Asymptotics of symmetric groups representations, random matrices and free probability

(joint work with Valentin Féray)

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Outline

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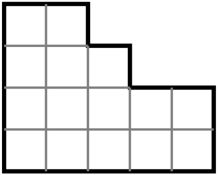

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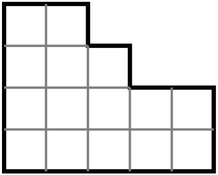

What happens with representations of S_n when $n \to \infty$?

Characters of symmetric groups

For a Young diagram λ and irreducible representation ρ^λ we define the character $\chi^\lambda : S_n \to \mathbb{R}$ by

$$\chi^\lambda(\pi) = \operatorname{tr} \rho^\lambda(\pi) \qquad \text{for } \pi \in S_n.$$

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A lot of questions concerning (representations of) S_n can be reduced to questions on characters.
Main problem: asymptotics of characters of S_n when $n \to \infty$.

Motivation 1: non-commutative Fourier transform...

If $f \in \mathbb{C}[S_n]$ we define a function $\hat{f}$ on Young diagrams with n boxes:

$$\hat{f}(\lambda) = \rho^{\lambda}(f) \in \operatorname{End}(V_\lambda).$$

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Non-commutative Fourier transform depends on characters.

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Proof: use non-commutative Fourier transform.

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ask me about it during coffee break!

Roichman's inequality

Theorem (Roichman 1996)

There exist constants $0 < q < 1$ *and* $b > 0$ *such that for any* $\pi \in S_n$

$$|\chi^\lambda(\pi)| \leq \left[\max\left(\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, q\right)\right]^{b\ |\pi|}$$

Notation:

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Roichman's estimate is not good enough for asymptotics of quantum computers.

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Normalized characters

For a Young diagram λ with n boxes and $\pi \in S_l$ ($l \leq n$) we define normalized character

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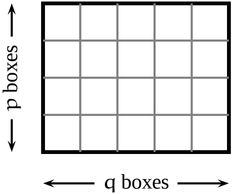
Important: we can think that $l = |\operatorname{supp} \pi|$.

Stanley's character formula

$p \times q =$ 

Stanley's character formula

Theorem (Stanley 2001)

For a rectangular Young diagram $p \times q$ and $\pi \in S_l$ (where $l \leq pq$)

$$\Sigma^{p\times q}(\pi) = \sum_{\substack{\sigma_1,\sigma_2 \in S_l,\\ \sigma_1\sigma_2=\pi}} (-1)^{|\sigma_1|}\ q^{|C(\sigma_1)|}\ p^{|C(\sigma_2)|},$$

where $|C(\sigma_i)|$ is the number of cycles of σ_i.

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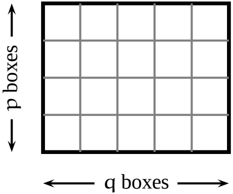

Stanley-Féray character formula

Theorem (Féray 2006)

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$$\Sigma^\lambda(\pi) = \sum_{\substack{\sigma_1,\sigma_2 \in S_l, \\ \sigma_1\sigma_2=\pi}} (-1)^{|\sigma_1|} \; N^\lambda(\sigma_1, \sigma_2),$$

where $N^\lambda(\sigma_1, \sigma_2)$ is decribed in the following.

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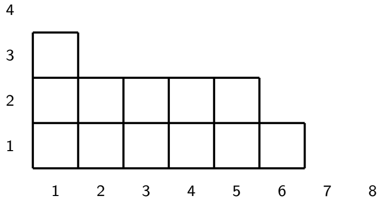
  the box in column $h_1(c_1)$ and row $h_2(c_2)$ belongs to λ.
- $N^\lambda(\sigma_1, \sigma_2)$ denotes the number of the colorings of σ_1, σ_2 which are compatible with λ.

Colorings: toy example

Factorization $(1,2) = \underbrace{(1)(2)}_{\sigma_1} \cdot \underbrace{(1,2)}_{\sigma_2}$. Coloring compatible with λ:



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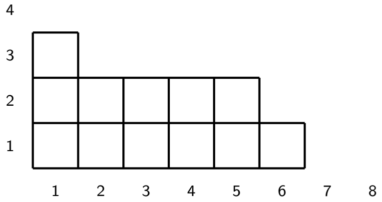

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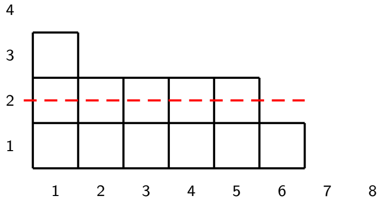

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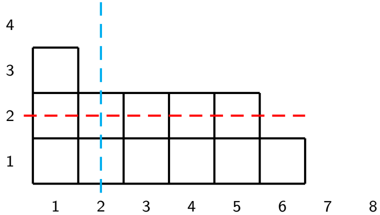

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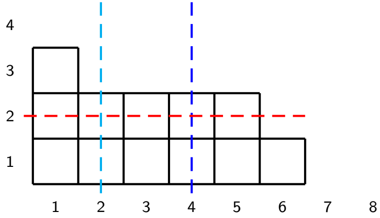

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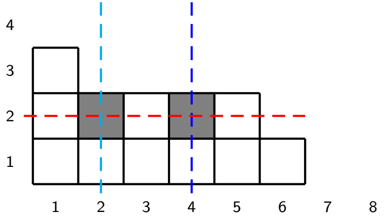

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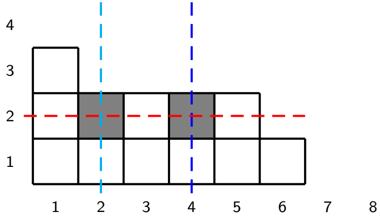

$$N^{\lambda}\big((1)(2),(1,2)\big) = \sum_i (\lambda_i)^2,$$

where λ_i is the number of boxes in i-th row.

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Theorem (Féray 2006)

For any Young diagram λ and a permutation $\pi \in S_l$ (where $l \leq n$)

$$\Sigma^{\lambda}(\pi) = \sum_{\substack{\sigma_1, \sigma_2 \in S_l, \\ \sigma_1 \sigma_2 = \pi}} (-1)^{|\sigma_1|} \, N^{\lambda}(\sigma_1, \sigma_2),$$

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- free probability (next section);

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$$\begin{bmatrix} 0 & \rho^\lambda(1,2) & \cdots & \rho^\lambda(1,n) & 1 \\ \rho^\lambda(2,1) & 0 & \cdots & \rho^\lambda(2,n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^\lambda(n,1) & \rho^\lambda(n,2) & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

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Like in the random matrix theory free cumulants are the right quantities.

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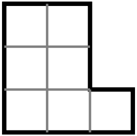
Minimal factorizations of $(1,\dots,k)$ = planar rooted trees with $k+1$ vertices!

Random matrices. . .

For a Young diagram λ we consider a random matrix T_λ.

$\lambda =$ 

$$T_\lambda = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots \\ g_{3,1} & g_{3,2} & 0 & 0 & \cdots \\ g_{2,1} & g_{2,2} & 0 & 0 & \cdots \\ g_{1,1} & g_{1,2} & g_{1,3} & 0 & \cdots \end{bmatrix}$$

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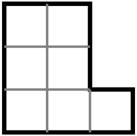
$\lambda =$ 

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Therefore the asymptotics of characters on long permutations $(l_1, l_2, \cdots \to \infty)$ is related to asymptotics of the largest eigenvalues of $T_\lambda T_\lambda^\star$.

Random matrices and circular operator

If λ is big then random matrix T_λ can be approximated by a circular operator T:

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For a Young diagram λ with n boxes

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Stanley-Féray formula: new one-line proof and estimate for error term

New bounds for characters

Theorem (Roichman 1996)

There exist constants $0 < q < 1$ and $b > 0$ such that for any Young diagram λ with n boxes

$$|\chi^\lambda(\pi)| \leq \left[\max\left(\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, q\right)\right]^{b\ |\pi|}$$

Theorem (Féray–Śniady 2007)

There exists a constant C such that for any Young diagram λ with n boxes

$$|\chi^\lambda(\pi)| \leq \left[C \max\left(\frac{r(\lambda)}{n}, \frac{c(\lambda)}{n}, \frac{|\pi|}{n}\right)\right]^{|\pi|}$$

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