Kerov character polynomials: recent progress in asymptotic representation theory of symmetric groups

(joint work with Maciej Dołęga and Valentin Féray)

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Outlook

- What can we say about the asymptotics of characters of symmetric groups $S(n)$ in the limit $n \to \infty$?
- Exact values of characters can be calculated from free cumulants thanks to Kerov polynomials.
- The main result: explicit combinatorial interpretation of the coefficients of Kerov polynomials.
- Open problems: relations to Schubert calculus, Toda hierarchy, ...

Plan

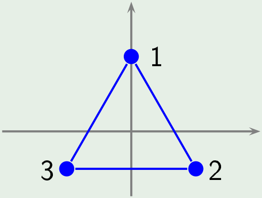
Representations

representation of a group G is a homomorphism from G to invertible $n \times n$ matrices

$$\rho : G \to M_{n\times n}(\mathbb{C}).$$

Example

Representation of $S(3)$ as symmetries of a triangle on a plane.



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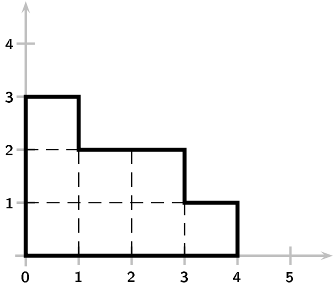
Motivations:

- irreducible representations $\longleftrightarrow$ Fourier transform,
- harmonic analysis on groups,
- random walks on groups,
- . . .

Irreducible representations of symmetric groups

Irreducible representations ρ^λ of symmetric group $S(n)$ are indexed by Young diagrams λ having n boxes.



Very combinatorial object, not good for asymptotic problems.

Dilations of diagrams

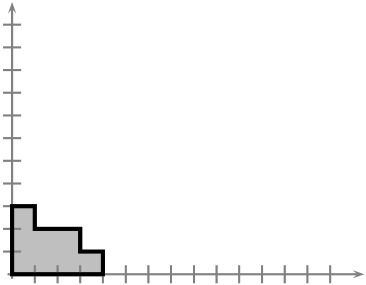
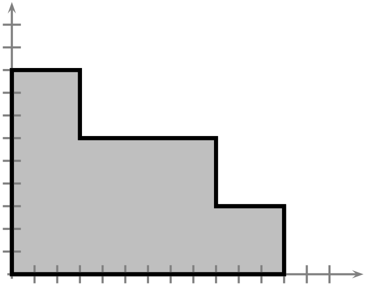


diagram λ



dilated diagram $s\lambda$ for $s = 3$

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Problem

For fixed $k \geq 1$ what can we say about $\Sigma_k^{s\lambda}$ for $s \to \infty$?

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Free cumulants are homogeneous with respect to dilations:
$R_k^{s\lambda} = s^k R_k^\lambda$.

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Kerov polynomials

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$$\begin{aligned}
\Sigma_1 &= R_2, \\
\Sigma_2 &= R_3, \\
\Sigma_3 &= R_4 + R_2, \\
\Sigma_4 &= R_5 + 5R_3, \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3.
\end{aligned}$$

Kerov conjecture

Theorem/Conjecture (Kerov)

For each $k \geq 1$ there exists a universal polynomial $K_k(R_2, R_3, \dots)$ with integer coefficients called Kerov character polynomial such that

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Féray: Kerov's conjecture is true, coefficients have a complicated combinatorial interpretation.

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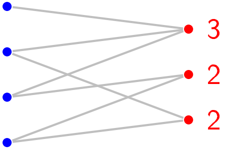
Marriage interpretation



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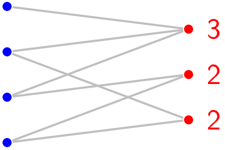

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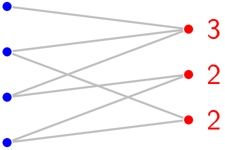

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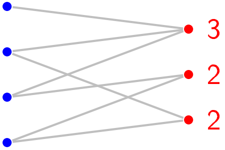

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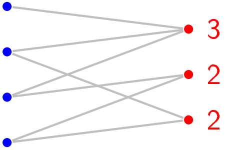

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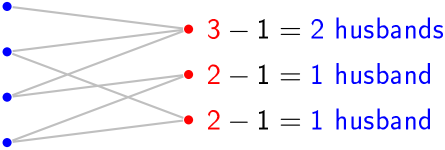

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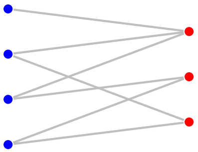

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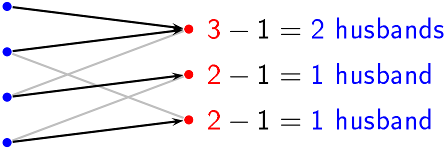

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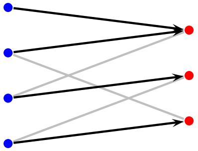

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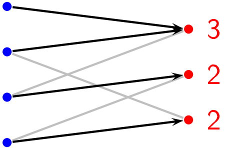

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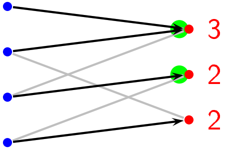

Example: coefficient $[R_2^2 R_3] K_k$. For given σ_1, σ_2 we consider a bipartite graph $\mathcal{V}_{\sigma_1,\sigma_2}$ with the vertices corresponding to cycles of σ_1 (boys) and cycles of σ_2 (girls). We draw an edge if two cycles intersect (boy is allowed to marry a girl). Each boy wants to marry one girl and each girl $g \in C(\sigma_2)$ wants to marry $q(g) - 1$ boys. We require that it is possible to arrange marriages and that for each non-trivial set of girls the set of their husbands is not uniquely determined.

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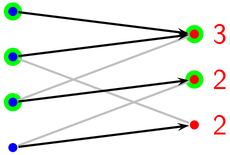

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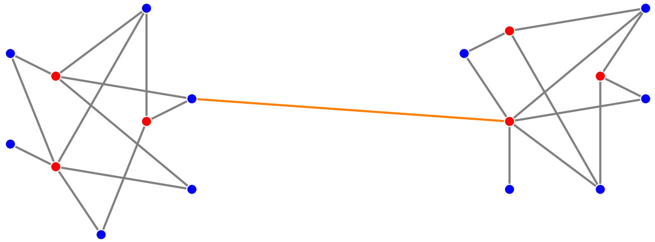
Restriction on graphs



Corollary

If there exists an disconnecting edge with at least one girl in both components then the factorization cannot contribute (no matter which labeling we choose).

Application: coefficients of Kerov polynomials are small.

Applications of the main result

- positivity: Kerov polynomials give characters as simple sums without too many cancellations,
- optimal estimates for characters,
- more information on the structure of Kerov polynomials (Lassalle's conjectures)

Plan

Exotic interpretations of Kerov polynomials

Conjecture

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- *are equal to something related to moduli space of analytic maps on Riemann surfaces? or ramified coverings of a sphere? [conjecture of Śniady]*
- *are algebraic solutions to some integrable hierarchy (Toda?) and their coefficients are related to the tau function of the hierarchy? [conjecture of Jonathan Novak]*

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- is it possible to study Kerov polynomials in such a scaling that phenomena of universality of random matrices occur?
- the structure of Kerov polynomials is still not clear (Goulden–Rattan conjecture, Lassalle's conjectures)

Conjecture: C-expansion of characters

Subdominant term of the character:

$$C^{\lambda}_{k-1} = \lim_{s\to\infty} \frac{1}{s^{k-1}} \left(\Sigma^{s\lambda}_{k} - R^{s\lambda}_{k+1} \right) = [s^{k-1}] \left(\Sigma^{s\lambda}_{k} - R^{s\lambda}_{k+1} \right)$$

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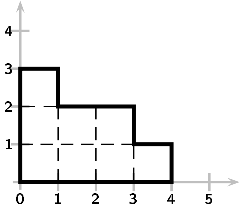
Conjecture (Goulden and Rattan)

For each $k \geq 1$ there exists a universal polynomial L_k called Goulden–Rattan polynomial with rational (non-negative?) coefficients (with relatively small denominators?) such that

$$\Sigma_k - R_{k+1} = L_k(C_2, C_3, \dots).$$

Plan

Fundamental functionals $S_2, S_3, \ldots$ of shape



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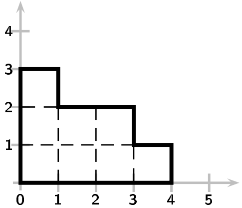

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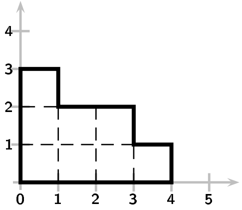

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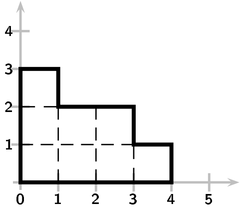

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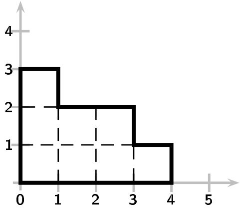

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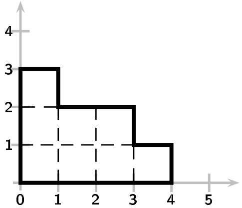

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- there are explicit formulas which express functionals $S_2, S_3, \ldots$ in terms of free cumulants $R_2, R_3, \ldots$ and conversely... therefore free cumulants can be explicitly calculated from the shape of a Young diagram!

Relation between functionals $S_2, S_3, \dots$ and free cumulants $R_2, R_3, \dots$

$$S_n = \sum_{l \geq 1} \frac{1}{l!} (n-1)_{l-1} \sum_{\substack{k_1,\dots,k_l \geq 2 \\ k_1+\cdots+k_l=n}} R_{k_1} \cdots R_{k_l},$$

$$R_n = \sum_{l \geq 1} \frac{1}{l!} (-n+1)^{\overline{l-1}} \sum_{\substack{k_1,\dots,k_l \geq 2 \\ k_1+\cdots+k_l=n}} S_{k_1} \cdots S_{k_l},$$

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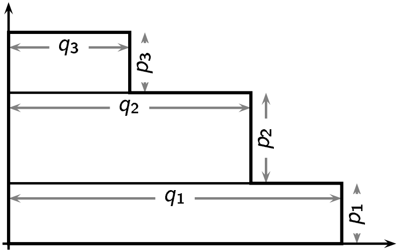
Example:

$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} \mathcal{F} = \frac{\partial^2}{\partial S_{k_1} \partial S_{k_2}} \mathcal{F} + (k_1 + k_2 - 1) \frac{\partial}{\partial S_{k_1+k_2}} \mathcal{F}.$$

All derivatives at $R_2 = R_3 = \cdots = S_2 = S_3 = \cdots = 0$.

Stanley polynomials



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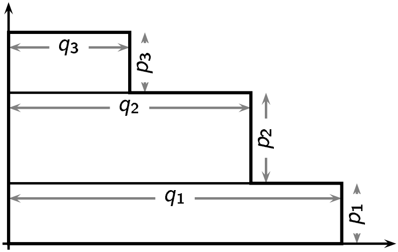

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Theorem (conjectured by Stanley, proved by Féray)

For any permutation π the normalized character $\Sigma_\pi^{\mathbf{p} \times \mathbf{q}}$ is a polynomial in $p_1, p_2, \dots, q_1, q_2, \dots$, called Stanley polynomial, for which there is an explicit formula.

Stanley-Féray character formula

Theorem (conjectured by Stanley, proved by Féray)

For $\pi \in S(n)$

$$\Sigma_{\pi}^{\mathbf{p}\times\mathbf{q}} = \sum_{\substack{\sigma_1,\sigma_2\in S(n)\\ \sigma_1\circ\sigma_2=\pi}} \sum_{\phi_2:C(\sigma_2)\to\mathbb{N}} (-1)^{\sigma_1}\cdot \prod_{b\in C(\sigma_1)} q_{\phi_1(b)} \cdot \prod_{c\in C(\sigma_2)} p_{\phi_2(c)},$$

where coloring $\phi_1 : C(\sigma_1) \to \mathbb{N}$ *is defined by*

$$\phi_1(c) = \max_{\substack{b\in C(\sigma_2),\\ b \text{ and } c \text{ intersect}}} \phi_2(b) \qquad \text{for } c \in C(\sigma_1)$$

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Stanley-Féray character formula, toy version

Corollary

For $\pi \in S(n)$

$$(-1)[p_1 q_1^i p_2 q_2^j] \Sigma_\pi^{\mathbf{p}\times\mathbf{q}}$$

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Stanley polynomials and functionals $S_2, S_3, \dots$

Theorem

If $\mathcal{F}$ is a sufficienly nice function on the set of generalized Young diagrams then it as a polynomial in $S_2, S_3, \dots$.

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- Stanley polynomials are explicitly given by Stanley-Féray formula and depend on geometry of bipartite graphs $\mathcal{V}_{\sigma_1,\sigma_2}$.
- Once we know the expansion of Σ_π in terms of $S_2, S_3, \dots$ we can find expansion of Σ_π in terms of free cumulants $R_2, R_3, \dots$.

Free cumulants vs fundamental functionals

Free cumulants $R_2, R_3, \dots$

Functionals $S_2, S_3, \dots$

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- describe Young diagram in language of representation theory

Functionals $S_2, S_3, \dots$

- describe Young diagram in language of its shape

Free cumulants vs fundamental functionals

Free cumulants $R_2, R_3, \dots$

- describe Young diagram in language of representation theory
- best quantities for calculating characters

Functionals $S_2, S_3, \dots$

- describe Young diagram in language of its shape
- directly related to Stanley polynomials

Toy example: $[R_{k_1} R_{k_2}]\Sigma_n$

$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} \mathcal{F} =$$

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$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} \mathcal{F} = \frac{\partial^2}{\partial S_{k_1} \partial S_{k_2}} \mathcal{F} + (k_1 + k_2 - 1) \frac{\partial}{\partial S_{k_1+k_2}} \mathcal{F} =$$

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