Characters of symmetric groups in terms of free cumulants

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joint work with:

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Valentin Féray

Dilations of Young diagrams

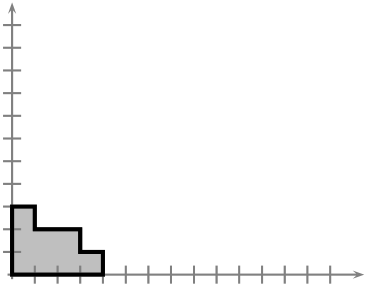
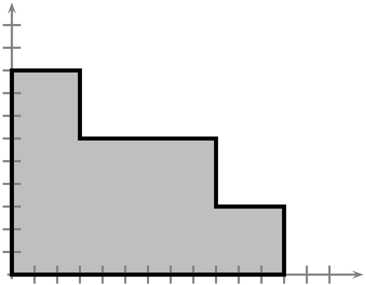


diagram λ



dilated diagram $s\lambda$ for $s = 3$

Problem

What happens to irreducible characters of symmetric groups corresponding to $s\lambda$ for $s \to \infty$?

Normalized characters

For $\pi \in S(k)$ and irreducible representation ρ^λ of $S(n)$ (assume $k \leq n$) we define the normalized character

$$\Sigma^\lambda_\pi = \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\text{dimension of } \rho^\lambda}.$$

Most interesting case: characters on cycles

$$\Sigma^\lambda_k = \Sigma^\lambda_{(1,2,\dots,k)}.$$

The same problem, concretely:

For fixed $k \geq 1$ what can we say about $\Sigma^{s\lambda}_k$ for $s \to \infty$?

Free cumulants

The map $s \mapsto \Sigma^{s\lambda}_{k-1}$ is a polynomial of degree k.
We define free cumulants $R^\lambda_2, R^\lambda_3, \dots$ of diagram λ to be asymptotically the dominant terms of the character on cycles:

$$R^\lambda_k = \lim_{s\to\infty} \frac{1}{s^k} \Sigma^{s\lambda}_{k-1} = [s^k] \Sigma^{s\lambda}_{k-1}.$$

Advertisement

Free cumulants are very nice quantities describing a Young diagram.

Free cumulants are homogeneous with respect to dilations:
$R^{s\lambda}_k = s^k R^\lambda_k$.

Kerov polynomials

Free cumulants give approximations of characters:

$$\Sigma_k \approx R_{k+1},$$

but they can also give exact values of characters thanks to Kerov character polynomials:

$$\begin{aligned}
\Sigma_1 &= R_2, \\
\Sigma_2 &= R_3, \\
\Sigma_3 &= R_4 + R_2, \\
\Sigma_4 &= R_5 + 5R_3, \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3.
\end{aligned}$$

Studied by: S. Kerov, Ph. Biane, R. Stanley, I. Goulden, A. Rattan, M. Lassalle,. . .

The main result: combinatorial interpretation of Kerov polynomials

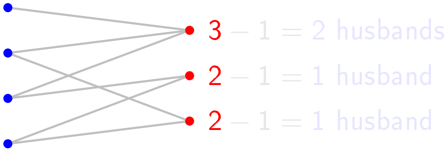
For a permutation π we denote by $C(\pi)$ the set of cycles of π.

Theorem (Dołęga, Féray, Śniady)

The coefficient $[R_2^{s_2} R_3^{s_3} \cdots]\Sigma_k$ *is equal to the number of triples* (σ_1, σ_2, q) *such that*

- $\sigma_1, \sigma_2 \in S(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$,
- $|C(\sigma_1)| + |C(\sigma_2)| = 2s_2 + 3s_3 + 4s_4 + \cdots$,
- $q : C(\sigma_2) \to \{2, 3, \dots\}$ *is a labeling such that each label* $i \in \{2, 3, \dots\}$ *is used* s_i *times,*
- *for every nontrivial set* $\emptyset \subsetneq A \subsetneq C(\sigma_2)$ *of cycles of* σ_2 *there are more than* $\sum_{c \in A} \big(q(c) - 1\big)$ *cycles of* σ_1 *which intersect* $\bigcup A$.

Marriage interpretation



Example: coefficient $[R_2^2 R_3]\Sigma_k$. For given σ_1, σ_2 we consider a bipartite graph $\mathcal{V}_{\sigma_1,\sigma_2}$ with the vertices corresponding to cycles of σ_1 (boys) and cycles of σ_2 (girls). We draw an edge if two cycles intersect (boy is allowed to marry a girl). Each boy wants to marry one girl and each girl $g \in C(\sigma_2)$ wants to marry $q(g) - 1$ boys. We require that it is possible to arrange marriages and that for each non-trivial set of girls the set of their husbands is not uniquely determined.

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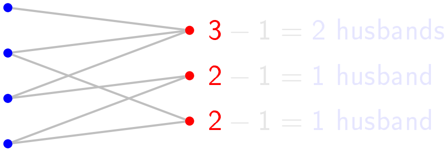

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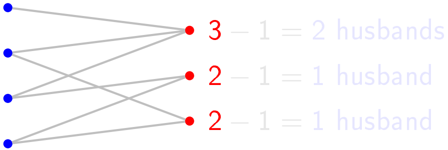

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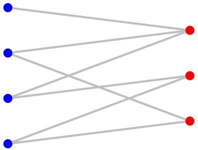

$3 - 1 = 2$ husbands

$2 - 1 = 1$ husband

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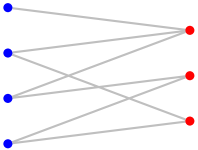

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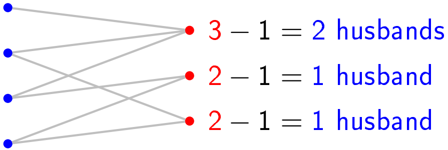

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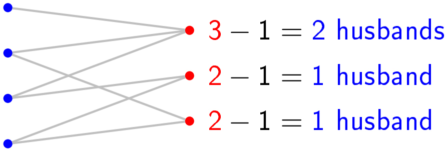

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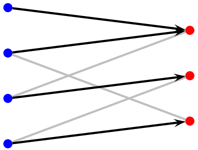

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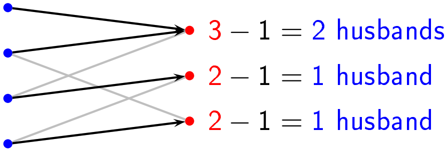

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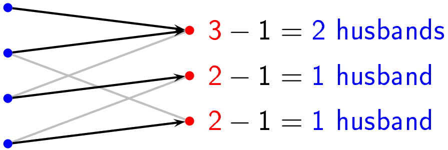

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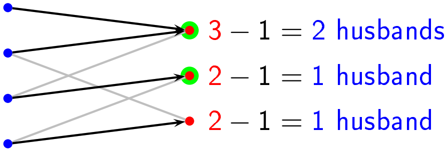

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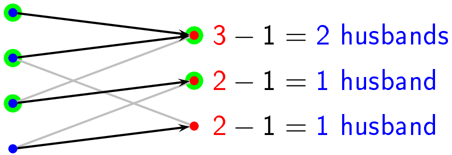

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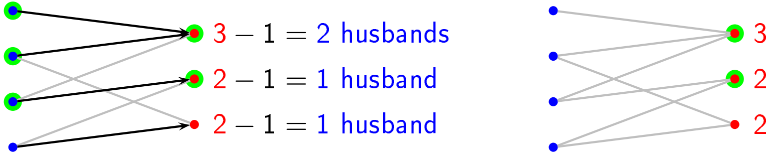

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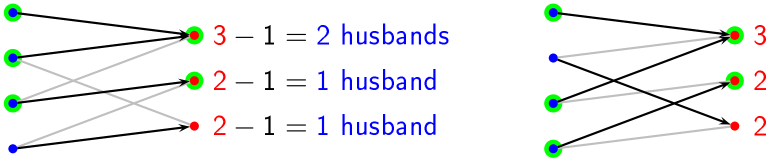

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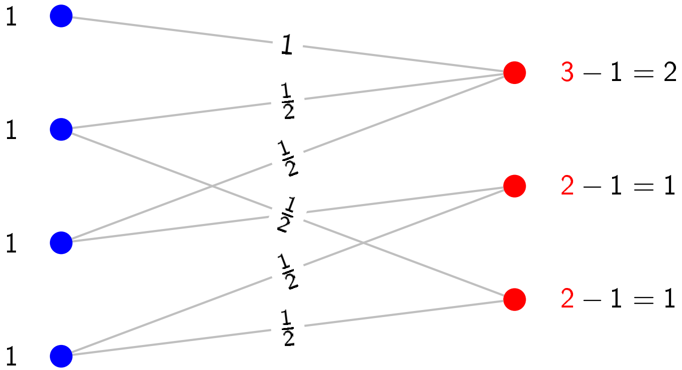
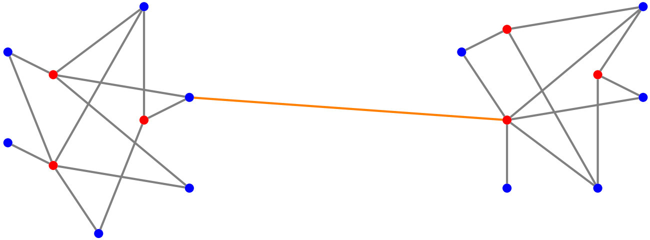
Transportation interpretation



Each blue factory produces 1 unit.
Each red consumer g uses $q(g) - 1$ units.
We require that there is a way to arrange transportation so that every edge of the graph has a positive number.

Restriction on graphs



Corollary

If there exists an disconnecting edge with at least one red vertex in both components then the factorization cannot contribute (no matter which labeling we choose).

Application: coefficients of Kerov polynomials are small.

Applications & exotic conjectures

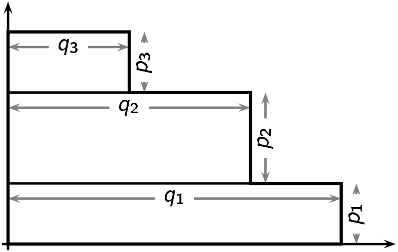
- positivity: Kerov polynomials give characters as simple sums without too many cancellations,
- optimal estimates for characters,
- more information on the structure of Kerov polynomials (Lassalle's conjectures)

Conjecture

Maybe coefficients of Kerov polynomials

- *are equal to dimensions of some intersection (co)homologies of Schubert varieties? [conjecture of Philippe Biane]*
- *are equal to something related to moduli space of analytic maps on Riemann surfaces? or ramified coverings of a sphere?*

Stanley polynomials



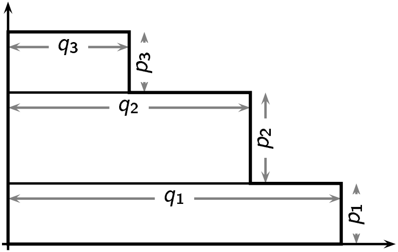
For numbers $p_1, p_2, \dots, q_1, q_2, \dots$ we consider multirectangular Young diagram $\mathbf{p} \times \mathbf{q}$.

Theorem (conjectured by Stanley, proved by Féray)

For any permutation π the normalized character $\Sigma_\pi^{\mathbf{p}\times\mathbf{q}}$ is a polynomial in $p_1, p_2, \dots, q_1, q_2, \dots$, called Stanley polynomial, for which there is an explicit formula.

Idea: now we can do differential calculus on the set of Young diagrams.

Stanley-Féray character formula, toy version



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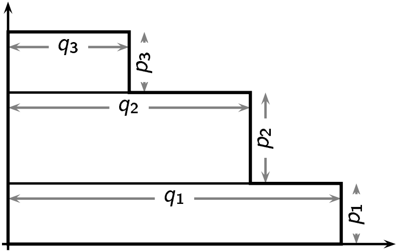


Corollary

For $\pi \in S(n)$

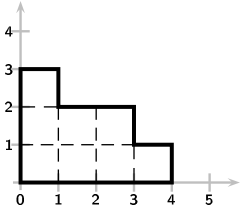
$$(-1)[p_1 q_1^i p_2 q_2^j] \Sigma_\pi^{\mathbf{p}\times\mathbf{q}}$$

is equal to the number of factorizations $\pi = \sigma_1 \circ \sigma_2$ *such that:*

- σ_1 *has* $i+j$ *cycles,*
- $\sigma_2 = \{c_1, c_2\}$ *has two labeled cycles,*
- *there are exactly* j *cycles of* σ_1 *which intersect* c_2.

Stanley polynomials give partial information about graphs $\mathcal{V}_{\sigma_1,\sigma_2}$.

Fundamental functionals $S_2, S_3, \dots$ of shape



$$\text{contents}_{(x,y)} = x - y$$

Fundamental functionals of shape of λ:

$$S_n^\lambda = (n-1) \iint_{(x,y)\in\lambda} \left(\text{contents}_{(x,y)}\right)^{n-2} \, dx \, dy$$

Theorem

If $\mathcal{F}$ is a sufficiently nice function on the set of Young diagrams then it is a polynomial in $S_2, S_3, \dots$:

$$\frac{\partial}{\partial S_{k_1}} \cdots \frac{\partial}{\partial S_{k_l}} \mathcal{F} \Big|_{S_2=S_3=\cdots=0} = [p_1 q_1^{k_1-1} \cdots p_l q_l^{k_l-1}] \mathcal{F}^{\mathbf{p}\times\mathbf{q}}$$

Therefore expansion of Σ_π in terms of $S_2, S_3, \dots$ follows from Stanley polynomials, explicitly given by Stanley-Féray formula. Then we express $S_2, S_3, \dots$ in terms of $R_2, R_3, \dots$.

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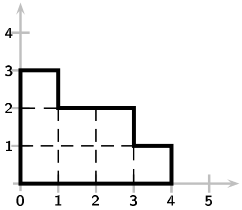

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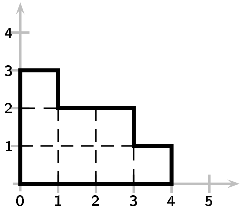

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Bibliography

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