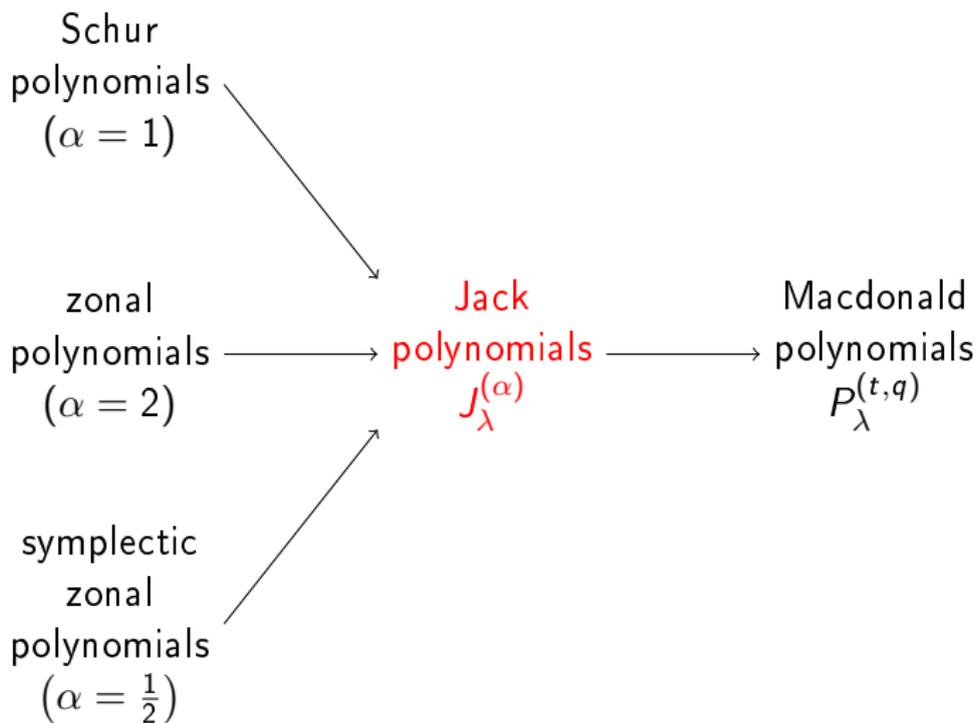# Jack deformation of characters of the symmetric groups

joint work with

Agnieszka Czyżewska-Jankowska,
Maciej Dołęga,
Valentin Féray
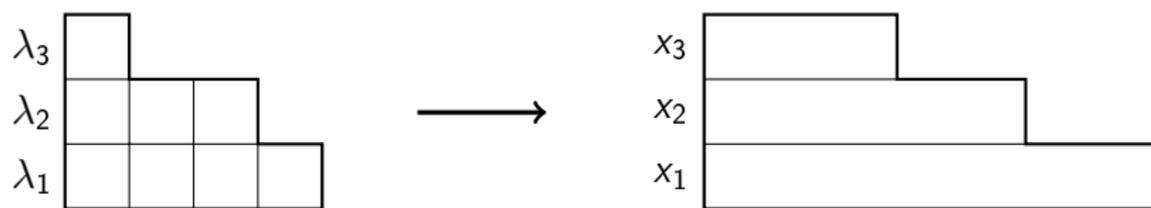
Piotr Śniady

Technische Universität München &
Polish Academy of Sciences &
University of Wrocław

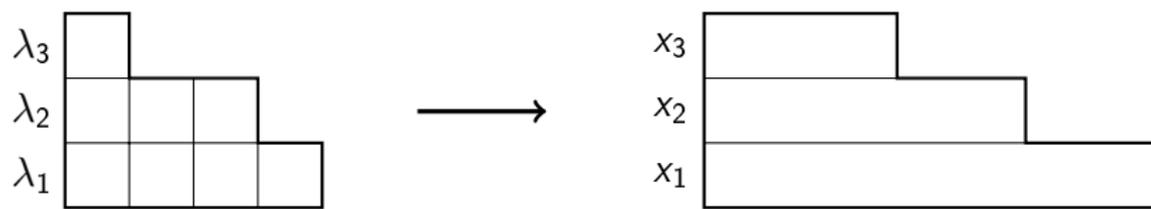# some famous symmetric polynomials

Schur
polynomials
$(\alpha = 1)$

zonal
polynomials
$(\alpha = 2)$

symplectic
zonal
polynomials
$\left(\alpha = \frac{1}{2}\right)$

Jack
polynomials
$J_{\lambda}^{(\alpha)}$

Macdonald
polynomials
$P_{\lambda}^{(t,q)}$

how to generalize Young diagrams?

$\lambda_1 \geq \lambda_2 \geq \cdots$ are natural numbers

$x_1 \geq x_2 \geq \cdots$ are natural numbers

how to generalize Young diagrams?

$\lambda_1 \geq \lambda_2 \geq \cdots$ are natural numbers

~~$x_1 \geq x_2 \geq \cdots$ are natural numbers~~

shifted-symmetric functions

$F(x_1, x_2, \dots)$ is an α-shifted-symmetric function of degree d
if for each k
function $F_k(x_1, \dots, x_k) := F(x_1, \dots, x_k, 0, 0, \dots)$
has the following properties:

- is a polynomial of degree (at most) d,
- is symmetric in variables $x_1 - \frac{1}{\alpha}, x_2 - \frac{2}{\alpha}, \dots, x_k - \frac{k}{\alpha}$

Example

α-shifted power-sum symmetric functions

$$p_k(x_1, x_2, \dots) := \sum_i \left(x_i - \frac{i}{\alpha} \right)^k - \left(-\frac{i}{\alpha} \right)^k,$$

$$p_{(\lambda_1, \dots)} := p_{\lambda_1} p_{\lambda_2} \cdots$$

shifted Schur and shifted Jack functions

$\longrightarrow$ KNOP, SAHI, . . .

shifted Jack function $J_\lambda^{(\alpha)}$ is uniquely characterized by:

- it is an α-shifted-symmetric function of degree $|\lambda|$,
- for any μ such that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$

$$J_\lambda^{(\alpha)}(\mu_1, \mu_2, \dots) = 0,$$

-

$$J_\lambda^{(\alpha)}(\lambda_1, \lambda_2, \dots) = (\text{some concrete constant}) \neq 0$$

case $\alpha = 1$ corresponds to shifted Schur functions

Fact

homogeneous top-degree part of shifted Jack/Schur function is equal to the usual (non-shifted) Jack/Schur function

dual approach to symmetric functions

traditional approach:

$$J_\lambda^{(\alpha)} = \sum_\pi \ ? \ p_\pi \qquad \text{or} \qquad [p_\pi]J_\lambda^{(\alpha)} = ?$$

λ is fixed, π varies, $|\lambda| = |\pi|$

---

dual approach:

define Jack character $\longrightarrow$ LASSALLE

$$\mathrm{Ch}_\pi^{(\alpha)}(\lambda) := (\text{normalizing factor})[p_{\pi \ 1^{|\lambda|-|\pi|}}]J_\lambda^{(\alpha)}$$

π is fixed, λ is arbitrary

special case $\alpha = 1$

normalized character of symmetric group $\longrightarrow$ Ivanov, Kerov, . . .

$$\text{Ch}_\pi^{(1)}(\lambda) = \underbrace{|\lambda| \cdots (|\lambda| - |\pi| + 1)}_{|\pi| \text{ factors}} \frac{\text{Tr } \rho^\lambda(\pi \, 1^{|\lambda| - |\pi|})}{\text{dimension of } \rho^\lambda}$$

where ρ^λ is the irreducible representation of the symmetric group

Jack characters $Ch_\pi^{(\alpha)}$

- are a deformation of
  the normalized characters of symmetric groups,

- describe dual combinatorics of
  the usual (non-shifted) Jack functions,

- also have an abstract description,
  not related to Jack functions
  (next slide)

abstract characterization of Jack characters

Jack character $\mathrm{Ch}_\pi^{(\alpha)}(x_1, x_2, \dots)$ is uniquely characterized by:

- it is an α-shifted-symmetric function of degree $|\pi|$,
- the top-degree part is equal to the power-sum symmetric function p_π,
- $\mathrm{Ch}_\pi^{(\alpha)}(\lambda) = 0$ for any Young diagram λ such that $|\lambda| < |\pi|$,

Kerov polynomials

$$\mathrm{Ch}_1^{(\alpha)} = R_2,$$

$$\mathrm{Ch}_2^{(\alpha)} = R_3 + \gamma R_2,$$

$$\mathrm{Ch}_3^{(\alpha)} = R_4 + 3\gamma R_3 + (1 + 2\gamma^2)R_2,$$

$$\mathrm{Ch}_4^{(\alpha)} = R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2)R_3 + (7\gamma + 6\gamma^3)R_2,$$

$$\mathrm{Ch}_5^{(\alpha)} = R_6 + 10\gamma R_5 + 5\gamma R_3 R_2 + 15 R_4 + 5R_2^2 + \gamma^2(35 R_4 + 10 R_2^2) +$$
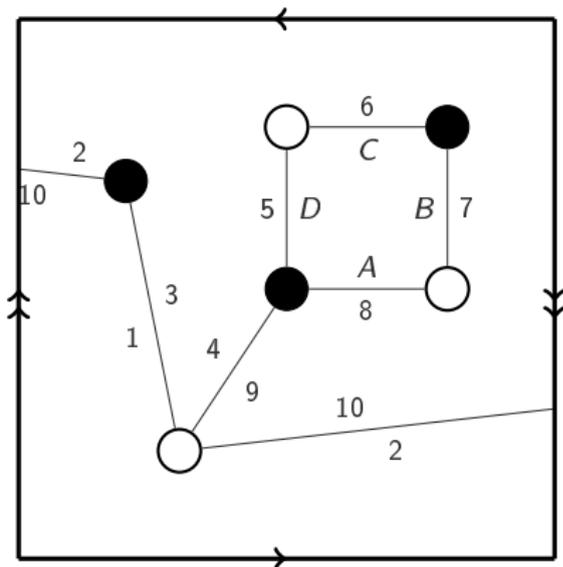$$+ (55\gamma + 50\gamma^3)R_3 + (8 + 46\gamma^2 + 24\gamma^4)R_2,$$

where
$R_i(x_1, x_2, \dots)$ are the α-shifted free cumulants,
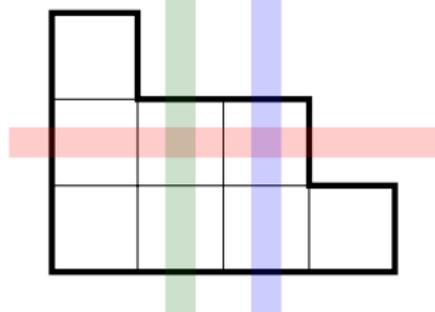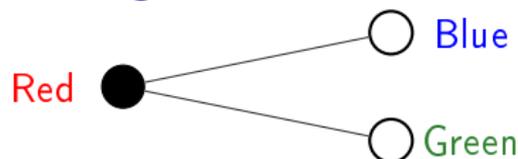$\gamma = \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}$

maps

(non-oriented) map
- is a bipartite graph,
- drawn on a (non-oriented) surface,
- labeled,

embeddings

Red ● — ○ Blue
         ○ Green

embedding of a map M to a Young diagram λ
is a function F which:

- maps black vertices of M to rows of λ,
- maps white vertices of M to columns of λ,
- if a black vertex b is connected
  with a white vertex w,

  we require that the box
  in the intersection of row $F(b)$ and column $F(w)$
  belongs to λ,

$N_M(\lambda)$ denotes the number of all embeddings of M to λ

$\alpha = 1$

$$\mathrm{Ch}_{\pi}^{(1)}(\lambda) = (-1)^{\ell(\pi)} \sum_{M} (-1)^{\#\text{black vertices}} N_M(\lambda)$$

over all oriented maps with the face-structure given by π

$\alpha = 2$

$\mathrm{Ch}_\pi^{(2)}(\lambda) =$

$$(-1)^{\ell(\pi)} \sum_M \left(-\frac{1}{\sqrt{2}}\right)^{\text{\# black vertices}} \left(\sqrt{2}\right)^{\text{\# white vertices}} \times$$

$$\times \left(-\frac{1}{\sqrt{2}}\right)^{|\pi|+\ell(\pi)-\text{\# vertices}} N_M(\lambda),$$

over all non-oriented maps with the face-structure given by π

general α

Conjecture:

$$\text{Ch}_\pi^{(\alpha)}(\lambda) =$$

$$(-1)^{\ell(\pi)} \sum_M \left(-\frac{1}{\sqrt{\alpha}} \right)^{\# \text{ black vertices}} \left(\sqrt{\alpha} \right)^{\# \text{ white vertices}} \times$$

$$\text{weight}_M \ N_M(\lambda),$$

$$\text{weight}_M = ?$$

three types of edges
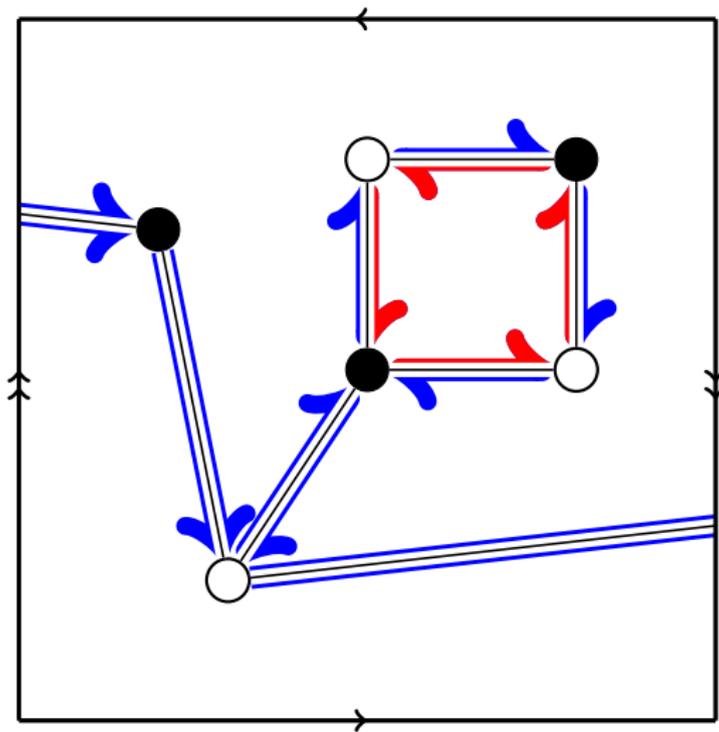
- straight edge
  $\text{weight}_E = 1$

- Möbius edge
  $\text{weight}_E = \gamma = \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}$

- interface edge
  $\text{weight}_E = \frac{1}{2}$

recipe for weight$_M$

- for a given map M...

- select a random edge E_1,
  calculate its weight$_{E_1}$,

- remove edge E_1 from M,
  warning! some edges change their type!

- select another random edge E_2, etc.

- iterate for edges $E_2, \ldots, E_\ell$,

$$\text{weight}_M \overset{?}{=} \mathbb{E}\ \text{weight}_{E_1} \cdots \text{weight}_{E_\ell}$$

is conjecture true?

- conjecture is true for rectangular Young diagrams,

- conjecture gives concrete predictions for Kerov polynomials,

- extensively tested numerically,

- would solve several open problems,

- or maybe the conjecture is not true?

further reading

📄 Piotr Śniady
Combinatorics of asymptotic representation theory.
Proceedings of 6th European Congress of Mathematics
arXiv:1203.6509

📄 Agnieszka Czyżewska-Jankowska, Maciej Dołęga,
Valentin Féray, Piotr Śniady
Non-orientability of maps
and dual combinatorics of Jack polynomials.
Soon available on arXiv