Asymptotics of symmetric groups representations

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Outline

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Representations

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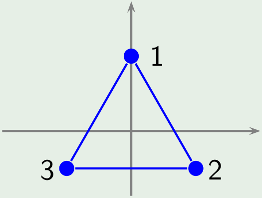
Example

Representation of S_3 as symmetries of a triangle on a plane.



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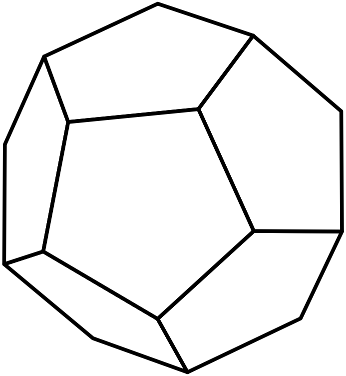

Into a dodecahedron we can inscribe a cube

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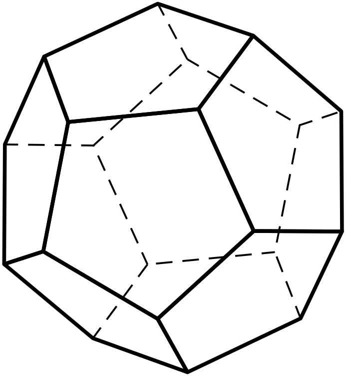

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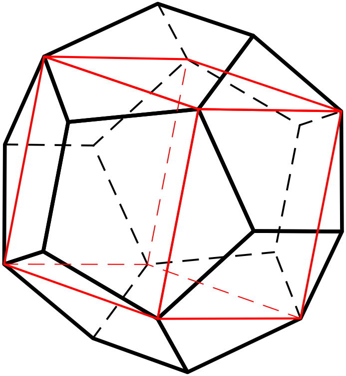

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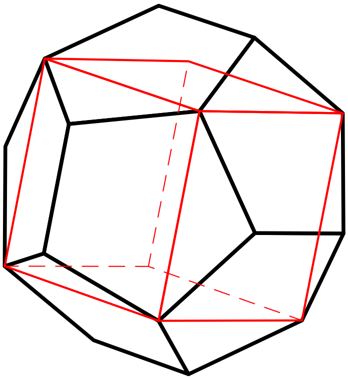

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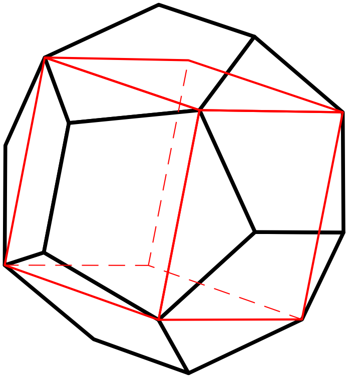

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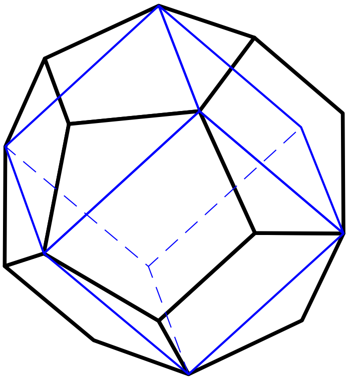

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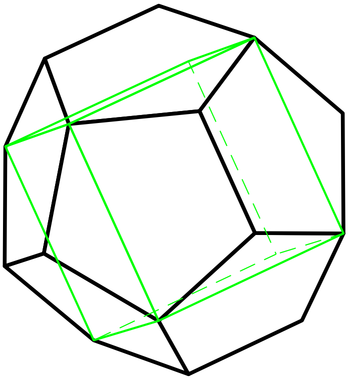

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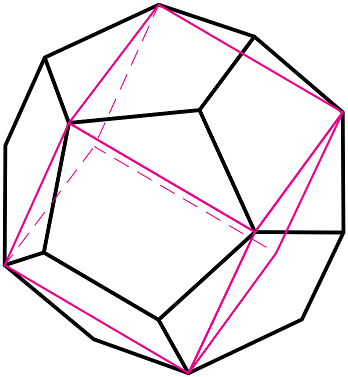

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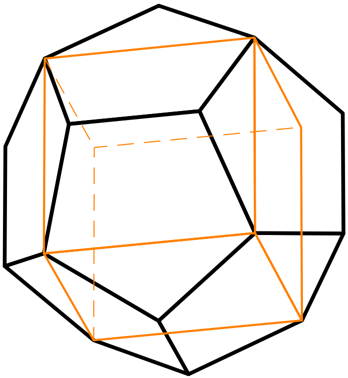

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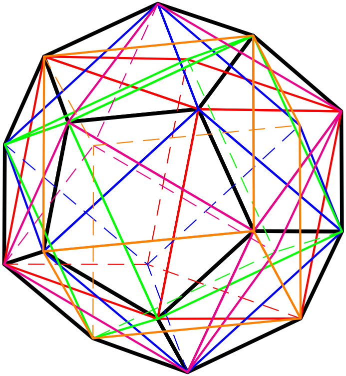

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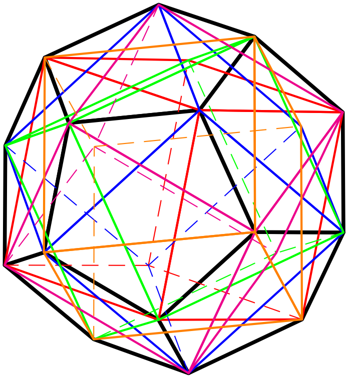

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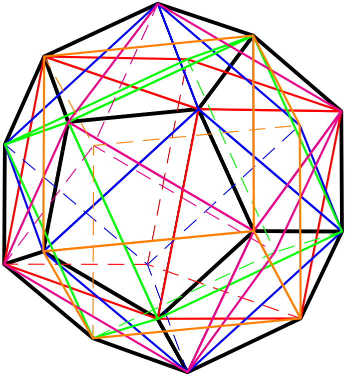

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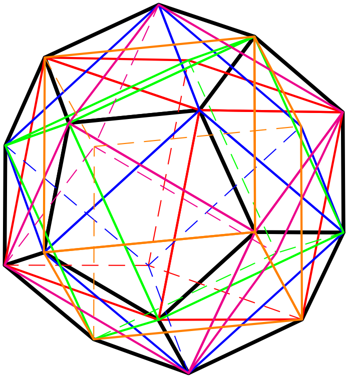

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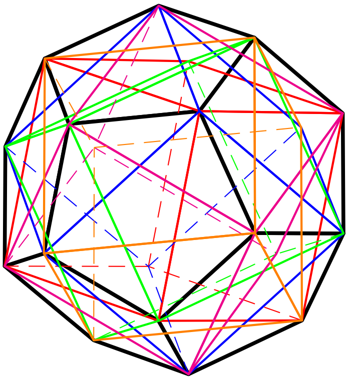

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This gives a representation of A_5 as rotations of the dodecahedron.

Asymptotic representation theory

Let $G_1, G_2, \dots$ be a fixed sequence of groups, and $\rho_1, \rho_2, \dots$ be a sequence of representations, where ρ_i is a representation of G_i.

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Irreducible representations

A representation $\rho : G \to \mathrm{End}(V)$ on a vector space V is reducible if there exists a nontrivial decomposition into subrepresentations: $V = V_1 \oplus V_2$ and $\rho = \rho_1 \oplus \rho_2$.

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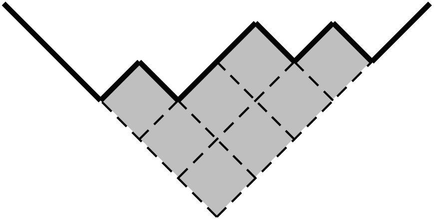
Irreducible representations ρ^λ of symmetric group S_n are in a one-to-one correspondence with Young diagrams λ having n boxes.



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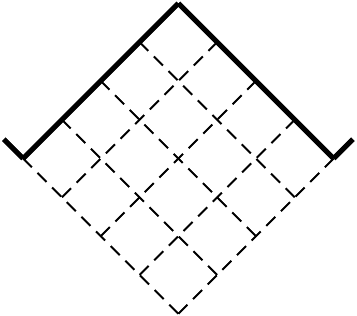
The main problem, concrete version

Suppose that some sequence of reducible representations ρ_n is given. What are the statistical properties of a randomly chosen Young diagram λ_n in the limit $n \to \infty$?

Outline

Example of a concrete problem: Restriction of irreducible representations



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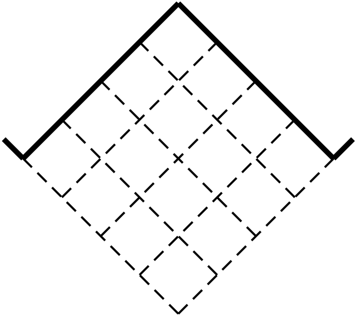

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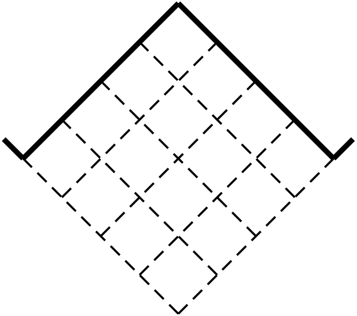
Problem

Let $0 < \alpha < 1$. What can we say about the restriction of the representation ρ^{ν} to a subgroup $S_{\alpha n^2}$?

Alternative description of the problem: Young tableaux



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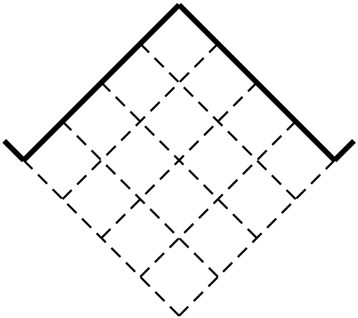

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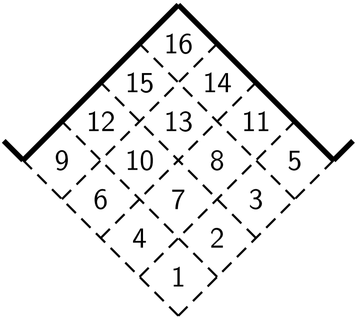

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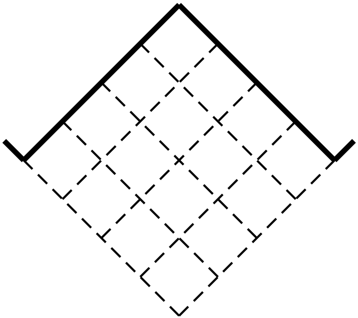

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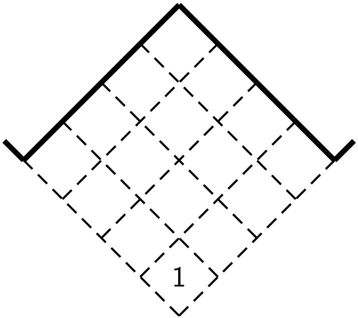

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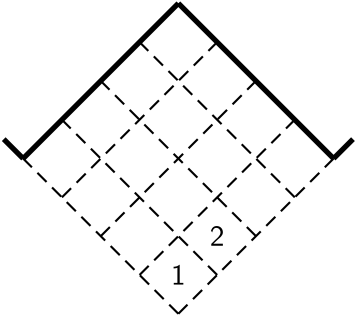

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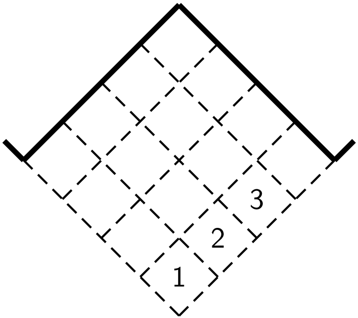

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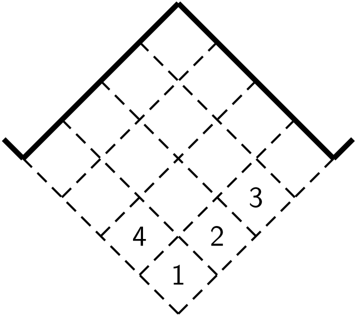

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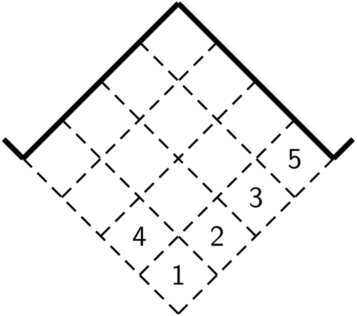

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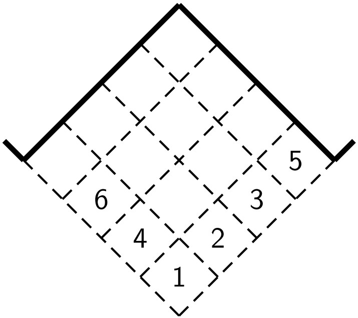

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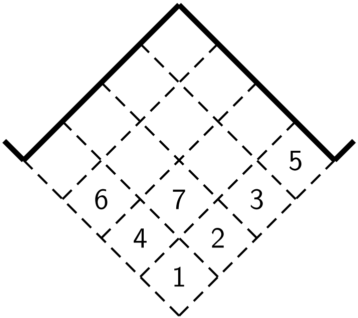

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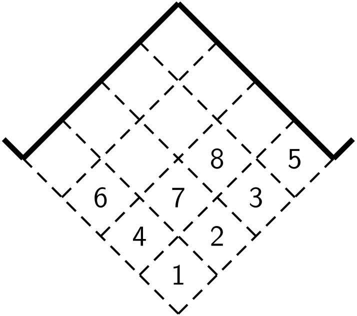

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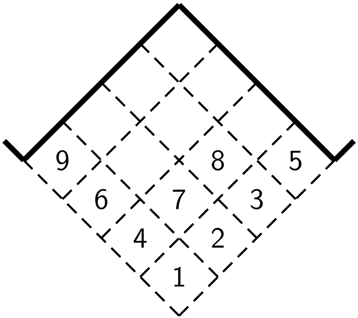

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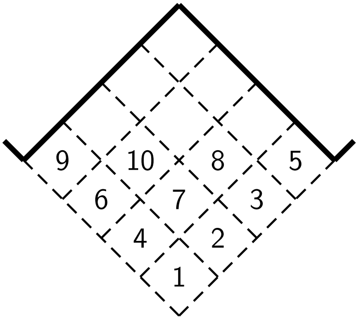

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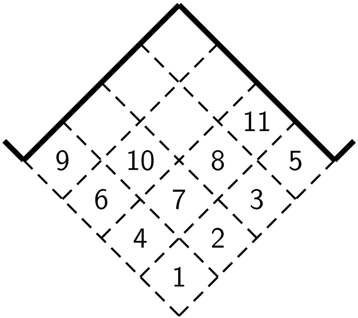

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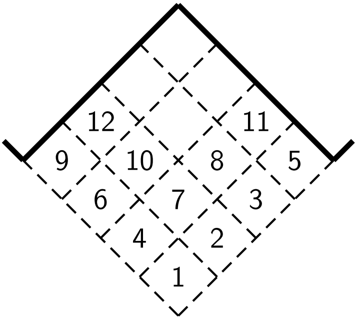

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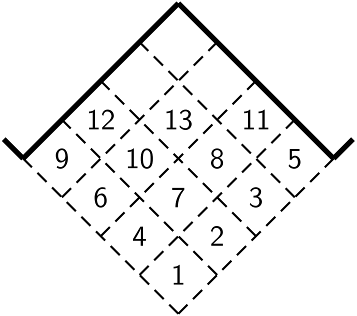

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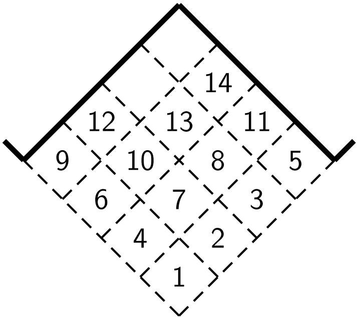

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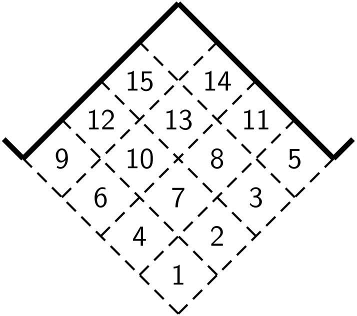

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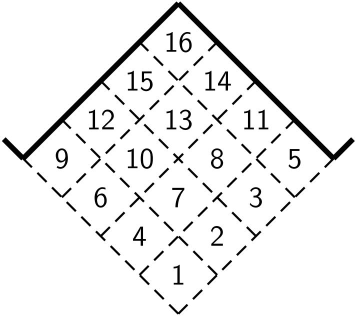

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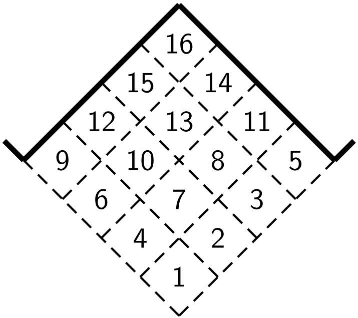

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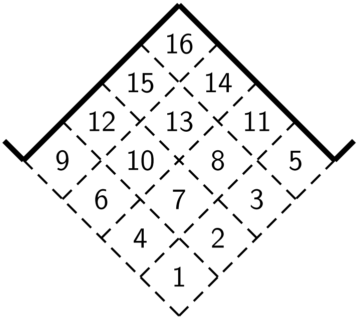

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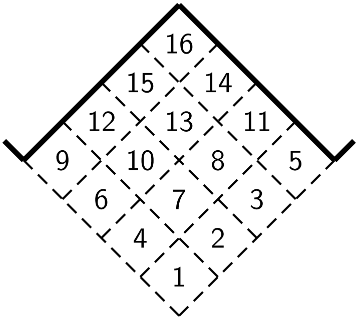

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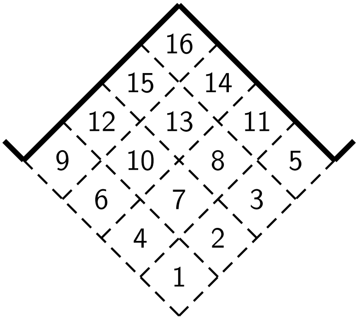

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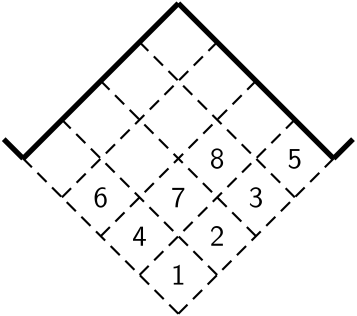

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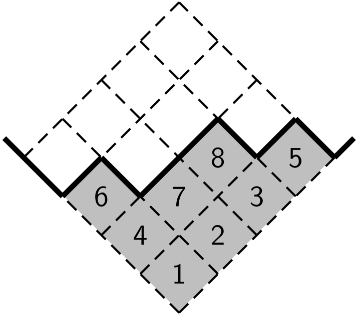

Theorem

The following random Young diagrams have the same distribution:

- *the random Young diagram associated to the restriction of the irreducible representation ρ^{ν} to a subgroup $S_{\alpha n^2}$;*
- *from a randomly chosen Young tableau we remove all boxes with numbers bigger than αn^2.*

What can we learn from this example?

- For any question concerning representations of S_n there is a well-known answer given by some *combinatorial* algorithm.

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  Analogy to random matrices: if M is a hermitian matrix, we can encode its eigenvalues in its spectral measure μ_M which is a probability measure on $\mathbb{R}$. When M is a random matrix, μ_M is a random probability measure on $\mathbb{R}$.

Outline

1. Representations of symmetric groups and Young diagrams
2. Asymptotic representation theory: Example of a problem
3. Main result: Gaussian fluctuations of Young diagrams

Main theorem: law of large numbers

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Theorem (law of large numbers, Philippe Biane 1998)

The sequence of rescaled random Young diagrams $(\frac{1}{\sqrt{n}}\lambda_n)$ converges in probability to some (generalized) Young diagram λ. The shape of this limit can be described by the free probability theory.

Main theorem: central limit theorem

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Generalization of Kerov's central limit theorem.

Notations

If ρ is a representation of the symmetric group, its *normalized character* χ_ρ is defined by

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Support of a permutation is the set of non-fixed points.

Approximate factorization of characters

We say that a sequence of representations (ρ_n) has the property of approximate factorization of characters if for any permutations $\pi_1, \dots, \pi_l$ with disjoint supports

$$\chi_{\rho_n}(\pi_1 \dots \pi_l) \approx \chi_{\rho_n}(\pi_1) \cdots \chi_{\rho_n}(\pi_l),$$

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We require that the classical cumulant $k(\pi_1, \dots, \pi_l)$ converges quickly enough to zero.

$$k(\pi_1, \dots, \pi_l) = O\left(n^{-\frac{|\pi_1|+\cdots+|\pi_l|+2(l-1)}{2}}\right).$$

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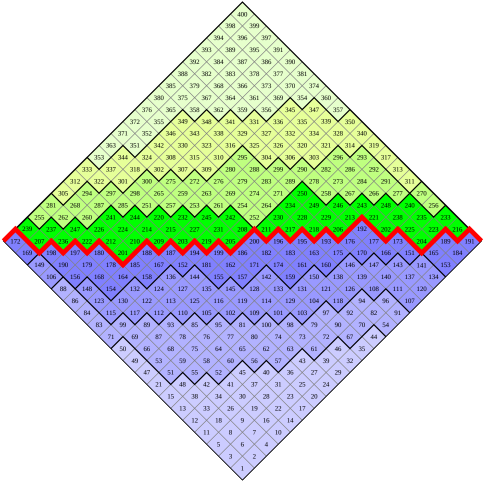
Corollary

What was the shape of the pile of stones? The answer for this problem is given by a certain Gaussian process.

Experimental verification



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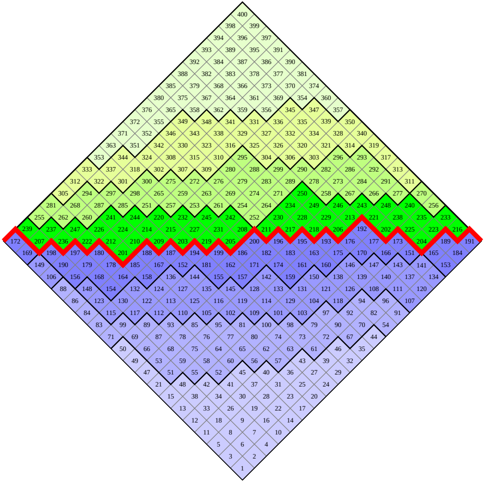
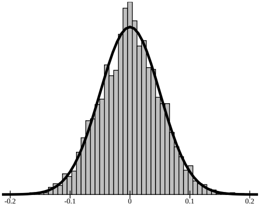
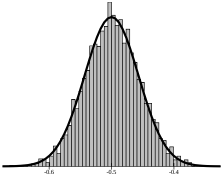




Distribution of free cumulant R_3



Distribution of free cumulant R_4

Characters factorization implies Gaussian fluctuations: How to prove the main result?

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To each such summand we associate a two-dimensional surface. The asymptotic behavior of a summand depends only on its topology. This technique is called genus expansion.

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We prove that the genus expansion can be applied for representations of S_n as well. In this way the proofs of the results for random matrices can be directly translated to symmetric groups.

Bibliography

- Piotr Śniady.
  Gaussian fluctuations of characters of symmetric groups and of Young diagrams.
  Probab. Theory Related Fields 136 (2006), no. 2, 263–297
- Piotr Śniady.
  Asymptotics of characters of symmetric groups, genus expansion and free probability.
  Discrete Math. 306 (2006), no. 7, 624–665
- Piotr Śniady.
  Gaussian fluctuations of representations of wreath products.
  Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006), no. 4, 529–546