Piotr Śniady
Top degree of Jack characters and enumeration of maps
Preprint arXiv:1506.06361

Piotr Śniady
Structure coefficients for Jack characters:
approximate factorization property
Preprint arXiv:1603.04268

Maciej Dołęga
Top degree part in *b*-conjecture for unicellular bipartite maps
Preprint arXiv:1604.03288

$$\mathrm{Ch}_1 = \underbrace{R_2}_{\mathrm{Ch}_1^{\mathrm{top}}},$$

$$\mathrm{Ch}_2 = \underbrace{R_3 + R_2\gamma}_{\mathrm{Ch}_2^{\mathrm{top}}},$$

$$\mathrm{Ch}_3 = \underbrace{R_4 + 3R_3\gamma + 2R_2\gamma^2}_{\mathrm{Ch}_3^{\mathrm{top}}} + R_2,$$

$$\mathrm{Ch}_4 = \underbrace{R_5 + 6R_4\gamma + R_2^2\gamma + 11R_3\gamma^2 + 6R_2\gamma^3}_{\mathrm{Ch}_4^{\mathrm{top}}} + 5R_3 + 7R_2\gamma.$$

$$\alpha t \frac{\partial}{\partial t} \log \left(\sum_{\lambda} \frac{J_\lambda(\mathbf{x})\ J_\lambda(\mathbf{y})\ J_\lambda(\mathbf{z})\ t^{|\lambda|}}{\langle J_\lambda, J_\lambda \rangle_\alpha} \right) =$$

$$\sum_{n \geq 1} t^n \left(\sum_{\mu,\nu,\tau \vdash n} h^{\tau}_{\mu,\nu}(\alpha - 1)\ p_\mu(\mathbf{x})\ p_\nu(\mathbf{y})\ p_\tau(\mathbf{z}) \right)$$

conjecture [Goulden & Jackson 1996]

there exists a function η such that

$$h^{\tau}_{\mu,\nu}(\beta) = \sum_{M} \beta^{\eta(M)}$$

where the summation runs over connected, rooted maps
with face-type τ,
blue vertex distribution μ, and red vertex distribution ν,
and $\eta(M) \in \{0, 1, 2, \dots\}$