Asymptotics of symmetric groups representations, random matrices and free probability

(joint work with Valentin Féray)

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University of Wroclaw

Outline

Introduction

Stanley-Féray character formula

Characters, free probability and random matrices

Representations of S_n

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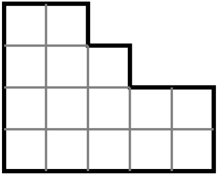

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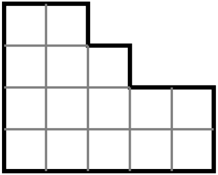

What happens with representations of S_n when $n \to \infty$?

Characters of symmetric groups

For a Young diagram λ and irreducible representation ρ^λ we define the character $\chi^\lambda : S_n \to \mathbb{R}$ by

$$\chi^\lambda(\pi) = \operatorname{tr} \rho^\lambda(\pi) \qquad \text{for } \pi \in S_n.$$

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Main problem: asymptotics of characters of S_n when $n \to \infty$.

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ask me about it during coffee break!

Roichman's inequality

Theorem (Roichman 1996)

There exist constants $0 < q < 1$ *and* $b > 0$ *such that for any* $\pi \in S_n$

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Not good enough for asymptotics of quantum computers. Can we have

$$|\chi^\lambda(\pi)| \leq \left(\frac{\text{const}}{\sqrt{n}}\right)^{|\pi|}?$$

Normalized characters

For a Young diagram λ with n boxes and $\pi \in S_l$ ($l \leq n$) we define normalized character

$$\Sigma^{\lambda}(\pi) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-l+1)}_{l \text{ factors}} \chi^{\lambda}(\pi)$$

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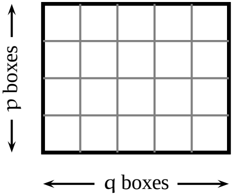
Important: we can think that $l = |\operatorname{supp} \pi|$.

Stanley's character formula

$$p \times q =$$



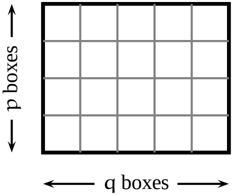
Stanley's character formula

Theorem (Stanley 2001)

For a rectangular Young diagram $p \times q$ and $\pi \in S_l$ (where $l \leq pq$)

$$\Sigma^{p\times q}(\pi) = \sum_{\substack{\sigma_1,\sigma_2 \in S_l,\\ \sigma_1\sigma_2=\pi}} (-1)^{|\sigma_1|}\ q^{|C(\sigma_1)|}\ p^{|C(\sigma_2)|},$$

where $|C(\sigma_i)|$ is the number of cycles of σ_i.

$p \times q =$ 

Stanley-Féray character formula

Theorem (Féray 2006)

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$$\Sigma^{\lambda}(\pi) = \sum_{\substack{\sigma_1,\sigma_2 \in S_l,\\ \sigma_1\sigma_2=\pi}} (-1)^{|\sigma_1|} \; N^{\lambda}(\sigma_1,\sigma_2),$$

where $N^{\lambda}(\sigma_1,\sigma_2)$ is decribed in the following.

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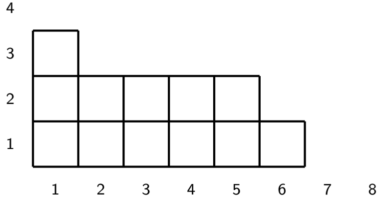
  the box in column $h_1(c_1)$ and row $h_2(c_2)$ belongs to λ.
- $N^\lambda(\sigma_1, \sigma_2)$ denotes the number of the colorings of σ_1, σ_2 which are compatible with λ.

Colorings: toy example

Factorization $(1,2) = \underbrace{(1)(2)}_{\sigma_1} \cdot \underbrace{(1,2)}_{\sigma_2}$. Coloring compatible with λ:



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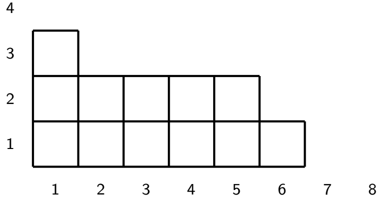

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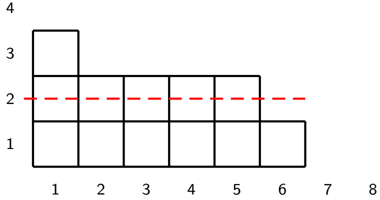

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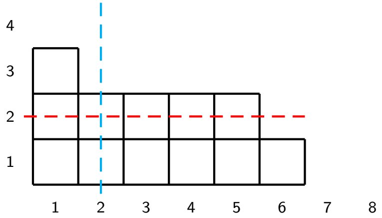

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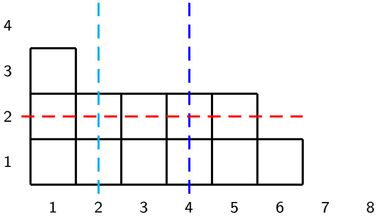

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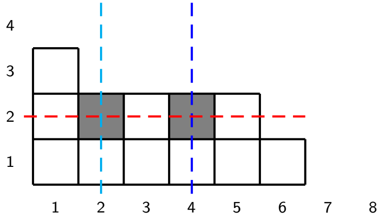

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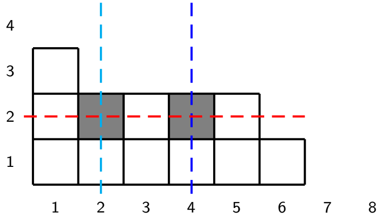

$$N^{\lambda}\big((1)(2),(1,2)\big) = \sum_i (\lambda_i)^2,$$

where λ_i is the number of boxes in i-th row.

Stanley-Féray character formula

Theorem (Féray 2006)

For any Young diagram λ and a permutation $\pi \in S_l$ (where $l \leq n$)

$$\Sigma^\lambda(\pi) = \sum_{\substack{\sigma_1,\sigma_2 \in S_l, \\ \sigma_1\sigma_2=\pi}} (-1)^{|\sigma_1|} \, N^\lambda(\sigma_1,\sigma_2),$$

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- free probability (next section);

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$$\begin{bmatrix} 0 & \rho^\lambda(1,2) & \cdots & \rho^\lambda(1,n) & 1 \\ \rho^\lambda(2,1) & 0 & \cdots & \rho^\lambda(2,n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^\lambda(n,1) & \rho^\lambda(n,2) & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

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Like in the random matrix theory free cumulants are the right quantities.

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New formula for free cumulants 1

Corollary

$$R^{\lambda}_{k+1} = \sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_1\sigma_2=(1,2,\dots,k) \\ |\sigma_1|+|\sigma_2|=|(1,2,\dots,k)|}} (-1)^{|\sigma_1|} \, N^{\lambda}(\sigma_1,\sigma_2),$$

where the sum runs over minimal factorizations of a cycle.

New formula for free cumulants 1

Corollary

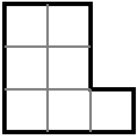
$$R^{\lambda}_{k+1} = \sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_1\sigma_2=(1,2,\dots,k) \\ |\sigma_1|+|\sigma_2|=|(1,2,\dots,k)|}} (-1)^{|\sigma_1|} \, N^{\lambda}(\sigma_1,\sigma_2),$$

where the sum runs over minimal factorizations of a cycle.

Minimal factorizations of $(1,\dots,k)$ = planar rooted trees with $k+1$ vertices!

Random matrices...

For a Young diagram λ we consider a random matrix T_λ.

$\lambda =$ 

$$T_\lambda = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots \\ g_{3,1} & g_{3,2} & 0 & 0 & \cdots \\ g_{2,1} & g_{2,2} & 0 & 0 & \cdots \\ g_{1,1} & g_{1,2} & g_{1,3} & 0 & \cdots \end{bmatrix}$$

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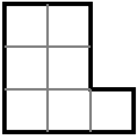
$\lambda =$ 

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Moments of random matrices:

$$\mathbb{E}\left[\operatorname{Tr}(T_\lambda T_\lambda^\star)^{l_1} \cdots \operatorname{Tr}(T_\lambda T_\lambda^\star)^{l_k} \right] = \sum_{\substack{\sigma_1,\sigma_2 \in S_l, \\ \sigma_1\sigma_2 = \pi}} N^\lambda(\sigma_1, \sigma_2).$$

where π is a permutation with a cycle structure $(l_1, \dots, l_k)$.

Random matrices and characters

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Therefore the asymptotics of characters on long permutations $(l_1, l_2, \cdots \to \infty)$ is related to asymptotics of the largest eigenvalues of $T_\lambda T_\lambda^\star$.

Random matrices and circular operator

If λ is big then the Gaussian band matrix T_λ can be approximated by a circular operator T with amalgamation:

$$\mathbb{E}\,\mathrm{tr}\left[(T_\lambda T_\lambda^\star)^n\right] \approx \phi\big[(TT^\star)^n\big].$$

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- state $\phi : \mathcal{D} \to \mathbb{C}$, $\phi(f) = \int_0^\infty f(t)dt$ corresponds to trace

Character and circular operator

covariance of T:

$$\big[k(T, f\ T^\star)\big](s) = \int_{(t,s)\in\lambda} f(t)\ dt,$$

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Theorem

$$R^\lambda_{k+1} = \phi\left[(TT^\star)^k\right]$$

Estimates for characters

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Theorem (Vershik–Kerov 1985)

For a Young diagram λ with n boxes

$$\chi^{\lambda}(1,2,\dots,k) \approx \sum_j \left(\frac{\lambda_j}{n}\right)^k - \sum_j \left(-\frac{\lambda'_j}{n}\right)^k$$

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Stanley-Féray formula: new one-line proof and estimate for error term

New bounds for characters

Theorem (Roichman 1996)

There exist constants $0 < q < 1$ *and* $b > 0$ *such that for any Young diagram* λ *with* n *boxes*

$$|\chi^\lambda(\pi)| \leq \left[\max \left(\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, q \right) \right]^{b\ |\pi|}$$

Theorem (Féray–Śniady 2007)

There exists a constant C *such that for any Young diagram* λ *with* n *boxes*

$$|\chi^\lambda(\pi)| \leq \left[C \max \left(\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, \frac{|\pi|}{n} \right) \right]^{|\pi|}$$

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Question

Suppose that λ has n boxes,
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We do not know, please help!

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