Characters of symmetric groups and free cumulants

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joint work with:

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Representations

representation of a group G is a homomorphism from G to invertible $n \times n$ matrices

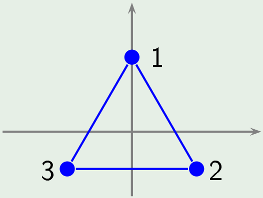
$$\rho : G \to M_{n\times n}(\mathbb{C}),$$

in other words,

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) \qquad \text{for any } g_1, g_2 \in G.$$

Example

Representation of $S(3)$ as symmetries of a triangle on a plane.



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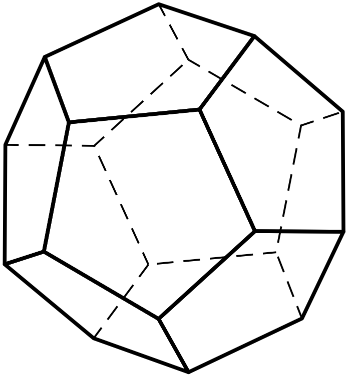
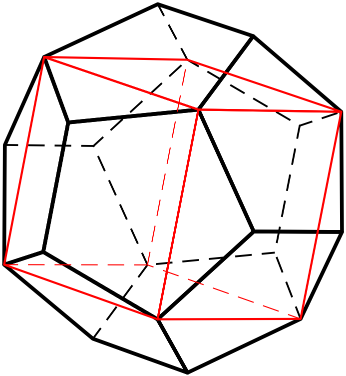


Into a dodecahedron we can inscribe a cube in five ways.

Example



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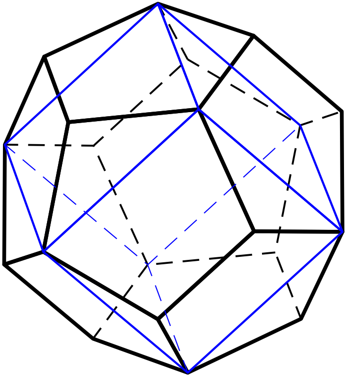

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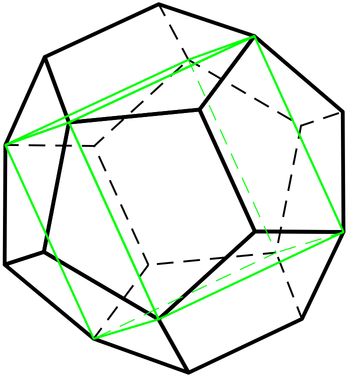

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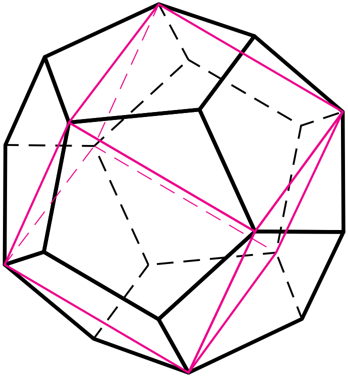

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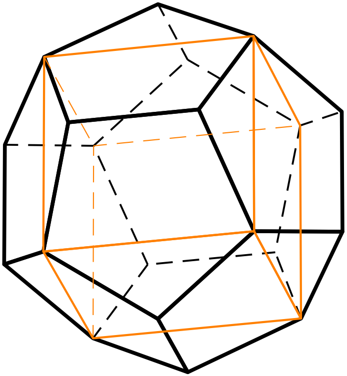

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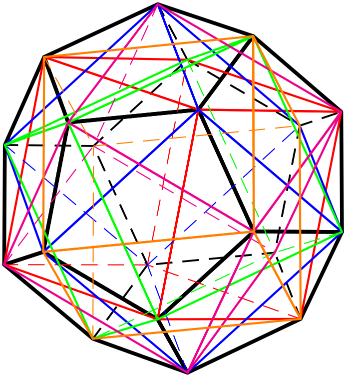

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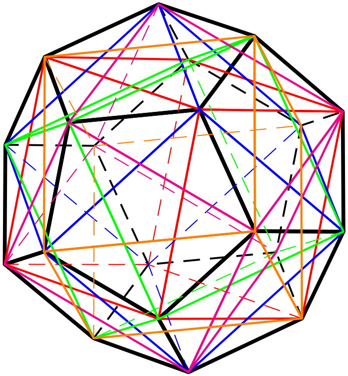

Into a dodecahedron we can inscribe a cube in five ways.

Every rotation of the dodecahedron defines a permutation of the five cubes.

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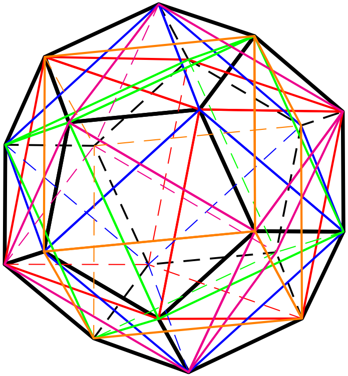

Into a dodecahedron we can inscribe a cube in five ways.

Every rotation of the dodecahedron defines an even permutation of the five cubes.

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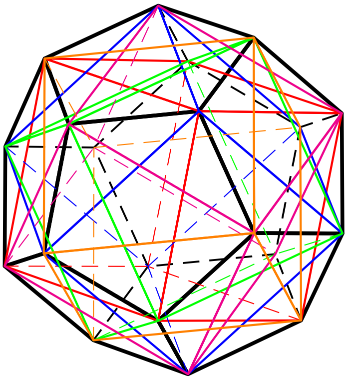

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In fact, there is an isomorphism between the group of rotations of the dodecahedron and the group $A(5)$ of even permutations.

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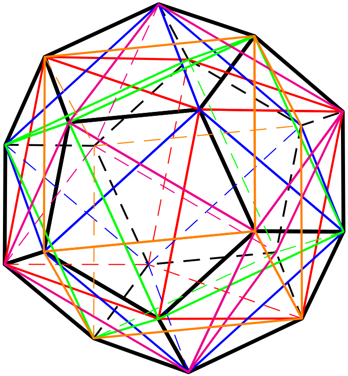

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This gives a representation of $A(5)$ as rotations of the dodecahedron.

Irreducible representations and characters

A representation $\rho : G \to \mathrm{End}(V)$ on a vector space V is called reducible if there exists a nontrivial decomposition into subrepresentations: $V = V_1 \oplus V_2$ and $\rho = \rho_1 \oplus \rho_2$.

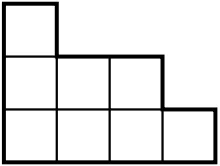
Otherwise, a representation is called irreducible.

If ρ is an irreducible representation, we define its character $\chi^\rho : G \to \mathbb{C}$ given by

$$\chi^\rho(g) = \frac{\mathrm{Tr}\, \rho(g)}{\text{dimension of } \rho}.$$

Irreducible representations of symmetric groups

Irreducible representations ρ^λ of symmetric group $S(n)$ are indexed by Young diagrams λ having n boxes.



Problem

What is the relation between the shape of a Young diagram and the corresponding irreducible character?

Dilations of Young diagrams

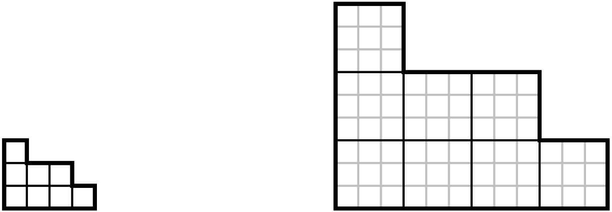


diagram λ dilated diagram $s\lambda$ for $s = 3$

Problem

What happens to irreducible characters of symmetric groups corresponding to $s\lambda$ for $s \to \infty$?

Normalized characters

For $\pi \in S(k)$ and irreducible representation ρ^λ of $S(n)$ (assume $k \leq n$) we define the normalized character

$$\Sigma_\pi^\lambda = \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\text{dimension of } \rho^\lambda}.$$

Most interesting case: characters on cycles

$$\Sigma_k^\lambda = \Sigma_{(1,2,\dots,k)}^\lambda.$$

The same problem, concretely:

For fixed $k \geq 1$ what can we say about $\Sigma_k^{s\lambda}$ for $s \to \infty$?

Free cumulants

The map $s \mapsto \Sigma_{k-1}^{s\lambda}$ is a polynomial of degree k.
We define free cumulants $R_2^\lambda, R_3^\lambda, \dots$ of diagram λ to be asymptotically the dominant terms of the character on cycles:

$$R_k^\lambda = \lim_{s\to\infty} \frac{1}{s^k} \Sigma_{k-1}^{s\lambda} = [s^k] \Sigma_{k-1}^{s\lambda}.$$

Advertisement

Free cumulants are very nice quantities describing a Young diagram.

$\Sigma_{k-1}^{\lambda} \approx R_k^\lambda$ has a lot of implications in the representation theory

Free cumulants are homogeneous with respect to dilations:
$R_k^{s\lambda} = s^k R_k^\lambda$.

There are relatively simple explicit formulas for free cumulants of Young diagrams.

Free cumulants for free probability people 1

Denote $\star = n+1$. Jucys-Murphy element is defined by

$$J = (1\star) + \cdots + (n\star) \in \mathbb{C}\big(S(n+1)\big).$$

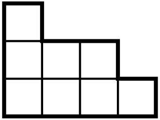
Let ρ^λ be an irreducible representation of $S(n)$.
We equip $\mathbb{C}\big(S(n+1)\big)$ with an expected value:

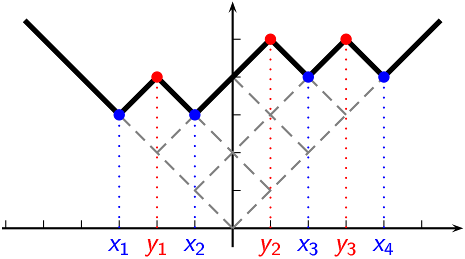
$$\mathbb{E}X = \chi^\lambda\left(X\downarrow^{S(n+1)}_{S(n)}\right).$$

Free cumulants of Young diagram λ are just free cumulants of Jucys-Murphy element with respect to this expected value:

$$R_k^\lambda = R_k(J).$$

Free cumulants for free probability people 2





Cauchy transform of a Young diagram:

$$G^{\lambda}(z) = \frac{(z - y_1) \cdots (z - y_{s-1})}{(z - x_1) \cdots (z - x_s)}.$$

Free cumulants of λ are the free cumulants related to G^{λ}.

Kerov polynomials

Free cumulants give approximations of characters:

$$\Sigma_k \approx R_{k+1},$$

but they can also give exact values of characters thanks to Kerov character polynomials:

$$\begin{aligned}
\Sigma_1 &= R_2, \\
\Sigma_2 &= R_3, \\
\Sigma_3 &= R_4 + R_2, \\
\Sigma_4 &= R_5 + 5R_3, \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3.
\end{aligned}$$

Studied by: S. Kerov, Ph. Biane, R. Stanley, I. Goulden, A. Rattan, M. Lassalle,...

The main result: combinatorial interpretation of Kerov polynomials

For a permutation π we denote by $C(\pi)$ the set of cycles of π.

Theorem (Dołęga, Féray, Śniady)

The coefficient $[R_2^{s_2} R_3^{s_3} \cdots]\Sigma_k$ *is equal to the number of triples* (σ_1, σ_2, q) *such that*

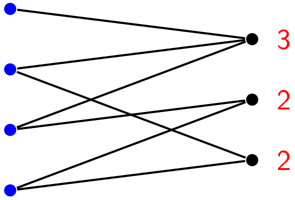
- $\sigma_1, \sigma_2 \in S(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$,
- $q : C(\sigma_2) \to \{2, 3, \dots\}$ *is a labeling such that each label* $i \in \{2, 3, \dots\}$ *is used* s_i *times,*
- *consider a bipartite graph* $C(\sigma_1) \sqcup C(\sigma_2)$, *where* $c_1 \in C(\sigma_1)$ *is connected by an edge with* $c_2 \in C(\sigma_2)$ *iff they are not disjoint; we require that it is a q-admissible graph.*

Admissible bipartite graphs 1

Let V_1 and V_2 be the vertices of a bipartite connected graph and let $q : V_2 \to \{2, 3, \dots\}$ be a labelling of the red vertices.
We say that this graph is *q-admissible* if...

it is possible to choose orientations on the edges in such a way that:

- each blue vertex has exactly one outgoing edge,
- each red vertex v has exactly $q(v) - 1$ incoming edges,
- for any red vertices v_1, v_2 it is possible to find a path from v_1 to v_2.



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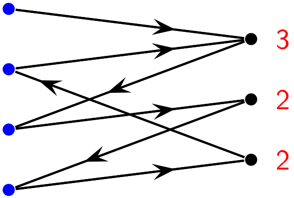
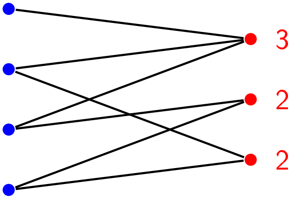


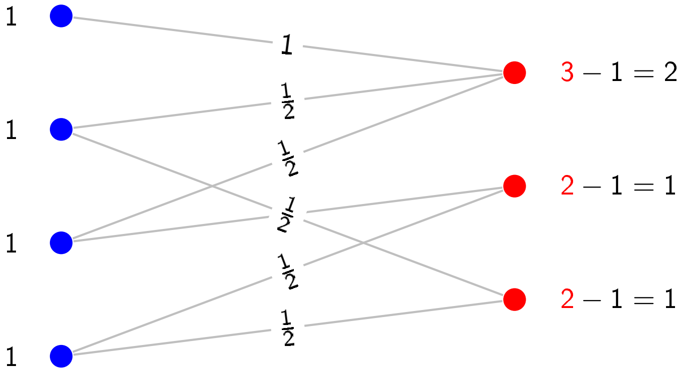
Admissible bipartite graphs 2

Let V_1 and V_2 be the vertices of a bipartite connected graph and let $q : V_2 \to \{2, 3, \dots\}$ be a labelling of the red vertices.
We say that this graph is *q*-admissible if...

for every nontrivial set $\emptyset \subsetneq A \subsetneq V_2$ of red vertices there are more than $\sum_{c \in A} \big(q(c) - 1\big)$ blue vertices are connected with at least one vertex in A.

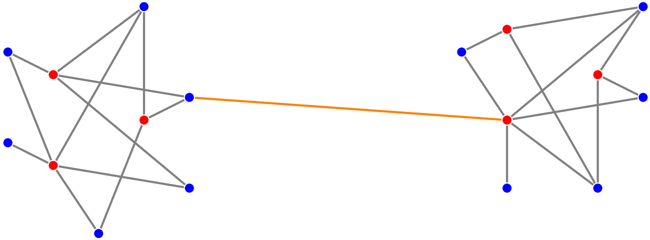


Admissible bipartite graphs 3



Each blue factory produces 1 unit.
Each red consumer g uses $q(g) - 1$ units.
We require that there is a way to arrange transportation so that every edge of the graph has a positive number.

Restriction on graphs



Corollary

If there exists an disconnecting edge with at least one red vertex in each of the components then the graph cannot be admissible (no matter which labeling we choose).

"No part of the graph can look like a tree."
Application: coefficients of Kerov polynomials are small.

The main result: combinatorial interpretation of Kerov polynomials

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Applications

- positivity: Kerov polynomials give characters as simple sums without too many cancellations,
- optimal estimates for characters,
- more information on the structure of Kerov polynomials (solution to Lassalle's conjectures)

What is behind positivity?

Conjecture

Maybe coefficients of Kerov polynomials

- *are equal to dimensions of some intersection (co)homologies of something?*
- *are equal to something related to moduli space of analytic maps on Riemann surfaces? or ramified coverings of a sphere?*
- *are algebraic solutions to some integrable hierarchy (Toda?) and their coefficients are related to the tau function of the hierarchy?*

Current definition of Kerov polynomials is rather implicit. Can we find an explicit definition of the coefficients of Kerov polynomials (using some exotic interpretation)?

Open problems

- strange: Kerov polynomials also show up in the random matrix theory for some bizzare random matrices (ask about it after the talk!);
- free cumulants originally come from Voiculescu's free probability theory / random matrix theory...
  is there some analogue of Kerov character polynomials in the random matrix theory / respresentation theory of the unitary groups $U(d)$?
- is it possible to study Kerov polynomials in such a scaling that phenomena of universality of random matrices occur?
- the structure of Kerov polynomials is still not clear: Goulden–Rattan conjecture (ask me after the talk!), Lassalle's conjectures

Character formula

For any Young diagram λ with n boxes and a permutation $\pi \in S_k$

$$\Sigma_\pi^\lambda = \sum_{\substack{\sigma_1,\sigma_2 \in S_k, \\ \sigma_1\sigma_2=\pi}} (-1)^{|\sigma_1|} \; N^\lambda(\sigma_1,\sigma_2),$$

where $N^\lambda_{(\sigma_1,\sigma_2)}$ is the number of colorings (next slides).

Up to the $\pm$ sign, the same formula gives moments of some bizzare random matrix.

Colorings

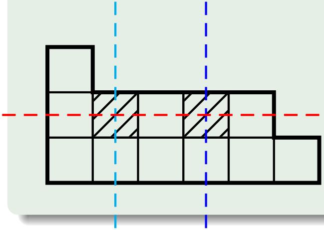
Assume $\pi = \sigma_1 \sigma_2$. Coloring (f_1, f_2) of the cycles of (σ_1, σ_2)

- $f_1 : C(\sigma_1) \to \mathbb{N}$ maps the cycles of σ_1 to columns of λ;
- $f_2 : C(\sigma_2) \to \mathbb{N}$ maps the cycles of σ_2 to rows of λ;
- if c_1 is a cycle of σ_1, c_2 is a cycle of σ_2 and $c_1 \cap c_2 \neq \emptyset$ then $\big(f_1(c_1), f_2(c_2)\big) \in \lambda$.

We denote the number of colorings of (σ_1, σ_2) by $N^\lambda(\sigma_1, \sigma_2)$.

Example

Factorization $(1,2) = \underbrace{(1)(2)}_{\sigma_1} \cdot \underbrace{(1,2)}_{\sigma_2}$.



$$N^\lambda\big((1)(2), (1,2)\big) = \sum_i (\lambda_i)^2,$$

where λ_i is the number of boxes in i-th row.

Number of colorings is homogeneous

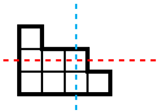
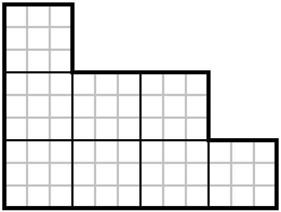


diagram λ



dilated diagram $s\lambda$ for $s = 3$

$$N^{s\lambda}(\sigma_1, \sigma_2) = s^{|C(\sigma_1)|+|C(\sigma_2)|} N^{\lambda}(\sigma_1, \sigma_2)$$

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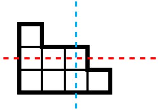

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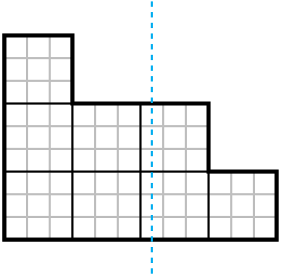

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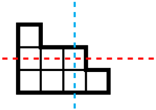

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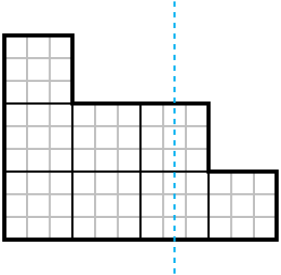

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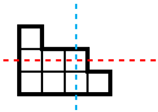

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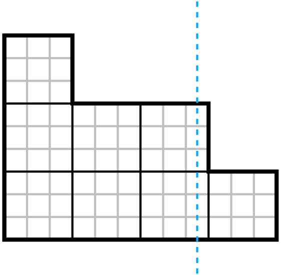

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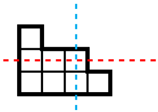

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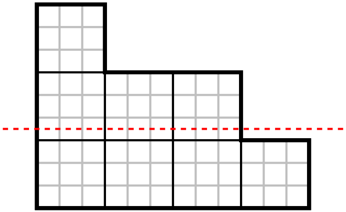

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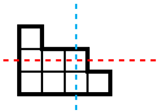

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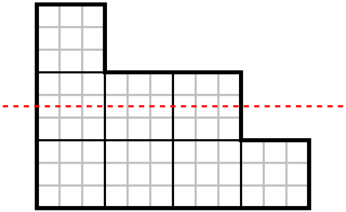

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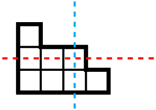

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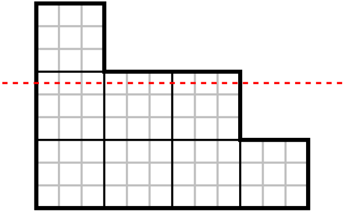

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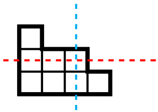

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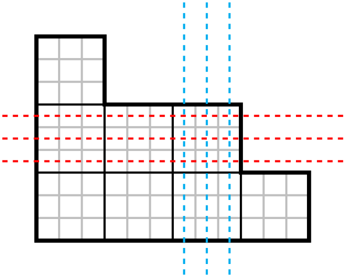

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$$\Sigma_\pi^\lambda = \sum_{\substack{\sigma_1,\sigma_2 \in S_k,\\ \sigma_1\sigma_2=\pi}} (-1)^{|\sigma_1|}\, N^\lambda(\sigma_1,\sigma_2).$$

It is nice because:

- small number of summands if π is fixed;
- each summand is directly related to the shape of λ;
- shows that $s \mapsto \Sigma_\pi^{s\lambda}$ is a polynomial function.

Example

$\pi = (1,2) = (1)(2)\cdot(1,2) = (1,2)\cdot(1)(2)$, so

$$\Sigma^\lambda_{(1,2)} = N^\lambda_{(1)(2),(1,2)} - N^\lambda_{(1,2)\cdot(1)(2)} = \sum_i \lambda_i^2 - \sum_i \lambda_i'^2$$

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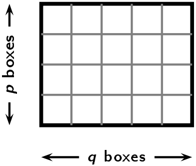
Toy example: linear terms of Kerov polynomials 1



$$\Sigma_\pi^{p\times q} = \sum_{\substack{\sigma_1,\sigma_2\in S_k,\\ \sigma_1\sigma_2=\pi}} (-1)^{|\sigma_1|} \underbrace{q^{|C(\sigma_1)|}\; p^{|C(\sigma_2)|}}_{N^{p\times q}_{(\sigma_1,\sigma_2)}}.$$

Corollary

- $[q^i p]\Sigma_\pi^{p\times q}$ *is equal (up to the sign) to the number of factorizations* $\pi = \sigma_1\sigma_2$ *such that* σ_1 *has* i *cycles and* σ_2 *has only one cycle.*
-
$$[q^i p]R_{k+1}^{p\times q} = [q^i p](\textit{degree } k+1 \textit{ part of } \Sigma_k) = \begin{cases} 1 & \textit{if } i=k, \\ 0 & \textit{otherwise} \end{cases}$$

Toy example: linear terms of Kerov polynomials 2

$$R_{k+1}^{p\times q} = q^k p + (\text{terms containing higher powers of } p),$$

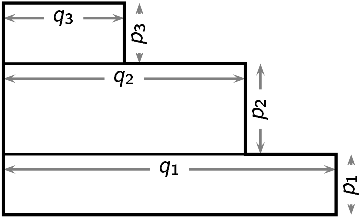
$$\Sigma_\pi = X\ R_{k+1} + (\text{other monomials in free cumulants})$$

$$\Sigma_\pi^{p\times q} = X\ q^k p + (\text{other monomials in } p, q)$$

Corollary

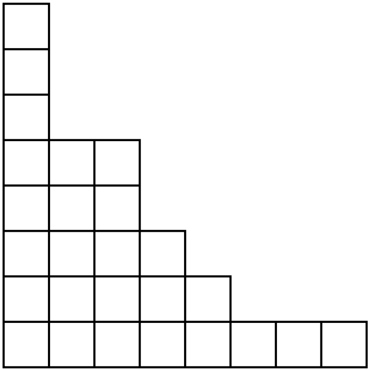
$$X = [R_{k+1}]\Sigma_\pi = [q^k p]\Sigma_\pi^{p\times q}$$
$$= \pm \left(\textit{the number of factorizations } \pi = \sigma_1\sigma_2 \textit{ such that } \sigma_1 \textit{ has } k \textit{ cycles and } \sigma_2 \textit{ has only one cycle}\right)$$

Kerov polynomials, general case



To get information about general coefficients of Kerov polynomials, one has to consider more complex shapes.

Application: random Young diagrams



$n = 25$

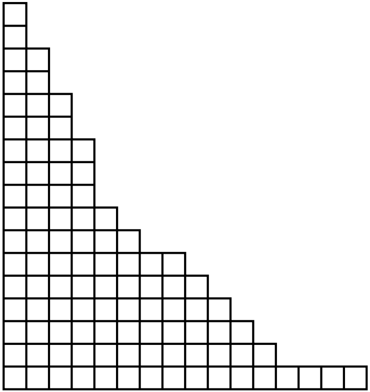
Let ρ be, for example, left-regular representation of $S(n)$.

We decompose it into irreducible components and we randomly select an irrecible component ρ^{λ}.

What can we say about λ?

What can we say about $\frac{1}{\sqrt{n}}\lambda$?

Application: random Young diagrams



$n = 100$

Let ρ be, for example, left-regular representation of $S(n)$.

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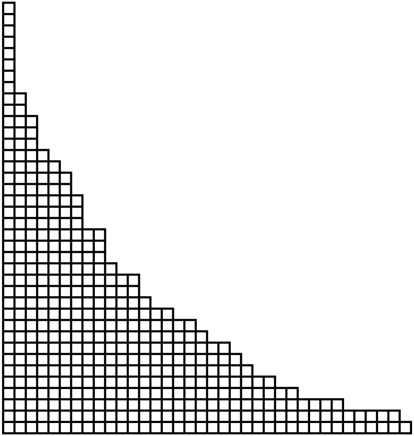
Application: random Young diagrams



$n = 400$

Let ρ be, for example, left-regular representation of $S(n)$.

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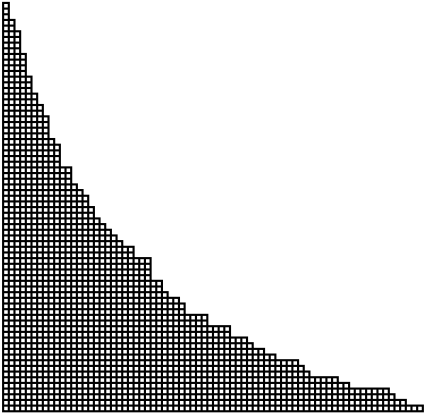
Application: random Young diagrams



$n = 1600$

Let ρ be, for example, left-regular representation of $S(n)$.

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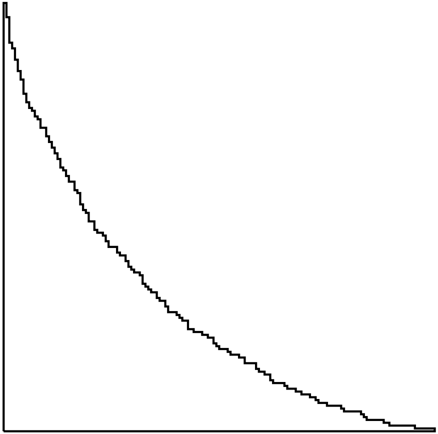

$n = 6400$

Let ρ be, for example, left-regular representation of $S(n)$.

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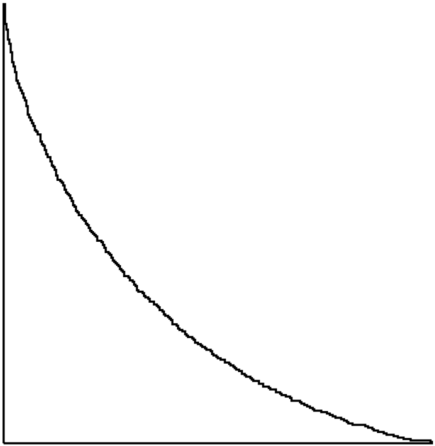
Application: random Young diagrams



$n = 25600$

Let ρ be, for example, left-regular representation of $S(n)$.

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What can we say about $\frac{1}{\sqrt{n}}\lambda$?

Free cumulants: shape of rescaled diagram vs characters

Shape of $\frac{1}{\sqrt{n}}\lambda$ is determined by free cumulants:

$$R_k^{\frac{1}{\sqrt{n}}\lambda} = \frac{1}{\sqrt{n^k}} R_k^\lambda \approx \frac{1}{\sqrt{n^k}} \Sigma_{k-1}^\lambda \approx n^{\frac{k-2}{2}} \chi^\lambda_{(1,\dots,k-1)}$$

In the case of random Young diagrams we know the typical value of characters:

$$\mathbb{E}\chi^\lambda_\pi = \chi^\rho_\pi$$

General phenomenon: random Young diagrams behave like eigenvalues of random matrices.

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