Combinatorics of (Jack-deformed) characters of the symmetric groups

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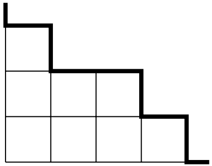
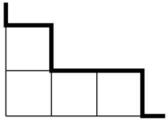
irreducible representations of the symmetric groups

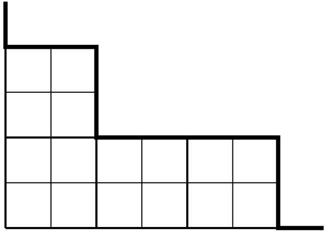
irreducible representation ρ^λ of the symmetric group $\mathfrak{S}(n)$ $\longleftrightarrow$ Young diagram λ with n boxes



dilations of Young diagrams



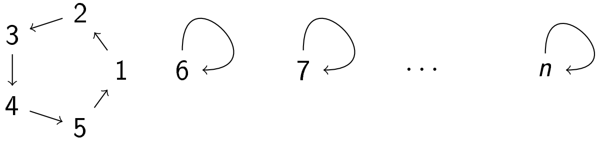
Young diagram λ



dilated diagram 2λ

dual combinatorics of the representation theory of $\mathfrak{S}(n)$

classical combinatorics	dual combinatorics
λ is fixed	conjugacy class is fixed
character $\chi^\lambda(\pi)$ — function of π	character $\mathrm{Ch}_k(\lambda)$ — function of λ

$[5] :=$ 

normalized character: → Kerov & Olshanski

$$\mathrm{Ch}_5(\lambda) := \underbrace{n(n-1)\cdots(n-4)}_{5 \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda([5])}{\operatorname{Tr} \rho^\lambda(\mathrm{Id})},$$

n — the number of boxes of λ

free cumulants

$s \mapsto \mathrm{Ch}_k(s\lambda)$ is a polynomial of degree $k+1$

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free cumulants $R_2(\lambda), R_3(\lambda), \dots$ are top-degree coefficients:

$$R_{k+1}(\lambda) := \lim_{s\to\infty} \frac{1}{s^{k+1}} \mathrm{Ch}_k(s\lambda)$$

Kerov polynomials

$$\overbrace{\mathrm{Ch}_2}^{\text{character}} = \overbrace{R_3}^{\text{shape}},$$

$$\mathrm{Ch}_3 = R_4 + R_2,$$

$$\mathrm{Ch}_4 = R_5 + 5R_3,$$

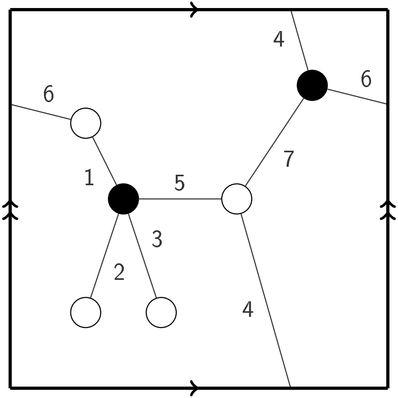
$$\mathrm{Ch}_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2,$$

$$\mathrm{Ch}_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3$$

Kerov positivity conjecture:
the coefficients are non-negative integers;
what is their combinatorial meaning?

Kerov polynomials for Ch_k count. . .

oriented,
labeled,
bicolored maps
with one face
and k edges



Kerov polynomials for Ch_k count...

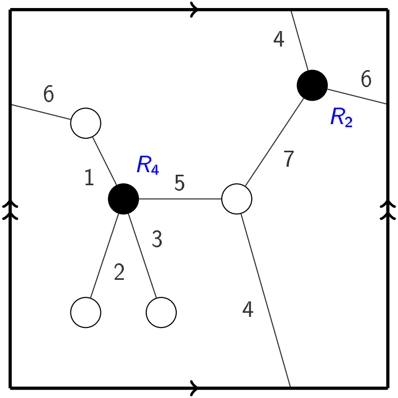
coefficient of $R_{i_1} \cdots R_{i_\ell}$ in Ch_k counts the number of maps with k edges

with black vertices labelled by $R_{i_1}, \dots, R_{i_\ell}$,

each black vertex R_i produces $i-1$ units of liquid,

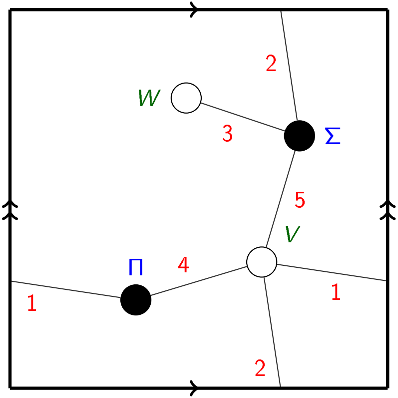
each white vertex demands 1 unit of the liquid,

each edge transports strictly positive amout of liquid from black to white vertex

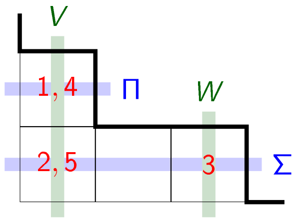


→ Féray, Dołęga & Śniady

Stanley's character formula

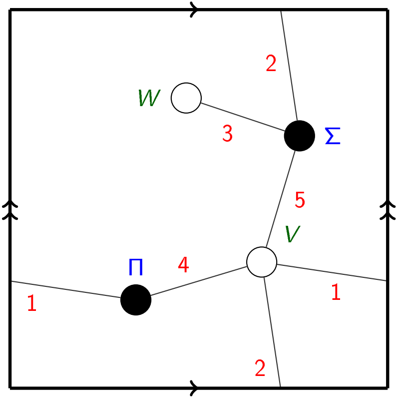


→Stanley, Féray, Śniady

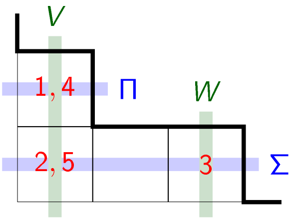


$N_M(\lambda) = \#$ embeddings of M to λ

Stanley's character formula



→Stanley, Féray, Śniady



$N_M(\lambda) = \#$ embeddings of M to λ

$$\mathrm{Ch}_k(\lambda) = \sum_M (-1)^{k - \#\text{white vertices}} N_M(\lambda),$$

where the sum runs over maps M with k edges

some famous symmetric polynomials

Schur polynomials ($\alpha = 1$) →

zonal polynomials ($\alpha = 2$) → Jack polynomials $J_\lambda^{(\alpha)}$ → Macdonald polynomials $P_\lambda^{(t,q)}$

symplectic zonal polynomials ($\alpha = \frac{1}{2}$) →

dual approach to symmetric functions

traditional approach:

$$J^{(\alpha)}_\lambda = \sum_\pi \,?\, p_\pi \qquad \text{or} \qquad [p_\pi] J^{(\alpha)}_\lambda = \,?$$

λ is fixed, π varies, $|\lambda| = |\pi|$

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dual approach:

define Jack character $\longrightarrow$ LASSALLE

$$\mathrm{Ch}^{(\alpha)}_\pi(\lambda) := (\text{normalizing factor}) [p_{\pi\ 1^{|\lambda|-|\pi|}}] J^{(\alpha)}_\lambda$$

π is fixed, λ is arbitrary

Kerov polynomials for Jack characters

$$\begin{aligned}
\mathrm{Ch}_1^{(\alpha)} &= R_2, \\
\mathrm{Ch}_2^{(\alpha)} &= R_3 + \gamma R_2, \\
\mathrm{Ch}_3^{(\alpha)} &= R_4 + 3\gamma R_3 + (1 + 2\gamma^2) R_2, \\
\mathrm{Ch}_4^{(\alpha)} &= R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2) R_3 + (7\gamma + 6\gamma^3) R_2,
\end{aligned}$$

→Lassalle

integrality? positivity?

$$\begin{aligned}
\gamma &= -A + \frac{1}{A}, \\
A &= \sqrt{\alpha}
\end{aligned}$$

content evaluation and Jack characters

$$\text{content}(\square) = A\ (x\text{-coordinate}) - \frac{1}{A}(y\text{-coordinate})$$

$$\mathrm{Ch}_k^{(\alpha)}(\lambda) = \sum_{i \geq 0} \sum_{\square_1, \dots, \square_i \in \lambda} \underbrace{P_i(\gamma, c_1, \dots, c_i)}_{\text{polynomial of degree } k+1-2i},$$

where

$$c_1 := \text{content}(\square_1), \quad \dots, \quad c_i := \text{content}(\square_i), \qquad \gamma := -A + \frac{1}{A}$$

Example

$$\mathrm{Ch}_3^{(\alpha)}(\lambda) = \sum_{\square_1 \in \lambda} \left(3(c_1 + \gamma)(c_1 + 2\gamma) + \frac{3}{2} \right) + \sum_{\square_1, \square_2 \in \lambda} \left(-\frac{3}{2} \right)$$

abstract characterization of Jack character $\mathrm{Ch}_k^{(\alpha)}$

- $$\mathrm{Ch}_k^{(\alpha)}(\lambda) = \sum_{i \geq 0} \sum_{\square_1, \dots, \square_i \in \lambda} \underbrace{P_i(\gamma, c_1, \dots, c_i)}_{\text{polynomial of degree } k+1-2i},$$

- $$\mathbb{Y} \ni (\lambda_1, \dots, \lambda_m) \mapsto \mathrm{Ch}_k^{(\alpha)}(\lambda_1, \dots, \lambda_m)$$

  is a polynomial of degree k, the top-degree part is equal to

  $$A^{n-1} \sum_j \lambda_j^k$$

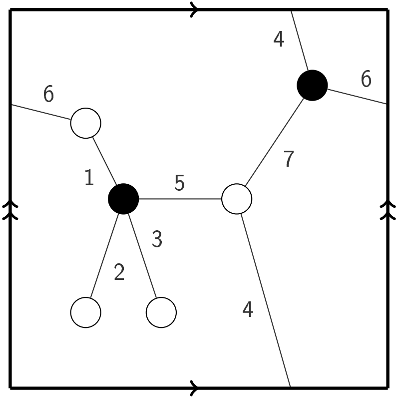
- for each $\lambda \in \mathbb{Y}$ such that $|\lambda| < k$ we have

  $$\mathrm{Ch}_k^{(\alpha)}(\lambda) = 0.$$

special case $\alpha = 1$ thus $\gamma = 0$

oriented,
labeled,
bicolored maps
with one face
and k edges

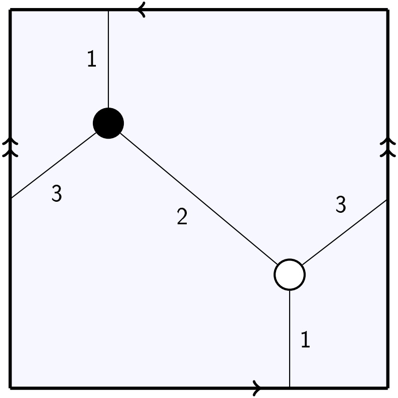
weight: 1



special cases $\alpha = 2$ and $\alpha = \frac{1}{2}$; thus $\gamma = \mp\frac{1}{\sqrt{2}}$

non-oriented,
labeled,
bicolored maps
with one face
and k edges

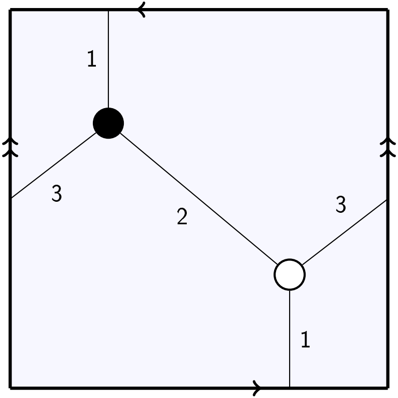
weight: $\gamma^{k+1-\#\text{vertices}}$



generic case?

non-oriented,
labeled,
bicolored maps
with one face
and k edges

some mysterious weight $w(\gamma)$
which measures
non-orientability
of the surface?



top-degree of Jack characters

$$\deg R_k = k,$$
$$\deg \gamma = 1$$

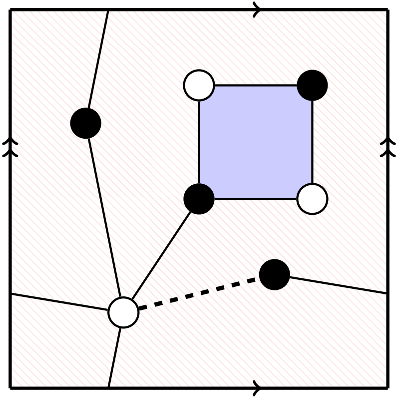
Example

$$\mathrm{Ch}_3^{(\alpha)} = \underbrace{R_4 + 3\gamma R_3 + 2\gamma^2 R_2}_{\mathrm{Ch}_3^{\mathrm{top}}} + R_2,$$

Kerov polynomials for $\mathrm{Ch}_k^{\mathrm{top}}$ count...

oriented,
unlabeled, rooted
bicolored maps
with arbitrary face structure
and k edges,

weight: $\gamma^{k+1-\#\text{vertices}}$



further reading

- Piotr Śniady.
  Combinatorics of asymptotic representation theory.
  In *European Congress of Mathematics Kraków, 2–7 July, 2012*, pages 531–545.
  European Mathematical Society Publishing House, 2014.
  arXiv:1203.6509

- Maciej Dołęga, Valentin Féray, Piotr Śniady
  Jack polynomials and orientability generating series of maps.
  arXiv:1301.6531

- Piotr Śniady
  Top degree of Jack characters and enumeration of maps.
  Work in progress.